Important Parameters for Characterizing a Random Process

In the example we gave for the Gaussian Process we were able to completely determine the n-dimensional distributions. In many practical engineergin applications this is neither possible—nor is it necessary. The three most important parameters that help us characterize a random process are its:

- mean
- autocorrelation
- autocovariance

From our study of jointly distributed random variables it turns out that we already have the *tools* to understand and work with these parameters. Conceptually, they are basically the same as *correlation* and *covariance* as we've studied previously for jointly distributed random variables—the difference being that they are applied to the random variables from the same random process. The term 'auto' means 'self',hence, how are the random variables from the "collection of random variables" that we define as a random process, related to themselves?

For our purposes we shall consider a special (and practical) class of random processes: Define a *second-order random process* $X(t)$ as a random process that satisfies the following inequality:

$$E[X(t)^2] < \infty \quad \forall t$$

What does this tell us? Assume that $X(t)$ represents some physical signal (e.g. voltage). The value of $E[X(t)^2]$ is actually the expected value of the instantaneous power of that signal (one-ohm basis), equivalently, $\sqrt{E[X(t)^2]}$ is the rms value. For real systems it is reasonable for us to assume that the average power is indeed finite; hence we focus on *second-order* random processes.

MEAN: Define the mean of a second-order random process $X(t),\ t \in \mathcal{T}$ as follows:

$$\mu_X(t) = E[X(t)]$$

We can think of the mean of the random process as the average value (weighted in the 'expected value' sense) of the process at time=t;

AUTOCORRELATION: Define the autocorrelation of a second-order random process $X(t),\ t \in \mathcal{T}$ as follows:

$$R_X(t_1, t_2) = E[X(t_1), X(t_2)]$$

AUTOCOVARIANCE: Define the autocovariance of a second-order random process $X(t),\ t \in \mathcal{T}$ as follows:

$$C_X(t_1, t_2) = E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))]$$

The autocorrelation $R_X(t_1, t_2)$ and autocivariance $C_X(t_1, t_2)$ are quantitative measures of the statistical "coupling" between $X(t_1)$ and $X(t_2)$. Hence, the measure the relation between random variables of different time indeces drawn from the same random process. If $X(t_1)$ and $X(t_2)$ are independent then they will be uncorrelated, hence, $C_X(t_1, t_2) = 0$.

Note the similarity to the parameter we used to relate jointly distributed random variables... Note also that $R_X(t_1, t_2)$ and $C_X(t_1, t_2)$ are themselves random process with finite power; hence they too are second-order random processes. The reason we are interested in these parameters is that they play a crucial role in the analysis of linear systems and very important in the design of communications devices such as filters which much be able to detect desired signals in the presense of *random noise* and other random signals (e.g. the multipath signals we modeled in the random phasor problem).

Noise

Assume that the signal $X(t)$ as described above represents some form of "noise" in our system. That is, $X(t)$ has random qualities, it is unwanted in the sense that it makes it difficult to 'extract' the desired signal from the combined signal that contains the noise, and, since it is a second-order process we know what its power is (which is kind of cool)... Note that the autocorrelation here is taken for $t_1 = t_2 = t$—check out your expression for covariance and correlation and you'll see how this works out! The noise power is:

$$E[X(t)^2] = R_X(t, t)$$

Example

Find the mean and the the autocorrelation function of the random process $X(t) = \alpha sin(2\pi f_0 + \theta)$, where, α is a constant, f_0 is the frequency and is also a constant, and θ is the phase and is a random variable.

The difficulty of a problem like this lies mostly in manipulating the expressions via trigonometric identitites. First, let $\omega_0 = 2\pi f_0$.

$$X(t) = \alpha[sin(\theta)cos(\omega_0 t) + cos(\theta)sin(\omega_0 t)$$

The solution for the mean will depend on the random variable θ and t of course, hence, cannot be fully evaluated without knowing the distribution of θ (we consider an example below)

$$\mu_X(t) = \alpha(E[sin(\theta)]cos(\omega_0 t) + E[cos(\theta)]sin(\omega_0 t))$$

For the autocorrelation we fix two times: t_1 and t_2:

$$\begin{aligned} R_X(t_1, t_2) &= \alpha^2 E[sin(\omega_0 t_1 + \theta)sin(\omega_0 t_2 + \theta) \\ &= \frac{\alpha^2}{2}(E[cos(\omega_0(t_1 - t_2)) - cos(\omega_0(t_1 + t_2) + 2\theta)]) \end{aligned}$$

With some further manipulation it is found that:

$$R_X(t_1,t_2) = (\frac{\alpha^2}{2})(cos(\omega_0(t_1-t_2)) + E[sin(2\theta)]sin(\omega_0(t_1+t_2)) - E[cos(2\theta)]cos(\omega_0(t_1+t_2)))$$

Solution with a particular distribution for the phase angle: Let θ be distributed uniformly $(0, 2\pi)$. This simplifies things quite a bit!

$$E[sin(\theta)] = E[cos(\theta)] = E[sin(2\theta)] = E[cos(2\theta)] = 0$$

Thus, we have immediately that $\mu_X(t) = 0$ for any t, and the autocorrelation becomes:

$$R_X(t_1,t_2) = (\frac{\alpha^2}{2})cos(\omega_0(t_1-t_2))$$

If we let $t_1 = 0$ and $t_2 = T$, wherein T is the period of the signal, that is $T = 1/f_0$, then $\omega_0 T = 2\pi$ and we have:

$$R_X(0,T) = (\frac{\alpha^2}{2})cos(\omega_0 T) = \frac{\alpha^2}{2}$$

This is a nice result as it shows the average power of a sinusoid of amplitude α is $\alpha^2/2$.

Normalized Autocovariance

Continuing the previous example we saw that is $\theta \rightarrow U(0,2\pi)$, then, since $X(t)$ has zero mean $R_X(t_1,t_2) = C_X(t_1,t_2)$. Define the *normalized autocovariance* as follows, where, $\sigma_i = \sqrt{VAR[X_(t_i)]}$:

$$K_X(t_1,t_2) = \frac{C_X(t_1,t_2)}{\sigma_1\sigma_2}$$

Thus, for the example the normalize autocovarinace reduces to the following, which has a maximum value of unity as expected when comparing to the correlation coefficient:

$$K_X(t_1,t_2) = \frac{2R_X(t_1,t_2)}{\alpha^2} = cos(\omega_0(t_1-t_2))$$

Some Important Properties

Here we just briefy note some important properties of these parameters for any second-order process:

- $R_X(t,t) \geq 0$
- $R_X(t_1,t_2) = R_X(t_2,t_2)$
- $|R_X(t_1,t_2)| \leq sqrt R_X(t_1,t_1)R_X(t_2,t_2)$

The mean, autocorrelation and autocovariance are of particular importance for the Gaussian random process. In particular, it is possible to completely characterize a Gaussian process given its mean and autocovariance process. Recall that if $X(t)$ is a Gaussian process then

$$X(t) = [X(t_1), X(t_2), \ldots, X(t_n)]$$

Has an n-dimensional Gaussian density...