G205 Fundamentals of Computer Engineering

CLASS 18, Mon. Nov. 10 2003

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M-W, 9:50am-11:30am, 410 Ell

Shortest Paths

- How to find the shortest route between two points in a map
- INPUT:
  - A directed graph G=(V,E)
  - A weight function $w:E \rightarrow \mathbf{R}$
- Weight of path $p=<v_0, v_1,\ldots,v_k>$ is

$$w(p)=SUM(i=1,k)w(v_{i-1}, v_i)$$

Weight of a Shortest Path

- Shortest-path weight from u to v
  - $\delta(u,v)=\min\{w(p): p = u \leadsto v\}$ if there is a path from u to v
  - $\delta(u,v)=\infty$ otherwise
- Shortest-path from u to v
  - Any path $p = u \leadsto v$ with weight $w(p) = \delta(u,v)$

The Weight Function

- Can think of weights as something that:
  - Accumulate linearly along the path
  - We want to minimize
- Examples:
  - Times, costs, penalties, losses …
- Generalization of Breadth-First Search to weighted graphs

Variants, 1

- Single-Source Shortest Paths
  - Find a shortest path from a given source vertex s to every other vertices v
- Single-Destination Shortest Paths
  - Find a shortest path to a given destination vertex t from every other vertices v

Variants, 2

- Single-Pair Shortest Paths
  - Find a shortest path from a given vertex u and a given vertex v
- All-Pair Shortest-Paths
  - Find a shortest path from vertex u to vertex v for every pair of vertices u and v

Negative-weight Edges

- Ok, as long as no negative cycles are reachable from the source
- Negative cycle → we keep going around it obtaining $w(s \rightsquigarrow v) = -\infty$ for each v in the cycle
- Some algorithm do not tolerate negative-weight edges at all

Optimal Substructure

- Lemma: Every sub-path of a shortest path is a shortest path

  Proof: Assume $p = u \rightsquigarrow v$ is a shortest path such that $p = u \rightsquigarrow x \rightsquigarrow y \rightsquigarrow v$ and $w(p)=w(u \rightsquigarrow x)+w(x \rightsquigarrow y)+w(y \rightsquigarrow v)$.

  Now suppose that $x \rightsquigarrow y$ is a path shorter than $x \rightsquigarrow y$. Hence, $w(x \rightsquigarrow y)<w(x \rightsquigarrow y)$. But then

  $p'=u \rightsquigarrow x \rightsquigarrow y \rightsquigarrow v$ is shorter than p. A contradiction.

Cycles

- Shortest paths cannot contain cycles
  - Negative-weight cycles are already ruled out
  - Positive-weight → we can obtain a shortest path by omitting the cycle
  - Zero-weight. No reason to use them (this we will assume)

Single-Source Shortest Paths Output

- For $v \in V$ the output is $d[v] = \delta(s,v)$
  - Initially $d[v]=\infty$
  - Reduces as algorithm progresses, but always $d[v] \geq \delta(s,v)$
  - Call $d[v]$ a **shortest path estimate**
- $\pi[v]$=the predecessor of v in a path to s
  - If no predecessor, $\pi[v]$=NIL
  - π induces a tree—**Shortest-path tree**

Initialization

- All shortest-paths algorithms start with

```
Init-Single-Source(V,s)
  for each v ∈ V do
    d[v]=∞
    π[v]=NIL
   d[s]=0
```

Relaxation

- Can we improve the shortest-path estimated for v going through u and taking (u,v)?

```
Relax(u,v,w)
  if d[v]>d[u]+w(u,v)
    then    d[v]=d[u]+w(u,v)
            π[v]=u
```

Scheme for Single-Source Shortest-Paths Algorithms

- Start by calling Init-Single-Source
- Relax edges
- Different algorithms differ on
  - Number of relaxations
  - Order of relaxations
- Bellman-Ford
- Dijkstra

Shortest-Path Properties

- Based on calling Init-Single-Source once and Relax zero or more times
- Lemma: Triangle inequality

  For all $(u,v) \in E$: $\delta(s,v) \leq \delta(s,u) + w(u,v)$

  Proof: Weight of shortest path $s \rightsquigarrow v$ is $\leq$ weight of any path $s \rightsquigarrow v$. Path $s \rightsquigarrow u \rightarrow v$ is a path from s to v, and if $s \rightsquigarrow u$ is a shortest path its weigh is $\delta(s,u) + w(u,v)$

Upper-bound Property

- Lemma: Always have $d[v] \geq \delta(s,v)$ for all v. When $d[v] = \delta(s,v)$ it never changes.

  Proof: Initially true. Suppose v such that $d[v] < \delta(s,v)$, and wlog v is the first vertex for which this happens. Let u the vertex that updates d[v] to $d[u]+w(u,v)$. So …

Upper-bound Property, 2

$d[v] < \delta(s,v)$

$\leq \delta(s,u) + w(u,v)$ (triangle inequality)

$\leq d[u] + w(u,v)$ (v is first violation)

→ $d[v] < d[u] + w(u,v)$ which contradicts $d[v] = d[u] + w(u,v)$.

Once d[v] reaches $\delta(s,v)$, it never goes lower. It never goes up, since relaxations only lower estimates.

Other properties

- No-path property
  - If $\delta(s,v)=\infty$ then $d[v]=\infty$ always
- Convergence property
  - If $s \leadsto u \rightarrow v$ is a shortest path, $d[u]= \delta(s,u)$ and we call Relax(u,v,w) then $d[v]= \delta(s,v)$ afterward
- Path-relaxation property
  - Let $p=\langle v_0, v_1,\ldots v_k\rangle$ be a shortest path from $s=v_0$ to $v=v_k$. If we relax in order $(v_0,v_1),(v_1,v_2),\ldots,(v_{k-1},v_k)$ then $d[v_k]=\delta(s,v)$

Assignments

- Textbook, Chapter 24, pages 580—592
- Updated information on the class web page:

  www.ece.neu.edu/courses/eceg205/2003fa