Hence

$$S_Z(f) = S_X(f) * S_T(f)$$

Problem 2-9

By definition, the autocorrelation function is

$$R_s(t_1, t_2) = \mathrm{E}[s(t_1, \underline{a}, T)s(t_2, \underline{a}, T)]$$

where the expectation is over $\underline{a}$ and T. For $|t_1 - t_2| > T_c$ the expectation over $\underline{a}$ gives zero, since t_1 and t_2 fall within different symbol intervals. For $|t_1 - t_2| \le T_c$, we obtain

$$R_s(t_1, t_2) = \mathrm{E}\left[\sum_{n=-\infty}^{\infty} a_n p(t_1 + T - nT_c) \sum_{m=-\infty}^{\infty} a_m p(t_2 + T - mT_c)\right]$$

$$= \mathrm{E}\left[\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_n a_m p(t_1 + T - nT_c)p(t_2 + T - mT_c)\right]$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathrm{E}[a_n a_m]\mathrm{E}[p(t_1 + T - nT_c)p(t_2 + T - mT_c)]$$

For $n \ne m$, $\mathrm{E}[a_n a_m] = 0$ which gives

$$R_s(t_1, t_2) = \sum_{m=-\infty}^{\infty} \mathrm{E}[p(t_1 + T - nT_c)p(t_2 + T - nT_c)]$$

For the waveform given, we are interested the value under the summation only for $0 \le T \le T_c$. Thus, there are only two terms of the summation of interest. Denote these by n_1 and $n_1 + 1$. Also, let $t_2 = t_1 + \tau$ with $0 \le \tau \le T_c$. The value of n_1 is such that $-T_c \le n_1T_c - t_1 \le 0$. Then

$$R_s(t_1, t_1 + \tau) = \frac{1}{T_c}\int_0^{T_c} p(t_1 + T - n_1T_c)p(t_1 + \tau + T - n_1T_c)dT$$

$$+ \frac{1}{T_c}\int_0^{T_c} p[t_1 + T - (n_1 + 1)T_c]p[t_1 + \tau + T - (n_1 + 1)T_c]dT$$

If $\tau \le t_1 - n_1T_c$, this can be simplified to

$$R_s(t_1, t_1 + \tau) = \frac{1}{2\pi} \int_{2\pi\tau/T_c}^{2\pi} \sin(x)\sin(x + 2\pi\tau/T_c)dx$$

$$= \frac{1}{2}\left(1 - \frac{\tau}{T_c}\right)\cos(2\pi\tau/T_c) - \frac{1}{8\pi}\left[\sin(2\pi\tau/T_c) - \sin(6\pi\tau/T_c)\right]$$

If $\tau > t_1 - n_1 T_c$, the result is identical. Since the autocorrelation function is even, it follows that

$$R_s(\tau) = \frac{1}{2}\left(1 - \frac{\tau}{T_c}\right)\cos(2\pi\tau/T_c) - \frac{1}{8\pi}\left[\sin(2\pi|\tau|/T_c) - \sin(6\pi|\tau|/T_c)\right], \; -T_c \le \tau \le T_c$$

with $R_s(\tau + T_c) = R_s(\tau)$. To obtain the power spectral density, take the Fourier transform.

Problem 2-10

(a) This is slow frequency hop because the data rate is 100 times faster than the hop rate. The FSK signals will be orthogonal if the hop spacing is a multiple of 2×10^6 Hz. Because the synthesizer is noncoherent hop-to-hop, and because of the slow frequency hop, the transmit signal power spectral density is the sum of the frequency translations of the power spectral density of the data modulated carrier. All frequency translations have equal weight if all synthesizer frequencies are equally likely. Let the power spectral density of the data modulated carrier be

$$S_d(f) = S_0(f - f_0) + S_0(f + f_0)$$

where f_0 is an arbitrary center frequency. The power spectral density of the transmitted signal is

$$S_t(f) = \frac{1}{64}\sum_{m=0}^{63}\left[S_0(f - f_0 - m\Delta f) + S_0(f + f_0 + m\Delta f)\right]$$

where $\Delta f = 2$ MHz.

(b) The error probability for binary noncoherent FSK in AWGN is

$$P_b = \frac{1}{2}e^{-E_b/2N_0}$$

Assume that the interference is due solely to a partial band jammer that jams a fraction ρ of the band with one-sided power spectral density $N_J = J/\rho B$ where J is the total power of the jammer and B is the transmission bandwidth. The average bit error probability is

Since the noise and the spreading code are independent, the power spectral density at the despreading mixer output is the convolution of these two separate spectra. Let $f_0 = f_1 + f_2$ where f_1 is the input center frequency and f_2 is the IF frequency. Due to the large processing gain, we are only interested in the magnitude of the convolution at $f = \pm f_2$. This gives the approximate result

$$S_{IF}(\pm f_2) \approx \frac{N_0 T_c}{2} \int_{-B/2}^{B/2} \text{sinc}^2(T_c u)\, du$$

For $B = 2/T_c$,

$$S_{IF}(\pm f_2) \approx 0.903 \frac{N_0}{2}$$

For $B = 3/T_c$ and $B = 3/T_c$, the results are $0.931N_0/2$ and $0.950N_0/2$, respectively.

Problem 2-13

The transmitted waveform is

$$x(t) = \sqrt{2P}\, d(t - T_0) c_1(t - T_1) c_2(t - \tau - T_1) \cos(\omega_0 t + \phi)$$

where T_0, T_1, and ϕ are random delays and phase included to make the processes stationary. c_1 and c_2 use the same random delay because they are clock synchronous. The autocorrelation function of the transmitted waveform is

$$\begin{aligned} R_X(\lambda) &= 2P E_{d,T_0}\{d(t - T_0) d(t - \lambda\, T_0)\} \\ &\times E_{c_1, c_2, T_1}\{c_1(t - T_1) c_1(t - \lambda - T_1) c_2(t - \tau - T_1) c_2(t - \tau - \lambda - T_1)\} \\ &\times E_\phi\{\cos(\omega_0 t + \phi) \cos[(\omega_0(t - \lambda) + \phi)]\} \\ &= P\Lambda(\lambda / T_d) R_{c_1 c_2}(\lambda) \cos(\omega_0 \lambda) \end{aligned}$$

where

$$\Lambda(u) = \begin{cases} 1 - |u|, & |u| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

and $R_{c1c2}(\lambda)$ is the autocorrelation function of the product of c_1 and c_2. This product can be viewed as a pair of time-interleaved pulse trains, the first with pulse width τ and the second with pulse width $T_c - \tau$. Both have periods T_c. The resultant autocorrelation function is

$$R_{c_1c_2}(\lambda) = \frac{\tau}{T_c}R_1(\lambda) + (1 - \tau/T_c)R_2(\lambda)$$

where $R_1(\lambda) = \Lambda(\lambda/\tau)$ and $R_2(\lambda) = \Lambda[\lambda/(T_c - \tau)]$. Thus, the complete autocorrelation function is

$$R_X(\lambda) = P\Lambda\left(\frac{\lambda}{T_d}\right)\left[\frac{\tau}{T_c}R_1(\lambda) + \left(1 - \frac{\tau}{T_c}\right)R_2(\lambda)\right]\cos(\omega_0\lambda)$$

The power spectral density is the Fourier transform of this.

Problem 2-14

During each signaling interval, a completely generic BPSK spread spectrum communication system would transmit

$$r_n(t) = s_m(t - nT_d)c(t),\ nT_d \le t \le (n+1)T_d,\ -\infty < m < \infty,\ m = 1,2,\cdots,M$$

where m is the message index and $s_m(t)$ is the data modulated signal waveform. The BPSK spreading waveform is $c(t)$ (it takes on the values ± 1 in T_c-second intervals). The minimum-probability-of-error receiver can be implemented in correlator or matched filter form with a correlator (matched filter) branch for each spread $s_m(t)$ followed by logic circuitry to choose the largest. The locally acquired spreading waveform can be multiplied with the incoming signal in each correlator branch or, equivalently, before the correlation with each $s_m(t)$.

Problem 2-15

The received signal is

$$r(t) = \sqrt{2P}\,d(t)c(t)\cos(\omega_0 t + \phi) + \sqrt{2J}\cos(\omega_0 t + \phi)$$

since the jammer is assumed coherent with the modulated signal. The integrator input is

$$[r(t)c(t)\times 2\cos(\omega_0 t + \phi)]_{LP} = \sqrt{2P}\left[d(t) + \sqrt{\frac{J}{P}}\,c(t)\right]$$

and the integrator output is

$$\sqrt{2P}\left[\pm 1 + \sqrt{\frac{J}{P}}\frac{1}{T}\int_0^T c(t)\,dt\right]$$

Write the spreading waveform as

$$c(t) = \sum_{n=-\infty}^{\infty} c_n p(t - nT_c)$$

where $c_n = \pm 1$ and $p(t)$ is a unit-amplitude square pulse of duration T_c. The integral then becomes

$$I = \frac{1}{N}\sum_{n=0}^{N} c_n$$

If the choice of the value of c_n is purely random, this sum is a binomially distributed random variable with distribution

$$P(I = 2k - N) = \binom{N}{k}\frac{1}{2^N},\ k = 0, 1, \cdots, N$$

Consider the normalized integrator output

$$X = \sqrt{\frac{J}{P}}\frac{I}{N}$$

It has probability distribution

$$P\left(IX = \frac{2k - N}{N}\sqrt{\frac{J}{P}}\right) = \binom{N}{k}\frac{1}{2^N},\ k = 0, 1, \cdots, N$$

When $d(t) = -1$ is transmitted, and error is made whenever $X > 1$. Thus

$$\Pr[\text{error}\,|\,d(t) = -1] = \sum_{k=\ell}^{N}\binom{N}{k}\frac{1}{2^N}$$

where ℓ is the smallest k for which

$$\frac{2k - N}{N}\sqrt{\frac{J}{P}} > 1$$

An identical result is found for $d(t) = 1$ so that the above expression is the average probability of error. For typical results, let $N = 7$ and 15 and $J/P = 0, 1, 2, \ldots 10$ dB. The results for ℓ and the probability of error are given in the table below.

J/P, dB	$N = 7$; ℓ	$N = 7$; P_e	$N = 15$; ℓ	$N = 15$; P_e
0	7	0.00781	15	3.052×10^{-5}
1	7	0.00781	15	3.052×10^{-5}
2	7	0.00781	14	4.883×10^{-4}
3	6	0.06250	13	0.00369
4	6	0.06250	13	0.00369
5	6	0.06250	12	0.01758
6	6	0.06250	12	0.01758
7	6	0.06250	11	0.05923
8	5	0.22656	11	0.05923
9	5	0.22656	11	0.05923
10	5	0.22656	10	0.15088