

G205

Fundamentals of Computer Engineering

CLASS 19, Mon. Nov. 15 2004

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M-W, 1:30pm-3:10pm

Maximum Flow, 1 (a Light Introduction)

- ◆ Another problem that can be modeled through graphs
- ◆ Answers questions about “material flows” along a network
 - Is given a graph $G=(V,E)$, a source and a sink
 - Source produces material (steady rate)
 - Sink receives it (same rate)
- ◆ The “flow” is the rate at which the material moves

Maximum Flow, 2

- ◆ Each directed edge = **conduit** for the material to go through
- ◆ Given the rate, each conduit has a stated **capacity**
- ◆ Maximum flow problem = compute the greatest rate material can be sent from source to sink within the stated capacities

Flow Networks

- ◆ A flow network $G=(V,E)$ is a directed graph such that
 - Each $(u,v) \in E$ has a capacity $c(u,v) \geq 0$
 - If $(u,v) \notin E$ then $c(u,v) = 0$
 - There is a source s and a sink t
 - Every other vertex v is such that $s \rightsquigarrow v \rightsquigarrow t$
→ the network is connected and $|E| \geq |V| - 1$

Flows, 1

- ◆ A flow in a flow network G is a function $f: V \times V \rightarrow \mathbf{R}$ that satisfies:
- **Capacity constraints:** For all $u, v \in V$
 $f(u, v) \leq c(u, v)$
 - **Skew symmetry:** For all $u, v \in V$
 $f(u, v) = -f(v, u)$
 - **Flow conservation:** For all $u \in V \setminus \{s, t\}$
 $\text{SUM}(v \in V) f(u, v) = 0$

Flows, 2

- ◆ The quantity $f(u,v)$ can be $=, >$ or < 0 and is called the **flow from u to v**
- ◆ The **value of a flow f** is defined as $|f| = \text{SUM}(v \in V) f(s,v)$ (total flow out of the source)
- ◆ **Maximum flow** = Given a flow network G with source s and sink t we want to find a flow f of maximum value

The Ford-Fulkerson Method, 1

- ◆ Method \neq algorithm
- ◆ Three basic ideas:
 - Residual networks
 - Augmenting paths
 - Cuts
- ◆ Iterative method: Start with all $f(u,v)=0$ and at each iteration the flow is augmented

The Ford-Fulkerson Method, 2

Ford-Fulkerson-Method(G,s,t)

initialize flow to 0

while exists an augmenting path p do

 augment flow f along p

return f

Residual Networks, 1

◆ Given a flow network G and a flow f , a **residual network** consists of edges that can admit more flow

◆ Residual capacity of (u,v) :

$$c_f(u,v) = c(u,v) - f(u,v)$$

(It is the additional flow we can push from u to v before exceeding the capacity $c(u,v)$)

Residual Networks, 2

- ◆ Given a flow network G and a flow f the residual network of G induced by f is $G_f = (V, E_f)$ such that:
 - $E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$
 - Careful about E_f : it can be tricky ;-)
- ◆ Edges in E_f are either in E or their reversals

Augmenting Paths

- ◆ An augmenting path in a flow network G with flow f is a simple path from s to t in G_f
- ◆ We can always increase a flow from s to t along an augmenting path
- ◆ The **residual capacity** of an augmenting path p is the maximum amount by which we can increase the flow on each of its edges

$$c_f(p) = \min\{c_f(u,v) : (u,v) \text{ is on } p\}$$

Cuts of Flow Networks

- ◆ A **cut** (S, T) of a flow network $G=(V, E)$ is a partition of V into S and $T=V \setminus S$ such that $s \in S$ and $t \in T$
- ◆ If f is a flow, the net flow across the cut (S, T) is: $f(S, T) = \sum_{x \in S} \sum_{y \in T} f(x, y)$
- ◆ The capacity of a cut is $c(S, T) = \sum_{x \in S} \sum_{y \in T} c(x, y)$
- ◆ A **minimum cut** of a network is a cut whose capacity is minimum over *all* cuts of the network

Max-Flow Min-Cut Theorem

- ◆ If f is a flow in a flow network $G=(V,E)$ with source s and sink t , then the following conditions are equivalent
 1. f is a maximum flow in G
 2. G_f contains no augmenting paths
 3. $|f| = c(S,T)$ for some cut (S,T) of G

The Basic Ford-Fulkerson Algorithm

Ford-Fulkerson(G, s, t)

for each $(u, v) \in E$ do

$f[u, v] = f[v, u] = 0$

while there is a $p = s \rightsquigarrow t$ in G_f do

$c_f(p) = \min\{c_f(u, v) : (u, v) \text{ in } p\}$

for each (u, v) in p do

$f[u, v] = f[u, v] + c_f(p)$

$f[v, u] = -f[u, v]$

Analysis

- ◆ Depends on how an augmenting path is determined
- ◆ If arbitrarily chosen: $O(E f^*)$, where f^* is the output maximum flow
- ◆ An augmenting path can be found with a BFS (Edmund-Karp algorithm): $O(V E^2)$

Assignments

- ◆ Textbook, Chapter 26, pages 643—664
- ◆ Updated information on the class web page:

www.ece.neu.edu/courses/eceg205/2004fa