Sequence of Bernoulli Trials:

The following is an explanation of the topic I was trying to get to on Friday (not a good day for me); I hope this will help to clarify what I was getting to in order to facilitate the Homework, which, I acknowledge is not all easy...

Bernoulli Trial: This is the simple, yet fundamental, piece of the puzzle. A Bernoulli Experiment is simply an action that has only two possible outcomes each time the experiment is run; we typically characterize these outcomes as either "S = success" or 'F = failure" with the following probabilities of occurrence:

$$Pr\{S\} = p \quad Pr\{F\} = q = (1 - p)$$

Clearly, the two outcomes: A and B are mutually exclusive and collectively exhaustive—ergo, the events partition the sample space, hence, the probabilities sum to unity. The second crucial aspect of Bernoulli trials is that the outcomes of a set of Bernoulli trials are collectively independent. That is, given the outcome of the first k experiments, the probability of each outcome remains the same for every subsequent experiment.

Now for the "so what" part of this—a single trial or experiment is rarely of much interest. What we are interested in are the collective results of a sequence, or set of these identical and independent experiments. Specifically, the most basic question we ask, and from which many difficult problems can be solved is the following:

What is the probability that outcome S occurs exactly k times in N trials?

This is the fundamental question—the answer requires the use of the *independence* of each trial, the probabilities of each outcome for a single trial, and *counting* the number of different ways S can possibly occur exactly k times in a sequence of N trials—this is where the *binomial coefficient* comes in. Before explaining the meaning of the binomial coefficient and why it fits this problem, let's try to understand why we would need to know more than just k and the probability of S.

Coin flipping provides a simple way to approach this—assume we have a fair coin, wherein, the probability of heads equals the probability of tails which equals $\frac{1}{2}$. The question we are interested in is: if we flip the coin three times, what is the probability that heads comes up exactly twice? The first mistake would be to assume that the order does not matter—it does—because the trials are all independent and each ordering of outcomes can be considered a distinct outcome over the set of trials. Lets go back to the coin flipping. Here are all the ways we get exactly two heads in three flips: {H,H,T}, {H,T,H} and {T,H,H}. Each of these represent a distinct outcome for the three trials, yet, with respect to our question they each satisfy the requirement, hence, must be counted in the probability computation.

Using the brute force method it is easy to determine the answer to the question. Here is how to do it: The probability of the first outcome—{H,H,T} is determined using the independence of each trial: $Pr\{H\}Pr\{H\}Pr\{T\} = \frac{1}{2}\frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{8}$. Similarly, the other two desired outcomes (orders) occur with probability $\frac{1}{8}$. Since these are mutually exclusive outcomes ({H,H,T} cannot occur at the same time as {T,H,H}) we can simply add the probabilities together. The final result is: Pr{2 Head in 3 Flips} = $\frac{3}{8}$.

In this case it was trivial to determine the number of ways to get two heads in three flips. Typically, the problem of counting the permutations is not as easy. The binomial coefficient provides the solution. It counts the precise number of times (or orders) that we will get k successes in N trials: N choose k:

$$\begin{pmatrix} N \\ k \end{pmatrix} = \frac{N!}{k!(N-k)!}$$

Given N Bernoulli trials (two possible outcome), the binomial coefficient tells us exactly how many ways one outcome will occur k times and the other occurs $(N-k)$ times. Now back to probability and the fundamental question: *What is the probability of k successes in N trials?* (Note: this implies $(N-k)$ failures!)... The answer is to first compute the probability of one way this could happen, as we did in the coin flip example. Lets assume that we got a success on the first k trials and a failure on the next $(N-k)$ trials (total N trials). By independence all we need to do is take the product of the probabilities for each trial: $p^k(1-p)^{N-k}$. However, since there are "N choose k" mutually exclusive distinct sequences to get k success all we need to do is multiply the probability of the "one way" by "the number of ways", hence the following fundamental result:

$$Pr\{k\ Successes\ in\ N\ Trials\} = \begin{pmatrix} N \\ k \end{pmatrix} p^k(1-p)^{N-k} = \frac{N!}{k!(N-k)!} p^k(1-p)^{N-k}$$

To show how this works let's return to the coin flipping problem. Here we have success defined as a toss of heads with $p = \frac{1}{2}$. We perform $N = 3$ trials and are looking for the probability of $k = 2$ successes. Plug in the numbers:

$$Pr\{2\ Heads\ in\ 3\ Flips\} = \frac{3!}{2!(3-2)!} \frac{1}{2}^2 (1-\frac{1}{2})^{3-2} = \frac{3}{8}$$

The result is the same as in the brute force method—as expected; hence the meaning and application is demonstrated via a simple example. Note that this gives the answer for *exactly* k successes.... In many practical problems we might be interested in *at least* or *no more than* k successes.

Problem 2.28:

Show that the events are independent: You are given two events, A and B with non-zero probabilities and the following condition:

$$Pr\{A|B\} + Pr\{\overline{A}|\overline{B}\} = 1$$

To prove they are independent you have to show that $Pr\{A|B\} = Pr\{A\}$. The most straight forward approach to achieve this requires using a couple of facts that come from the axioms of Probability and their extensions to the ideas of independence and total probability.

In this case we are going to need something to substitute into the given expression, so, start with the following observation:

$$Pr\{A|\overline{B}\} + Pr\{\overline{A}|\overline{B}\} = Pr\{\mathcal{S}|\overline{\mathcal{B}}\} = \infty$$

The above equation results from Axiom-2, Axiom-3 and Total Probability; Using this result and substitution with the given expression you can show that:

$$Pr\{A|B\} = Pr\{\overline{A}|\overline{B}$$

Problem #3:

Evaluate the following:

$$\sum_{n=1}^{n=\infty} np(1-p)^{n-1} \qquad where\ q = 1 - p$$

Problem #4:

Explain the concept of “probability” as you understand it—if it helps you may use an example. You may also extend the concept to include a mathematical model if that helps.