G205 Fundamentals of Computer Engineering

CLASS 7, Mon. Sept. 29 2003

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M-W, 9:50am-11:30am, 410 Ell

MERGE SORT, 1

- Follows the D&C approach
- To sort A[p…r]:
  - Divide the elements of A into two subarrays A[p…q] and A[q+1…r]
  - Conquer by recursively sorting the two subarrays
  - Combine by merging the two sorted subarrays to produce the sorted A[p…r]
- Recursion bottoms out when the subarray has just one element

MERGE SORT, 2

```
Merge-Sort(A, p, r)
  if p < r                                   check the base case
    then q = int((p+r)/2)                    divide
         Merge-Sort(A, p, q)                 conquer
         Merge-Sort(A, q+1, r)               conquer
         Merge(A, p, q, r)                   combine
```

- Initial Call: Merge-Sort(A,1,n)

Analyzing D&C Algorithms

- We use recurrence equations
- Base case: problem size is small enough ($n \leq c$). Costs constant time $\Theta(1)$
- Recursive case:
  - Divide the problem into a subproblems each 1/b the size of the original
  - Let D(n) be the time to divide a n-size problem
  - Each subproblem costs T(n/b) → all cost aT(n/b)
  - Let C(n) be the time to combine solutions

Recurrence for D&C

$T_{D\&C}(n)=\Theta(1)$ if $n \leq c$

$T_{D\&C}(n)=aT_{D\&C}(n/b)+D(n)+C(n)$ otherwise

Analyzing Merge-Sort

- Base case: $n=1$ ($p \geq r$) → $T(1)$ in $\Theta(1)$
- When $n \geq 2$:
  - Divide: Compute q as the average of p and r → $D(n)$ in $\Theta(1)$
  - Conquer: Recursively solve two $n/2$-size subproblems → $2T(n/2)$
  - Combine: Merge on a n-element subarray takes $\Theta(n)$ → $C(n)$ in $\Theta(n)$

Recurrence for Merge-Sort

$T_{MS}(n)=\Theta(1)$ if $n = 1$

$T_{MS}(n)=2T_{MS}(n/2)+ \Theta(n)$ if $n > 1$

- By the MASTER THEOREM:

  $T_{MS}(n)$ is in $\Theta(n\log n)$

- Faster than IS and BS

Merge-Sort Recurrence

- Without the Master Theorem
- Rewrite the recurrence:
  - $T_{MS}(n)=c$ if $n = 1$
  - $T_{MS}(n)=2T_{MS}(n/2)+ c$ if $n > 1$
- Recursion Tree = successive expansion of the recurrence

Merge-Sort Recursion Tree

- Each level of the tree has cost cn
- There are log n + 1 levels
  - Prove it by induction
- Total cost is cn(log n +1) cn log n + cn
- $T_{MS}(n)$ is in $\Theta(n\log n)$ “<“ $O(n^2)$
- QUESTION:

HOW FAST CAN WE SORT?

Lower Bounds for Sorting

- Lower bound: A function or growth rate below which solving a problem is impossible
- A measure of how much has to be spent
- Natural lower bound for sorting: All elements must at least be read → $\Omega(n)$

Comparison-based Sorting

- The only operation that may be used to gain order information about a sequence is comparison of pairs of elements
- All sorts seen so far are comparison sorts: insertion sort, bubble sort, merge sort
- Other famous sorting algorithms are too: quicksort, heapsort, treesort

Decision Tree, 1

- Abstraction of any comparison sort
- Represents comparisons made by
  - a specific sorting algorithm
  - on inputs of a given size
- Abstracts away everything else: control and data movement
- We are counting *only* comparisons

Decision Tree, 2

- For any comparison-based sorting:
  - One tree for each n
  - The algorithm splits in two at each node, based on the information it has up to that point
  - The tree models all possible execution traces
- The length h of the longest root-leaf path:
  - Depends on the algorithm
    - Insertion sort: $\Theta(n^2)$
    - Merge sort: $\Theta(n \log n)$

Decision Tree, 3

- Lemma: Any binary tree of height *h* has $l \leq 2^h$ leaves (by induction)
- Theorem: *Any* decision tree that sorts n elements has height $\Omega(n \log n)$
- Proof
  - Every decision tree has $l \geq n!$ leaves (every permutation appears at least once)
  - By lemma, $n! \leq l \leq 2h$ or $2h \geq n! \rightarrow h \geq \log n!$
  - Stirling approximation: $n! \geq (n/e)^n \rightarrow h$ in $\Omega(n\log n)$

Lower Bound for Comparison-based Sorting

- The height of a decision tree indicates how many comparison at least have to be made to sort a sequence of n elements → lower bound for sorting
- Comparison-based sorting is in $\Omega(n\log n)$
- Merge-Sort is as good as it gets (asymptotically optimal)

Sorting in Linear Time

- We cannot go faster than $\Omega(n)$
- Must be a non-comparison sorting
- Works when assumptions on the number to be sorted are made
  - Counting sort → numbers in $\{0,1,\ldots,k\}$
  - Radix sort → numbers with a constant number of digits
  - Bucket sort → numbers drawn from a uniform distribution

Counting Sort, 1

- Numbers are integers in $\{0,1,\ldots,k\}$
- INPUT: $A[1\ldots n]$, $A[j] \in \{0,1,\ldots,k\}$ for all $j=1,2,\ldots,n$. Array A and values n and k are given as parameters
- OUTPUT: $B[1\ldots n]$, sorted. B is assumed to be already allocated and is given as a parameter
- Auxiliary storage: $C[0\ldots k]$

Counting Sort, 2

```
Counting-Sort(A,B,n,k)
  for i=0 to k do C[i] = 0
  for j=1 to n do C[A[ j ]]=C[A[j]]+1
  for i=1 to k do C[i]=C[i]+C[i-1]
  for j=n downto 1 do
    B[C[A[j]]]=A[j]
    C[A[j]]=C[A[j]]-1
```

Counting Sort, Example

- INPUT: A = $2_1, 5_1, 3_1, 0_1, 2_2, 3_2, 0_2, 3_3$
- OUTPUT: B = $0_1, 0_2, 2_1, 2_2, 3_1, 3_2, 3_3, 5_1$
- Counting-Sort is STABLE: keys with same value appear in same order in output as they did in input (because of how the last loop works)
- Analysis: $\Theta(n+k)$, which is $\Theta(n)$ if k is in $O(n)$

Radix Sort

- Key idea: Sort least significant digit of each number first
- To sort d digits:

```
Radix-Sort(A,d)
  for i = 1 to d do
    use a stable sorting to sort array A on
    digit i
```

Radix Sort, Example

329		720		720		329
457		355		329		355
657		436		436		436
839	→	457	→	839	→	457
436		657		355		657
720		329		457		720
355		839		657		839

Radix Sort: Correctness

- Induction on number of passes (i in pseudo-code)
- Assume digits 1, 2,…, i-1 are sorted
- Show that a stable sort on digit i leaves digits 1, 2, …, i sorted
  - If two digits in position i are different ordering by position i is correct (other digits are irrelevant)
  - If the digits are the same, numbers are already in the right order (ind. hyp.)

Radix Sort, Analysis

- Use Counting Sort as stable sorting
- $\Theta(n+k)$ per pass
- d passes
- $\Theta(d(n+k))$ total
- If k is in O(n) the $T_{RS}(n)$ is in $\Theta(dn)$
- When d is $\Theta(1)$ Radix Sort is linear time

How to break a number into digits

- n b-bits numbers
- Break into r-bits digits, have d=ceil(b/r)
- Use Counting Sort $k = 2^r - 1$
- $T_{RS}(n)$ is in $\Theta((b/r)(n+2^r))$
- Exercise: Choose r and compare Radix Sort and Merge-Sort

Assignments

- Textbook, pages 165—173
- Updated information on the class web page:

  www.ece.neu.edu/courses/eceg205/2003fa