$$E(V) = n\left(1 + \frac{P}{N_0 B}\right)$$

For noise alone, the expected value is $E\{V\} = n = 80$. For signal plus noise, use the signal-to-noise ratio calculated in Problem 5-10, which gives

$$E\{V\} = 80\,(3.84 + 1) = 387$$

The variances for noise alone and signal plus noise cases are given by (5-85):

$$\sigma_V^2 = 2n\left[2\left(\frac{P}{N_0 B}\right) + 1\right] = \begin{cases} 160 \text{ for noise alone} \\ 1389, \text{ for signal plus noise} \end{cases}$$

Finally,

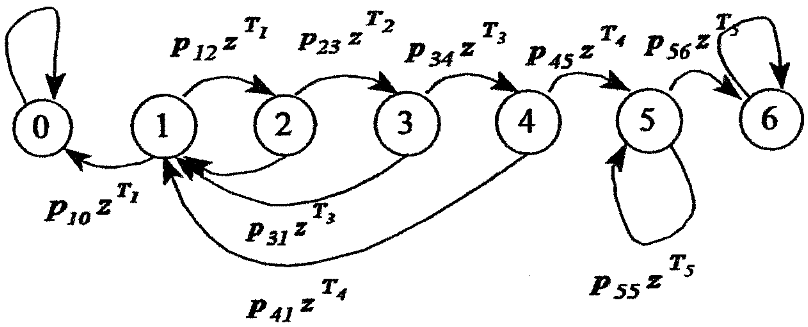
$$P_{fa} = Q\left(\frac{V_T - 6.6\times 10^{-14}}{3.05\times 10^{-14}}\right) = Q\left(\frac{V_T - 80}{12.65}\right)$$

$$P_d = Q\left(\frac{V_T - 3.176\times 10^{-13}}{3.05\times 10^{-14}}\right) = Q\left(\frac{V_T - 387}{37.3}\right)$$

The Q-function can be calculated using a rational approximation (see Abramowitz and Stegun or Appendix B). Using trial and error or a solve block in Mathcad, V_T can be found to achieve $P_{fa} = 10^{-3}$. The result is $V_T = 119.1$ V. Using this threshold, $P_d > 0.999$.

Problem 5-12

By inspection, the state diagram is as shown as below:



The transition matrix is given below:

state	0	6	1	2	3	4	5
0	1.0	0	0	0	0	0	0
6	0	1.0	0	0	0	0	0
1	$p_{10}z^{T_1}$	0	0	$p_{12}z^{T_1}$	0	0	0
2	0	0	$p_{21}z^{T_2}$	0	$p_{23}z^{T_2}$	0	0
3	0	0	$p_{31}z^{T_3}$	0	0	$p_{34}z^{T_3}$	0
4	0	0	$p_{41}z^{T_4}$	0	0	0	$p_{45}z^{T_4}$
5	0	$p_{56}z^{T_5}$	0	0	0	0	$p_{55}z^{T_5}$

Problem 5-13

(a) The transition matrix is:

state	0	4	1	2	3
0	1.0	0	0	0	0
4	0	1.0	0	0	0
1	$p_{10}z^{T_1}$	0	0	$p_{12}z^{T_1}$	0
2	0	0	$p_{21}z^{T_2}$	0	$p_{23}z^{T_2}$
3	0	$p_{34}z^{T_3}$	0	0	$p_{33}z^{T_3}$

The remaining matrices for solving the problem are:

$$Q = \begin{bmatrix} 0 & p_{12}z^{T_1} & 0 \\ p_{21}z^{T_2} & 0 & p_{23}z^{T_2} \\ 0 & 0 & p_{33}z^{T_3} \end{bmatrix}; \quad R = \begin{bmatrix} p_{10}z^{T_1} & 0 \\ 0 & 0 \\ 0 & p_{34}z^{T_3} \end{bmatrix}; \quad T = \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{bmatrix}$$

Use (5-113) to find T_{da}:

$$(I - Q) = \begin{bmatrix} 1.0 & -p_{12}z^{T_1} & 0 \\ -p_{21}z^{T_2} & 1.0 & -p_{23}z^{T_2} \\ 0 & 0 & 1 - p_{33}z^{T_3} \end{bmatrix}$$

The inverse of this matrix is

$$(I - Q)^{-1} = \frac{1}{D}\begin{bmatrix} 1 - p_{33}z^{T_3} & p_{12}z^{T_1}(1 - p_{33}z^{T_3}) & p_{12}p_{23}z^{T_1+T_2} \\ p_{21}z^{T_2}(1 - p_{33}z^{T_3}) & (1 - p_{33}z^{T_3}) & p_{23}z^{T_2} \\ 0 & 0 & 1 - p_{21}p_{12}z^{T_1+T_2} \end{bmatrix}$$

where

$$D = (1 - p_{33}z^{T_3})(1 - p_{12}p_{21}z^{T_1+T_2})$$

Continuing,

$$(I - Q)^{-1}T = \frac{1}{D}\begin{bmatrix} (1 - p_{33}z^{T_3})T_1 & p_{12}z^{T_1}(1 - p_{33}z^{T_3})T_3 & p_{12}p_{23}z^{T_1+T_2}T_3 \\ p_{21}z^{T_2}(1 - p_{33}z^{T_3})T_1 & (1 - p_{33}z^{T_3})T_2 & p_{23}z^{T_2}T_3 \\ 0 & 0 & (1 - p_{21}p_{12}z^{T_1+T_2})T_3 \end{bmatrix}$$

To simplify things, set $z = 1$; the matrix giving the results is:

$$[(I - Q)^{-1}T(I - Q)^{-1}R]_{z=1}$$

$$= \frac{1}{E}\begin{bmatrix} p_{10}(1 - p_{33})^2(T_1 + p_{21}p_{12}T_2) & p_{34}p_{12}p_{23}[(1 - p_{33})(T_1 + T_2) + (1 - p_{12}p_{21})T_3 \\ p_{10}p_{21}(1 - p_{33})^2(T_1 + T_2) & p_{34}p_{12}p_{23}p_{21}(1 - p_{33})T_1 + p_{34}p_{23}(1 - p_{33})T_2 + p_{34}p_{23}(1 - p_{12}p_{21})T_3 \\ 0 & p_{34}(1 - p_{12}p_{21})^2T_3 \end{bmatrix}$$

where

$$E = (1 - p_{33})^2(1 - p_{12}p_{21})^2$$

The desired result is the sum of the elements in the first row of the above matrix. Note that p_{ij} is evaluated with signal energy set equal to zero.

$$T_{da} = \frac{p_{10}(T_1 + p_{21}p_{12}T_2)}{(1 - p_{12}p_{21})^2} + \frac{p_{12}p_{23}p_{34}(T_1 + T_2)}{(1 - p_{33})(1 - p_{12}p_{21})^2} + \frac{p_{12}p_{23}p_{34}T_3}{(1 - p_{33})^2(1 - p_{12}p_{21})}$$

Use (5-115) to find P_d:

$$(I - Q)^{-1}R = \frac{1}{(1 - p_{33})(1 - p_{12}p_{21})}\begin{bmatrix} p_{10}(1 - p_{33}) & p_{34}p_{12}p_{23} \\ p_{10}p_{21}(1 - p_{33}) & p_{23}p_{34} \\ 0 & p_{34}(1 - p_{12}p_{21}) \end{bmatrix}$$

P_d is the element on the first row and second column with the transition probabilities evaluated for signal plus noise inputs. Thus

$$P_d = \frac{p_{12}p_{23}p_{34}}{(1 - p_{33})(1 - p_{12}p_{21})} = \frac{p_{12}p_{23}}{1 - p_{12}p_{21}}$$

(b) The state transition matrix is

$$Q' = \begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 \\ p_{10}z^{T_1} & 0 & 0 & p_{12}z^{T_1} & 0 \\ p_{20}z^{T_2} & 0 & 0 & 0 & p_{23}z^{T_2} \\ 0 & p_{34}z^{T_3} & 0 & 0 & p_{33}z^{T_3} \end{bmatrix}$$

The procedure is identical to that of part (a). The results are

$$T_{da} = p_{10}T_1 + p_{12}p_{20}(T_1 + T_2) + \frac{p_{12}p_{23}p_{34}(T_1 + T_2)}{(1 - p_{33})} + \frac{p_{12}p_{23}p_{34}T_3}{(1 - p_{33})^2}$$

and

$$P_d = \frac{p_{12}p_{23}p_{34}}{1 - p_{33}} = p_{12}p_{23}$$

Problem 5-14

Use the equation derived in Problem 5-13b for T_{da}, where all transition probabilities are calculated with noise alone at the input of the bandpass filter in front of the energy detector. The integration times are: $T_1 = 100/5000 = 20$ ms; $T_2 = 100/5000 = 20$ ms; $T_3 = 200/5000 = 40$ ms. Because the sum of the transition probabilities leaving any state is unity, it follows that $p_{10}' = 0.9$ and $p_{20}' = 0.99$. We must calculate p_{33}' to have enough information to find T_{da}. For this case, the time-bandwidth product is high so that the Gaussian approximation is valid. The threshold is calculated using given information about p_{33} with signal plus noise. The mean and variance of the integrator output must be calculated. They are:

$$E\{V\} = \begin{cases} n = 2BT_i = 400\text{, noise alone} \\ n\left(1 + \frac{C}{N_0B}\right) = 526.5\text{, signal plus noise} \end{cases}$$

and

$$\sigma_V^2 = \begin{cases} 2n = 800\text{, noise alone} \\ 2n\left(1 + 2\frac{C}{N_0B}\right) = 800\,(1 + 2\times 0.316) = 1305.6\text{, signal plus noise} \end{cases}$$

where we have used $C/N_0 = 32$ dB-Hz which implies that $C/N = -5$ dB.

Then

$$p_{33} = 0.999 = Q\left(\frac{V_T - E\{V\}}{\sigma_V}\right) = Q\left(\frac{V_T - 526.5}{36.16}\right)$$

Solve by trial and error or with a "solve" block in Mathcad to get $V_T = 414.5$. Using this threshold and the noise alone mean and variance calculated above, we find that

$$p'_{33} = 0.304 \text{ and } p'_{34} = 0.696$$

Then

$$T_{da} = (0.9)(20 \text{ ms}) + (0.1)(0.99)(40 \text{ ms}) + \frac{(0.1)(0.01)(0.696)}{1 - 0.304}(40 \text{ ms}) = 22 \text{ ms}$$

The probability of detection calculation requires p_{12} and p_{23} with signal plus noise present. The time-bandwidth products are given. For $B_N T_i = 100$,

$$E\{V\} = \begin{cases} 200, \text{ noise alone} \\ 263.3, \text{ signal plus noise} \end{cases}$$

and

$$\sigma_V^2 = \begin{cases} 400, \text{ signal plus noise} \\ 652.8, \text{ noise alone} \end{cases}$$

Using the Gaussian approximation, we find

$$p'_{12} = 0.1 = Q\left(\frac{V_T - 200}{20}\right) \rightarrow V_T = 225.6$$

Then

$$p_{12} = Q\left(\frac{225.6 - 263.3}{25.55}\right) = 0.93$$

Similarly,

$$p'_{23} = 0.01 = Q\left(\frac{V_T - 200}{20}\right) \rightarrow V_T = 246.5$$

and

$$P_{23} = Q\left(\frac{246.5 - 263.3}{25.55}\right) = 0.74$$

Thus

$$P_d = P_{12}P_{23} = (0.93)(0.74) = 0.69$$

Assume 1/2 chip steps. Then there are $2(2^{13} - 1) = 16{,}382$ steps. The average time to synchronize is given by (5-12), and is

$$\overline{T_s} = (C - 1)T_{da}\left(\frac{2 - P_d}{2P_d}\right) + \frac{T_i}{P_d} = (16{,}382 - 1)(22\ \text{ms})\left(\frac{2 - 0.692}{2(0.692)}\right) = 341\ \text{sec}$$

In the above calculation, the second term is small compared with the first term and has been neglected in the calculation.

Problem 5-15

Calculate the average time required to dismiss an incorrect phase cell using a sequential detector. Assume that the signal-to-noise ratio is low enough so that excess over boundaries can be ignored. In this case, the average dwell time on an incorrect phase cell is given by (5-164):

$$T_{da} = -2\left(\frac{N}{P}\right)^2 [P_{fa} \ln A + (1 - P_{fa}) \ln B] T_s + T_{fa} P_{fa}$$

where A and B are the thresholds. They are given by

$$A = \frac{P_d}{P_{fa}} \text{ and } B = \frac{1 - P_d}{1 - P_{fa}}$$

respectively. The signal-to-noise ratio is

$$\left(\frac{P}{N}\right)_{dB} = 32 - 10\log(B_N) = -5\ \text{dB or } \frac{P}{N} = 0.316$$

The minimum sampling interval possible to obtain independent samples is $T_s = 1/B_N = 200$ μs. Assume that the false alarm penalty is negligible. Then

$$T_{da} = -\frac{2(200\ \mu\text{s})}{(0.316)^2}\left[10^{-3}\ln\left(\frac{0.9}{10^{-3}}\right) + (1 - 10^{-3})\ln\left(\frac{1 - 0.9}{1 - 10^{-3}}\right)\right] = 9.18\ \text{ms}$$