ECEU574 Wireless Communication Circuits — Spring, 04

Assignment 5 — Prof. Brady

1. Transatlantic Cable. The transatlantic cable was laid in approximately 1865, and had a length $l = 3600km$. The following transmission line parameters hold for the cable over the frequency range of interest (10-1000Hz): $L = 460nHm^{-1},\ C = 75pFm^{-1},\ R = 0.007\Omega m^{-1}$, and $0 < G << C$.

a. Heaviside's Fix. Determine whether or not the cable may be considered to have no dispersion, using Heaviside's condition. Does the condition hold? Show your work.

Solution. Heaviside's fix requires that $\frac{R}{L} = \frac{G}{C}$. Since $\frac{R}{L} = 15 \times 10^3$ and $\frac{G}{C} \approx 0$ by assumption, Heaviside's fix does not apply.

b. High-Reactance Approximation. Determine whether or not the cable may be considered to have no dispersion using the high-reactance approximation. Does the high-reactance condition hold? Show your work.

Solution. Since $\frac{R}{L} = 15^3$, the transmission line is highly resistive, not reactive.

c. Attenuation and phase functions. Start with the expression for jk from the notes, and determine approximate formulas for its real part (attenuation α) and imaginary part ('phase constant' β). Use the fact that $0 < G << C$, and $\sqrt{\frac{r}{j}} = \sqrt{r}(1-j)$ for any real number r. Does the term 'phase constant' apply to the expression for β?

Solution. Since Heaviside's fix and the high-reactance approximation both do not apply, we must assume that we have a dispersive transmission line. Beginning with the general formula for jk

$$\begin{aligned} jk &= j\omega\sqrt{LC}\sqrt{\left(1+\frac{R}{j\omega L}\right)\left(1+\frac{G}{j\omega C}\right)}, \\ &\approx (1+j)\sqrt{\omega RC}, \\ &= \alpha + j\beta \ \ (Using\ hint\ in\ the\ homework). \end{aligned} \tag{1}$$

This method uses the hint in the homework, which is less accurate than the method in the book, for two reasons. First, I had a typo in the hint (replace $\sqrt{r}$ by $\sqrt{r/2}$). Second, the hint assumes that $\frac{R}{j\omega L} >> 1$, which is not quite true at $1000Hz$. A more accurate method is given in the book, and uses the approximation $\sqrt{z} \approx \sqrt{z_0} + \frac{z-z_0}{2\sqrt{z_0}}$. Further, the book uses $\sqrt{\frac{r}{j}} = \sqrt{\frac{r}{2}}(1-j)$. This second approach works well when z_0 is close to z. Here, we take $z_0 = \frac{R}{j\omega L}$ and $z = z_0 + 1$.

$$\begin{aligned} \alpha &= \omega\sqrt{LC}\left(\sqrt{\frac{R}{j\omega L}} + \frac{1}{2}\sqrt{\frac{j\omega L}{R}}\right), \\ &\approx \sqrt{\frac{\omega RC}{2}} \\ \beta &= \omega\sqrt{LC}\left(\sqrt{\frac{R}{j\omega L}} - \frac{1}{2}\sqrt{\frac{j\omega L}{R}}\right), \\ &\approx \sqrt{\frac{\omega RC}{2}} (More\ accurate\ approach). \end{aligned} \tag{2}$$

Due to the typo, I'll take either answer. Clearly, the 'phase constant' is not constant here, but depends on ω.

d. Phase velocity. Determine an expression for the phase velocity v.

Solution. The phase velocity is computed as

$$\begin{aligned} v &= \tfrac{\omega}{\beta}, \\ &= \sqrt{\tfrac{\omega}{RC}}(Using\ homework\ hint.) \end{aligned} \tag{3}$$

or

$$\begin{aligned} v &= \tfrac{\omega}{\beta}, \\ &= \sqrt{\tfrac{2\omega}{RC}}(Using\ method\ in\ book.) \end{aligned} \tag{4}$$

e. Total attenuation. Determine an expression for the total attenuation (over the entire cable length). What is the relative attenuation of a signal at $1kHz$ relative to one at $10Hz$ (in dB)?

Solution. The total attenuation is given by

$$\begin{aligned} attenuation &= \mathrm{e}^{-\alpha l} \\ &= \mathrm{e}^{-\sqrt{\omega RC}l}, (homework\ hint), \\ &= \mathrm{e}^{-\sqrt{\frac{\omega RC}{2}}l}, (method\ in\ the\ book), \end{aligned} \tag{5}$$

The relative attenuation at 1000 Hz versus 10 Hz is

$$\begin{aligned} relative\ attenuation &= \mathrm{e}^{-\sqrt{2\pi RC}(\sqrt{1000}-\sqrt{10})l}, (homework\ hint) \\ &= -\log_{10}(\mathrm{e})20\sqrt{2\pi RC}(\sqrt{1000}-\sqrt{10})l(homework\ hint, dB), \\ &= -31,447dB(homework\ hint), \\ &= \mathrm{e}^{-\sqrt{RC\pi}(\sqrt{1000}-\sqrt{10})l}, (method\ in\ the\ book), \\ &= -\log_{10}(\mathrm{e})20\sqrt{RC\pi}(\sqrt{1000}-\sqrt{10})l, (method\ in\ the\ book, dB), \\ &= -22,236dB(method\ in\ the\ book, dB). \end{aligned} \tag{6}$$

Either way, the signal components near 1000 Hz are strongly suppressed by the cable, relative to those near 10 Hz. Again, either calculation will be given full credit.

f. Ideal phase delay. The ideal phase transfer function (radians versus frequency) of any linear time-invariant system should be linear in f. How does the phase delay (seconds versus frequency) of the same system vary with f?

Solution. The ideal transmission line has a constant phase delay with f, so we should expect that the ideal system has a constant phase delay. If the ideal system is *real*, then the phase at $f = 0$ is zero, and the phase has the form af for some real a. Phase delay has units of *seconds*, while phase has units of *radians*. It is clear that to convert from one to the other, we must multiply the phase with something having units of *seconds / radian*. That 'something' is reciprocal ω.

$$phase\ delay = phase/\omega = af/\omega = a/(2\pi).$$

As this suggests, an ideal transmission line (constant phase delay) can be modeled as an ideal linear time invariant system with linear phase response.

g. Actual phase delay. The expression for the total phase delay of the transatlantic cable is $\theta = \frac{l}{v}$, where v is the phase velocity from part d, and l is the cable length. How far from ideal is the transatlantic cable at $1kHz$, using $10Hz$ as the reference? Express your answer as a percentage.

Solution. The higher frequency components lead the lower frequency components.

$$\begin{aligned} relative\ phase\ delay &= \frac{v(10)}{v(1000)} - 1, \\ &= \frac{\sqrt{10}}{\sqrt{1000}} - 1, \\ &= -90\%. \end{aligned} \tag{7}$$

The difference in phase delay is approximately 90% of the phase delay at $10Hz$. The negative sign indicates that the high frequency components arrive sooner than the lower frequency components.

h. Cable repair. Engineer a solution for the dispersion and distortion of the transatlantic cable which could be implemented prior to deployment, based on class notes. You may use only passive components. Support your proposal with calculations. Specify the implementation, and the resulting expressions for attenuation and phase delay.

Solution. Both Heaviside's solution and the high-inductance approximation required adding discrete inductors occasionally along the transmission line, so as to increase the total inductance per meter, L_{net}. For both the Heaviside fix and the high-inductance approximation, we would require $\frac{L_{net}}{R} > \omega$, for all ω in the range of interest. If we choose $\frac{L_{net}}{R} = K$, for example, then we must add $KR - L$ inductance per meter. If we insert an inductor every d meters, then the value of the discrete inductor we must insert is $d(KR - L)$. A correct answer is any combination of $d > 0$ and $K > 2 * \pi \times 1000$ with a discrete inductance of $d(KR - L)$. With line conditioning, the expressions for attenuation and phase delay become

$$\begin{aligned} & High\ reactance\ method: \\ Z_0 &= \sqrt{\frac{L_{net}}{C}}, \\ attenuation &= -8.686 \times 20 \times l \times \frac{R}{2Z_0}\ dB, \\ phase\ delay &= l \times \sqrt{L_{net}C}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} & Heaviside\ method: \\ Z_0 &= \sqrt{\frac{L_{net}}{C}}, \\ attenuation &= -8.686 \times 20 \times l \times \frac{R}{Z_0}\ dB, \\ phase\ delay &= l \times \sqrt{L_{net}C}. \end{aligned} \tag{9}$$

Note that as L_{net} increases, the characteristic impedance increases, the attenuation decreases, while the phase delay increases. From a practical perspective, one should not choose too high of a value of K (or L_{net}), as phase delay may be unacceptably high.