Expectation of a Binomial Random Variable

This is the example I was working in class—minus any 'typos'; unfortunately because it requires a couple of 'tricks' it tends to obscure the meaning of expectation, however, it does illustrate why one should committ the expected value and variance of common random variables to memory!! On the other hand, the 'tricks' tend to be common as well in solving a variety of probability problems. So here it goes.

Let's begin by defining X as a discrete random variable that is Binomially distributed [1]. Note that a Binomial random variable represents a sequence of n Bernoulli experiments, that is, an experiment with only 2 possible (disjoint) outcomes that is repeated n times. All the outcomes are pairwise and collectively independent of each other. The *parameters* of X are n and p, where n is the number of trials and p is the probability of success in any given trial. For any value of $i \leq n$ the Binomial (PMF) gives us the probability of k successful outcomes in n trials—note that it accounts for all possible ways (orders) that can occur:

$$Pr\{X = i\} = \left(\begin{array}{c} n \\ i \end{array} \right) p^i (1-p)^{n-i}$$

The mean or expected value of X is the weighted average number of successes in n trials, wherein, each possible number of successful outcomes is weighted by the probability of that number of successful outcomes. Hence, to compute the mean all we need to do is take the sum of the product of each possible number of successful outcomes, which range from 0 to n, and the probability of achieving that number of successful outcomes:

$$\begin{aligned}
E[X] &= \sum_{i=0}^{i=n} i Pr\{X = i\} = \sum_{i=0}^{i=n} i \left(\begin{array}{c} n \\ i \end{array} \right) p^i (1-p)^{n-i} && (1)\\
&= \sum_{i=1}^{i=n} \frac{in!}{(n-i)!i!} p^i (1-p)^{n-i} && (2)\\
&= \sum_{i=1}^{i=n} \frac{n!}{(n-i)!(i-1)!} p^i (1-p)^{n-i} && (3)\\
&= np \sum_{i=1}^{i=n} \frac{(n-1)!}{(n-i)!(i-1)!} p^{i-1} (1-p)^{n-i} && (4)\\
&= np \sum_{k=0}^{k=n-1} \frac{(n-1)!}{(n-1-k)!k!} p^k (1-p)^{n-1-k} && (5)\\
&= np \sum_{k=0}^{k=n-1} \left(\begin{array}{c} n-1 \\ k \end{array} \right) p^k (1-p)^{n-1-k} && (6)
\end{aligned}$$

[1] There is a generalization to the Binomial referred to as *Multinomial*, wherein the number of possible outcomes is k. The Binomial distribution is a special case of the Multinomial with $k = 2$

Explanation: the first step to solving this problem is shown in Equation-2 where the limits of the summation are changed ; this can be done because the case of $i = 0$ obviously comes to zero. In Equation-3 the i in the numerator cancels with the first term of the $i!$ in the denominator; thus the $(i-1)!$ in the denominator. The 'trick' used in Equation-4 is to pull np outside the summation, wherein, the n is taken from the $n!$ and the p is taken from the p^i terms in the numerator. Equation-5 is just a change of variables substitution, wherein, $k = i-1$. The fraction inside the summation is clearly $(n-1)$ choose k; hence the corresponding binomial coefficient is replaced in Equation-6.

For our purposes it is sufficient to apply a known solution to the remaining summation that is given by the binomial theorem. The 'known' result in general is the following:

$$\sum_{i=0}^{i=n} \left(\begin{array}{c} n \\ i \end{array} \right) p^i (1-p)^{n-i} = (p + (1-p))^n = 1$$

Note that this result *must* hold in order for the Binomial PMF to be a PMF. Why is this? Because the term in the above equation is precisely a Binomial PMF; since the sum of the probabilities over all possible outcomes of a discrete random variable (or the integral for the continuous case) must come to unity the result must be valid. Knowing this fact—one could complete the derivation of the expectation by observing that the $(n-1)$ term in Equation-6 is equivalent to the n term used above. Hence, we complete the solution for the expected value (first moment) of a Binomial random variable:

$$\begin{aligned} E[X] &= np \sum_{k=0}^{k=n-1} \left(\begin{array}{c} n-1 \\ k \end{array} \right) p^k (1-p)^{n-1-k} && (7) \\ &= np(p + (1-p))^{n-1} && (8) \\ &= np && (9) \end{aligned}$$

Expectation of a Geometric Random Variable

Define X as a random variable that is geomentrically distributed—the geometric distribution, like the binomial distribution, involves a sequence of indepedent trials with two disjoint outcomes. Each trial has a probability of success equal to p. However, for the geometric case we define X as being the number of trials that are performed until the first success occurs. Hence, the distribution of X gives the probability that it takes n trials before a success occurs. The only parameter for a geometric r.v. is p.

$$Pr\{X = n\} = (1-p)^{n-1}p \ , \quad n = 1, 2, \ldots$$

The distribution should be clear since the trials are independent (we can take the product of the individual probabilities) and there must be $n-1$ failures before a success if the first success occurs after n trials. Note that there is no need to consider different orders of outcomes since there is only one possible order, namely, $n-1$ failures, each with probability $1-p$, followed by a single success with probability p. To show that this is indeed a PMF take the sum over n from one to infinity. Since by power series expansion $\sum_{n=0}^{n=\infty} a^n = \frac{1}{1-a}$ if $|a| < 1$, and $|(1-p)| < 1$ we have:

$$\begin{aligned}
\sum_{n=1}^{n=\infty} (1-p)^{n-1}p &= p\sum_{n=1}^{n=\infty} (1-p)^{n-1} \\
&= p\sum_{n=0}^{n=\infty} (1-p)^{n} \\
&= p\frac{1}{(1-(1-p))} \\
&= 1
\end{aligned}$$

Next we derive the expected value of X:

$$\begin{aligned}
E[X] &= \sum_{n=1}^{n=\infty} np(1-p)^{n-1} && (10) \\
&= \sum_{n=1}^{n=\infty} npq^{n-1} && (11) \\
&= p\sum_{n=1}^{n=\infty} nq^{n-1} && (12) \\
&= p\sum_{n=1}^{n=\infty} \frac{d}{dq}q^{n} && (13) \\
&= p\frac{d}{dq}(\sum_{n=1}^{n=\infty} q^{n}) && (14) \\
&= p\frac{d}{dq}q(\sum_{n=0}^{n=\infty} q^{n}) && (15) \\
&= p\frac{d}{dq}\frac{q}{1-q} && (16) \\
&= \frac{p}{(1-q)}^{2} = \frac{1}{p} && (17)
\end{aligned}$$

Expectation of a Poisson Random Variable

The Poisson r.v. is very important with a wide range of applications. A random variable X is said to be Poisson distributed if it has the following discrete distribution with parameter $\lambda > 0$:

$$Pr\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!}\ , \quad n = 0, 1, 2, \ldots$$

To show that this function satisfies the requirements of a PMF the sum from zero to infinity must be equal to unity. Since $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{i=0}^{i=\infty} \frac{x^i}{i!}$ we have:

$$\begin{aligned}
\sum_{i=0}^{i=\infty} e^{-\lambda}\frac{\lambda^i}{i!} &= e^{-\lambda}\sum_{i=0}^{i=\infty}\frac{\lambda^i}{i!} \\
&= e^{-\lambda}e^{\lambda} \\
&= 1
\end{aligned}$$

Next we derive the expected value of X:

$$\begin{aligned}
E[X] &= \sum_{i=0}^{i=\infty}\frac{ie^{-\lambda}\lambda^i}{i!} && (18)\\
&= \sum_{i=1}^{i=\infty}\frac{ie^{-\lambda}\lambda^i}{i!} && (19)\\
&= \sum_{i=1}^{i=\infty}\frac{e^{-\lambda}\lambda^i}{(i-1)!} && (20)\\
&= \lambda e^{-\lambda}\sum_{i=1}^{i=\infty}\frac{\lambda^{i-1}}{(i-1)!} && (21)\\
&= \lambda e^{-\lambda}\sum_{k=0}^{i=\infty}\frac{\lambda^k}{(k)!} && (22)\\
&= \lambda e^{-\lambda}e^{\lambda} && (23)\\
&= \lambda && (24)
\end{aligned}$$

Expectation of an Exponential Random Variable

Let X be a continuously distributed r.v. with an exponential distribution which is given be the following PDF with one parameter λ:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} \ , & x \geq 0 \\ 0 \ , & x < 0 \end{cases}$$

The CDF of the exponential distribution is computed as follows:

$$\begin{aligned} F_X(a) &= \int_0^a \lambda e^{-\lambda x} dx \\ &= 1 - e^{-\lambda a} \ , \quad x \geq 0 \end{aligned}$$

To compute the expected value we must integrate by parts: (See solutions to the 'entrace quiz').