

ECEG105/ECEU646  
Optics for Engineers  
Course Notes  
Part 5: Polarization  
Prof. Charles A. DiMarzio  
Northeastern University  
Fall 2008

# Wave Nature of Light

- Failure of Raytracing
  - Zero- $\lambda$  Approximation
  - Incoherent Process
- Empirical Wave Thy.
  - Huygens' Principle
    - Derives Geom. Optics
    - Backward Wave
  - Fresnel: Sum Fields
- Maxwell's Equations
  - Vector Wave Equation
  - Polarization
  - Scalar Wave Field
  - Fresnel-Kirchoff Integral
    - No Backward Wave

# Some Wave Phenomena

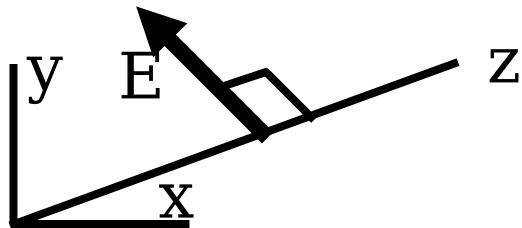
- Diffraction
  - Limiting Spot Size
  - Slits
  - Gratings
  - Fourier Optics
- Waveguides & Fibers
- Interference
  - Interferometry
  - Beat Signals
- Fabry-Perot Resonator
- Thin Films
- Polarization
  - Birefringence
  - Partial Polarization
  - E/O Effect
- Non-Linear Optics
  - Frequency Multiplying
  - Wave Mixing

# Remaining Topics

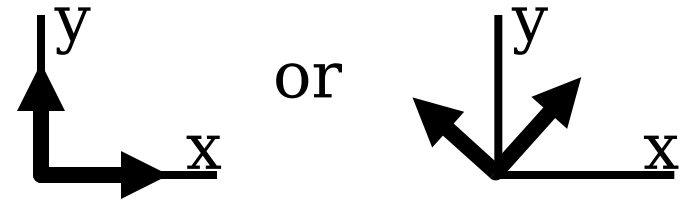
- Polarized Light
  - Jones, Mueller, Coherency Matrices
  - Poincare Sphere
  - Fresnel Reflection
- Interference
  - Interferometers, Optical Testing, Laser Cavities, Thin Films, Doppler Laser Radar
- Diffraction
  - Kirchoff Integral, Fraunhofer, Fresnel
  - Gratings, Gaussian Beams, Fourier Optics
- Radiometry
- Materials
  - Birefringence
  - Non-Linearity

# Polarized Light

- Starting Point
  - Maxwell's Equations
  - Homogeneous Medium
  - Isotropic Medium
- Solution
  - Plane Waves
  - Transverse Fields



- Eigenstates



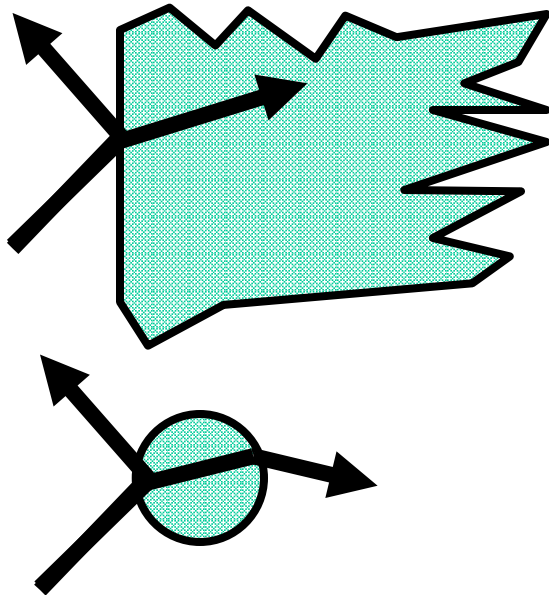
- General Solution

$$\mathbf{E} = \left( E_x \hat{\mathbf{E}}_x + E_y \hat{\mathbf{E}}_y \right) e^{i(\omega t - kz)}$$

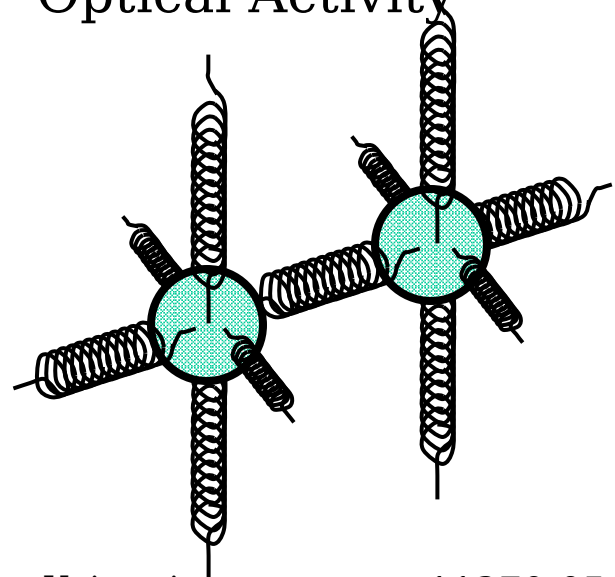
$E_x, E_y$  Complex

# Some Mechanisms of Interaction

- Dielectric Interface
  - Oblique Incidence
  - Scattering



- Material Properties
  - Linear Birefringence
  - Circular Birefringence, AKA Optical Activity



# Linear Polarization

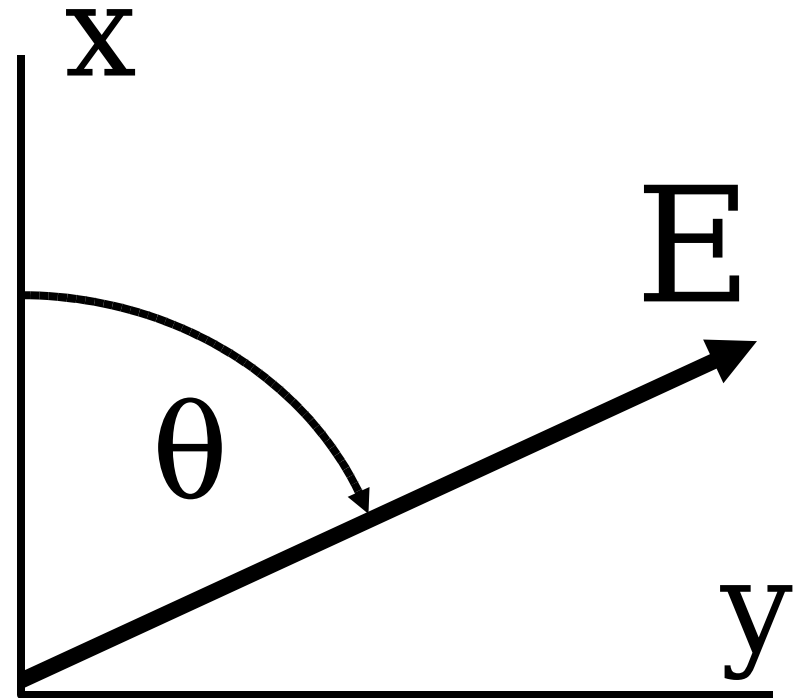
Components

$$E_x = E_0 \cos \theta \quad E_y = E_0 \sin \theta$$

Specific Example

$$z = 0$$

$$\mathbf{E} = E_0 \left( \hat{\mathbf{E}}_x \cos \theta + \hat{\mathbf{E}}_y \sin \theta \right) e^{i\omega t} = E_0 \hat{\mathbf{E}}' e^{i\omega t}$$



# Circular Polarization

## Components

$$E_x = E_0 \quad E_y = iE_0$$

## Specific Example

$$z = 0 \quad \mathbf{E} = E_0 (\hat{\mathbf{E}}_x + i\hat{\mathbf{E}}_y) e^{i\omega t}$$

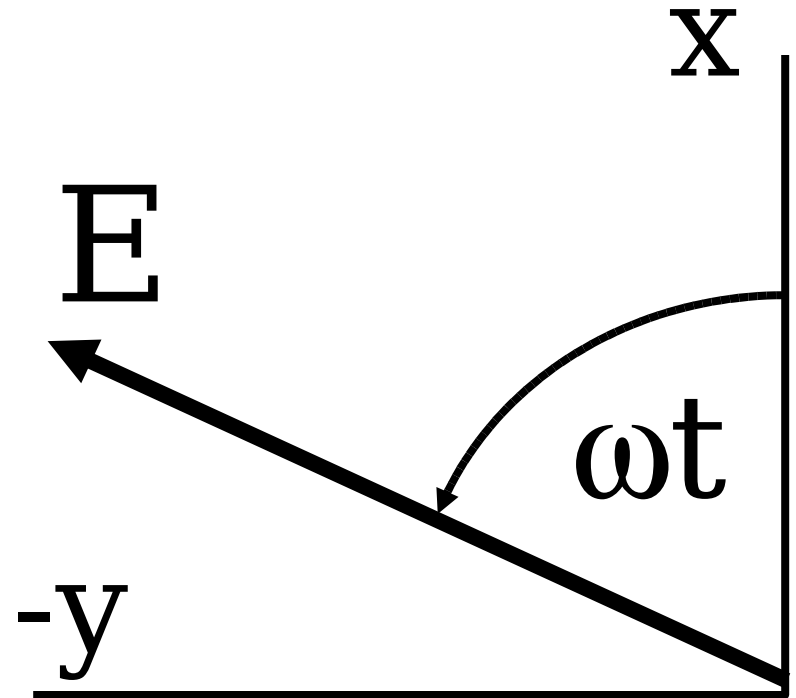
## Actual Field

$$\mathcal{E} = \Re \mathbf{E} = E_0 \hat{\mathbf{E}}_x \cos \omega t - E_0 \hat{\mathbf{E}}_y \sin \omega t$$

Vector Magnitude  $= E_0$

Vector Direction  $= -\omega t$

Name = Left-Hand Circular



# Unpolarized Light

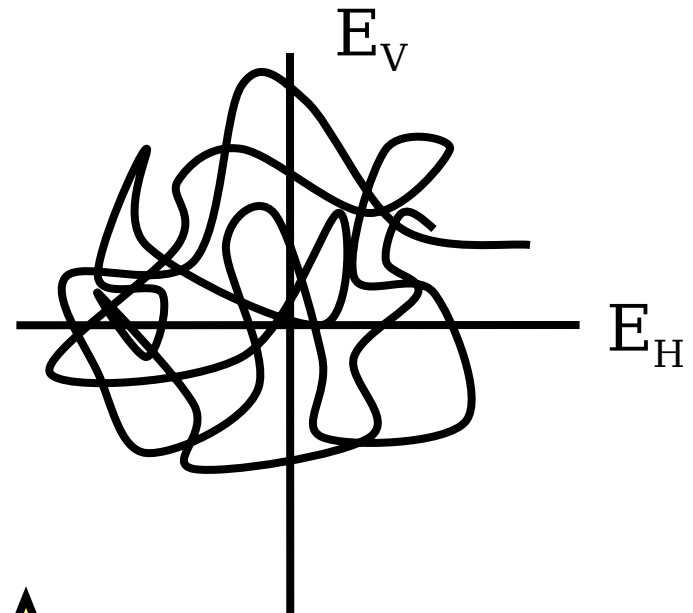
## Linear Representation

$$\mathbf{E} = V\hat{\mathbf{E}}_v + H\hat{\mathbf{E}}_h$$

$$\langle V \rangle = \langle H \rangle = 0$$

$$\langle |V|^2 \rangle = \langle |H|^2 \rangle$$

$$\langle VH \rangle = 0$$



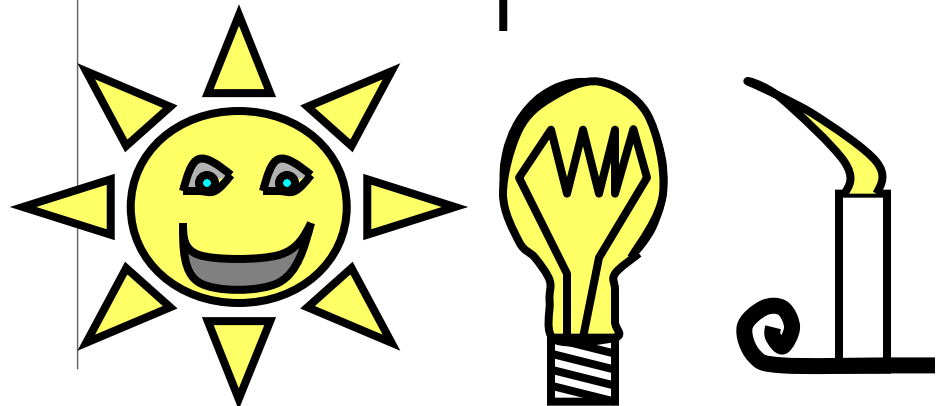
## Circular Representation

$$\mathbf{E} = L\hat{\mathbf{E}}_\ell + R\hat{\mathbf{E}}_r$$

$$\langle L \rangle = \langle R \rangle = 0$$

$$\langle |L|^2 \rangle = \langle |R|^2 \rangle$$

$$\langle LR \rangle = 0$$



# Polarizing Components

## Polarizers

Dielectric Interfaces

Dichroics

Polaroid

Wire Grids

Scattering

## Waveplates

Natural Birefringence

Stress-Induced

4061ppt5.tex-4

Electrically-Induced

Used in Modulators

Magnetically-Induced

Used in Sensors

## Rotators

Natural Optical Activity

Crystals and Molecules

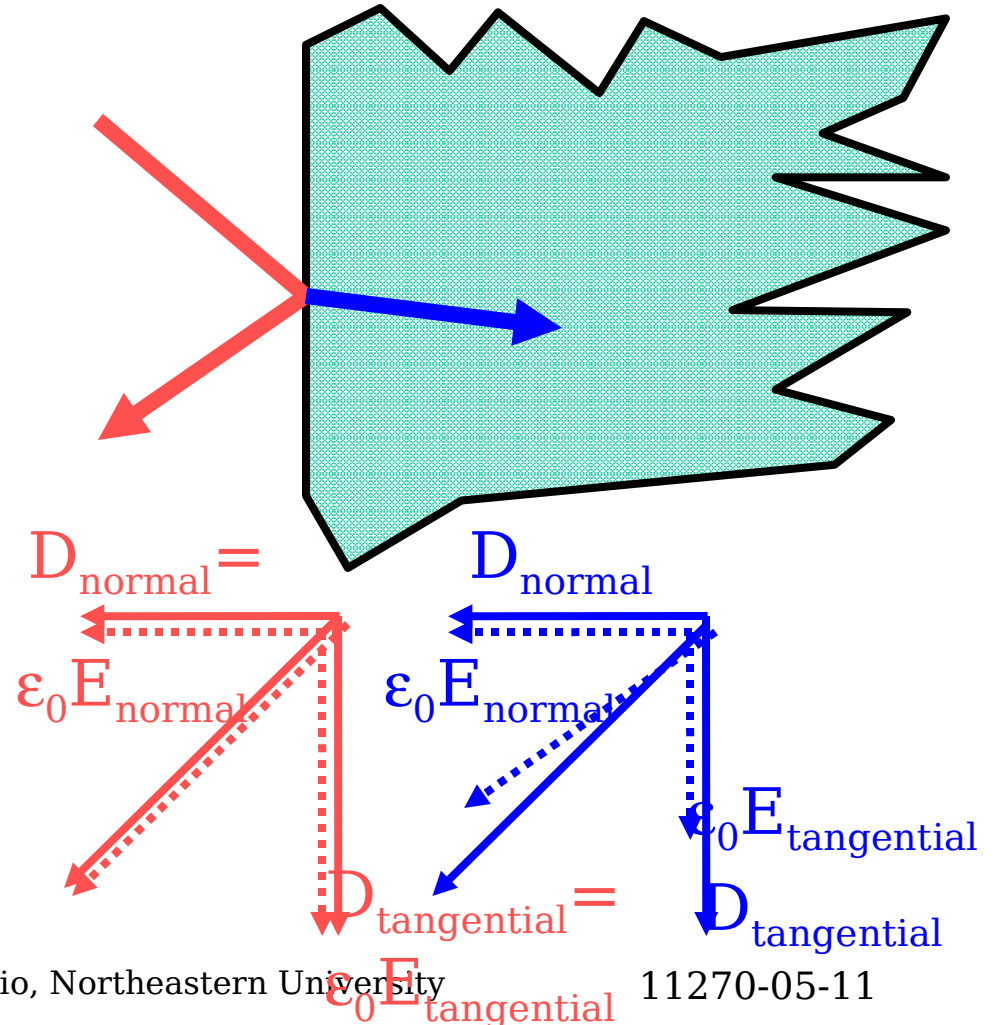
Faraday Effect

4061ppt5.tex-5

# Fresnel Reflection

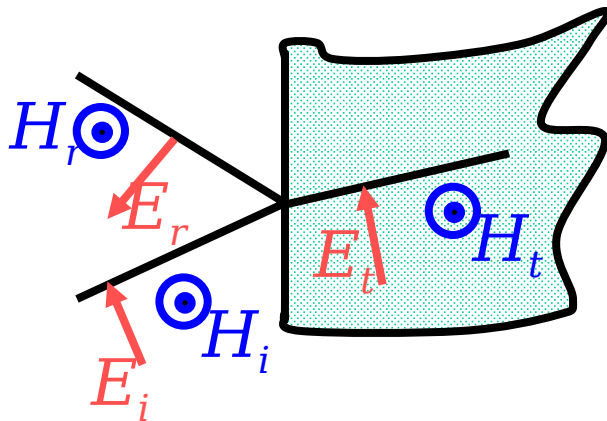
## Boundary Conditions

| Condition               | Justification                                                                    |
|-------------------------|----------------------------------------------------------------------------------|
| $\Delta D_{normal} = 0$ | $\nabla \bullet \mathbf{D} = \rho = 0$                                           |
| $\Delta E_{tang} = 0$   | $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$             |
| $\Delta B_{normal} = 0$ | $\nabla \bullet \mathbf{B} = 0$                                                  |
| $\Delta H_{tang} = 0$   | $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$ |

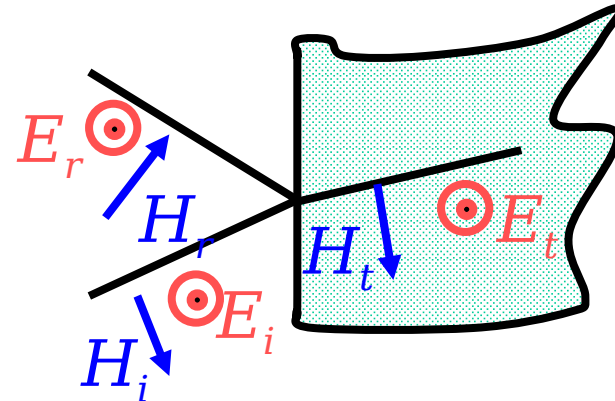


# Polarization Labels

- Plane of Incidence
  - Normal & Incident Ray
- P-Polarization (TM)
  - $E$  Parallel to Plane of Incidence

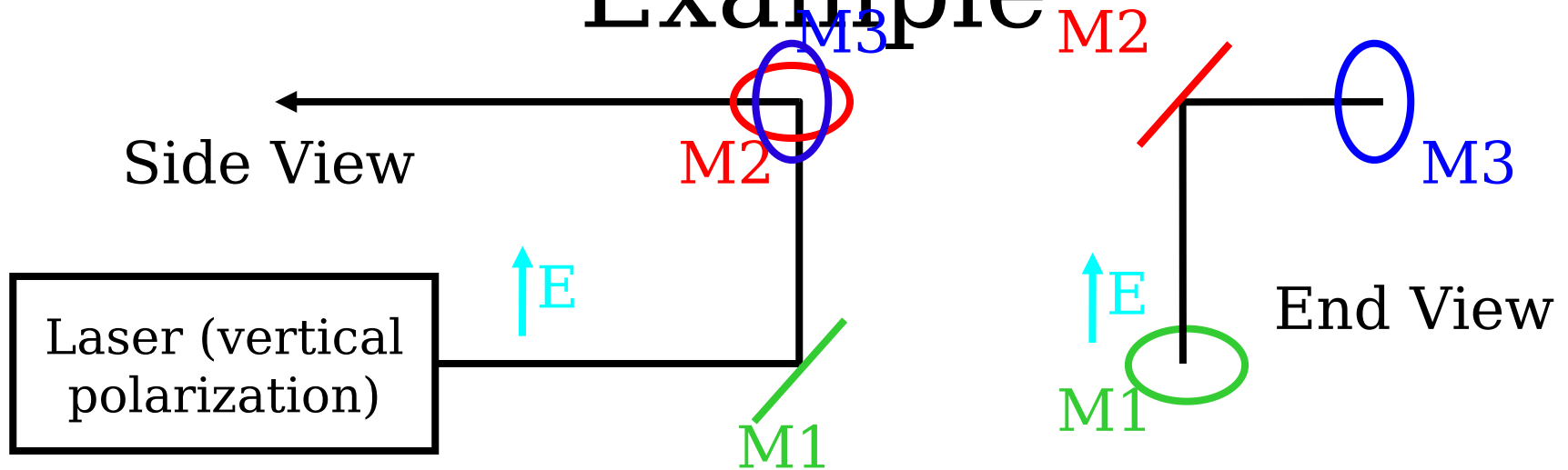


- S-Polarization (TE)
  - $E$  Senkrecht = Perpendicular to Plane of Incidence



# Polarization Labels

## Example



- Beam Path:
  - “North” to **M1**, Up to **M2**, “West” to **M3**, “South”
- Is the Output After **M3** Polarized Vertically or Horizontaly or Something Else?
- Is the Polarization *S* or *P* at **M1**? At **M2**? At **M3**?

# S-Polarization (1)

## Plane Waves

$$\mathbf{E}_i = E_i \hat{\mathbf{x}} \exp[-jk_1 (\sin \theta_i z + \cos \theta_i y)]$$

$$\mathbf{E}_r = E_r \hat{\mathbf{x}} \exp[-jk_1 (\sin \theta_r z - \cos \theta_r y)]$$

$$\mathbf{E}_t = E_t \hat{\mathbf{x}} \exp[-jk_2 (\sin \theta_t z + \cos \theta_t y)]$$

unknowns are  $E_r$ ,  $E_t$ ,  $\theta_r$ ,  $\theta_t$

## Boundary Condition

$$\Delta E_{tang} = 0 \quad (\text{True for all } z \text{ at } y = 0)$$

Snell's Law from constant  $k_{transverse}$

$$\theta_r = \theta_i \quad k_1 \sin \theta_i = k_2 \sin \theta_t$$

$$k = k_0 n \quad n_1 \sin \theta_i = n_2 \sin \theta_t$$

## Magnetic Field

$$\mathbf{H} = \hat{\mathbf{n}} \times \mathbf{E}/Z \quad \hat{\mathbf{n}} = \hat{\mathbf{k}}$$

## Plane Waves (at origin)

$$\mathbf{H}_i = (E_i/Z_1) (\sin \theta_i \hat{\mathbf{y}} - \cos \theta_i \hat{\mathbf{z}})$$

$$\mathbf{H}_r = (E_r/Z_1) (\sin \theta_r \hat{\mathbf{y}} + \cos \theta_r \hat{\mathbf{z}})$$

$$\mathbf{H}_t = (E_t/Z_2) (\sin \theta_t \hat{\mathbf{y}} - \cos \theta_t \hat{\mathbf{z}})$$

## Boundary Conditions

$$\Delta H_{tang} = 0 \quad \rightarrow \quad \Delta H_z = 0$$

## Solution

$$\cos \theta_i (-E_i + E_r)/Z_1 = -E_t \cos \theta_t/Z_2$$

4061ppt5.tex-37

4061ppt5.tex-36

# S-Polarization (2)

Magnetic B. C.

$$\cos \theta_i (-E_i + E_r) / Z_1 = -E_t \cos \theta_t / Z_2$$

Snell's Law

$$\cos \theta_t = \sqrt{1 - \sin^2 \theta_i \left( \frac{n_1}{n_2} \right)^2} = \frac{Z_2}{Z_1} \sqrt{\left( \frac{n_2}{n_1} \right)^2 - \sin^2 \theta_i}$$

Result (A)

$$E_i - E_r = E_t \sqrt{\frac{\epsilon_2}{\epsilon_1} - \sin^2 \theta_i} \frac{1}{\cos \theta_i}$$

Electric B. C. (B)

$$\Delta E_{tang} = 0 \rightarrow E_i + E_r = E_t$$

Solution

$$\frac{\text{eqn(B)} - \text{eqn(A)}}{\text{eqn(B)} + \text{eqn(A)}} = \frac{E_r}{E_i}$$

# Fresnel Coefficient Summary

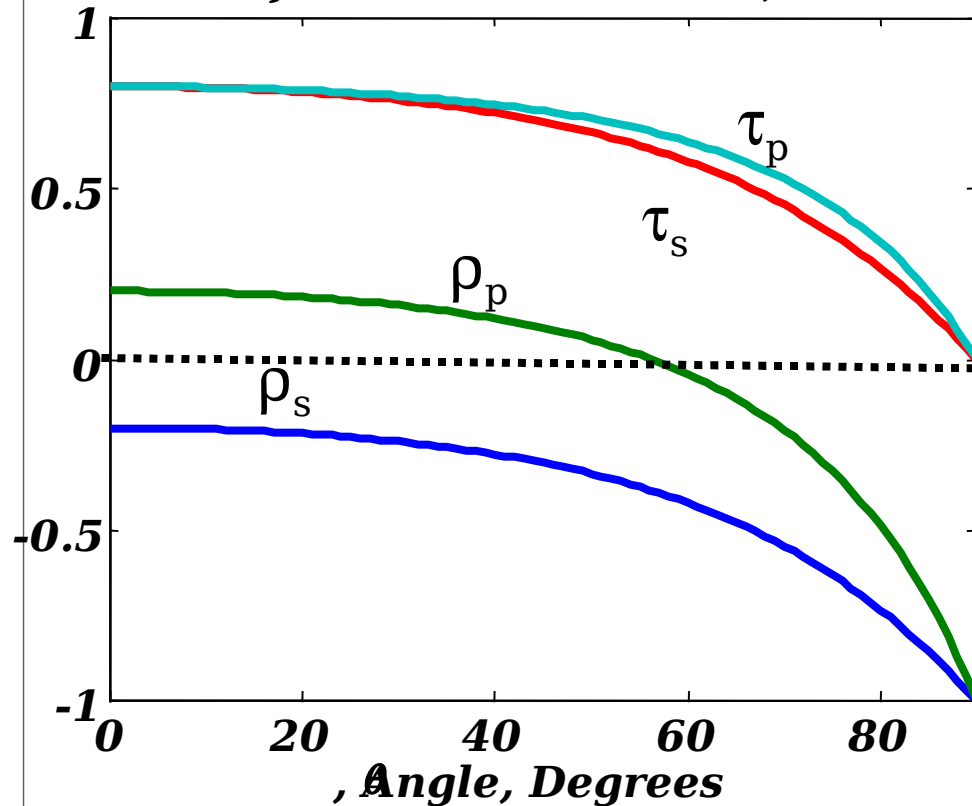
$$\rho_s = \frac{E_{sr}}{E_{si}} = \frac{\cos \theta_i - \sqrt{\frac{n_2^2}{n_1^2} - \sin^2 \theta_i}}{\cos \theta_i + \sqrt{\frac{n_2^2}{n_1^2} - \sin^2 \theta_i}}$$

$$\tau_s = \frac{E_{st}}{E_{si}} = 1 + \rho_s$$

$$\rho_p = \frac{E_{pr}}{E_{pi}} = \frac{\sqrt{\frac{n_2^2}{n_1^2} - \sin^2 \theta_i} - \frac{n_2^2}{n_1^2} \cos \theta_i}{\sqrt{\frac{n_2^2}{n_1^2} - \sin^2 \theta_i} + \frac{n_2^2}{n_1^2} \cos \theta_i}$$

$$\tau_p = \frac{E_{pt}}{E_{pi}} = \frac{1 + \rho_p}{\sqrt{\epsilon_2/\epsilon_1}} = (1 + \rho_p) \frac{n_1}{n_2}$$

Example  
**Fresnel Reflection: Air to Glass,  $n=1.5$**



# Special Angles

## Brewster's Angle

No P-Reflection

Numerator of  $\rho_p = 0$

$$\frac{n_2^2}{n_1^2} - \sin^2 \theta_i = \frac{n_2^4}{n_1^4} (1 - \sin^2 \theta_i)$$

$$\tan \theta_b = \frac{n_2}{n_1}$$

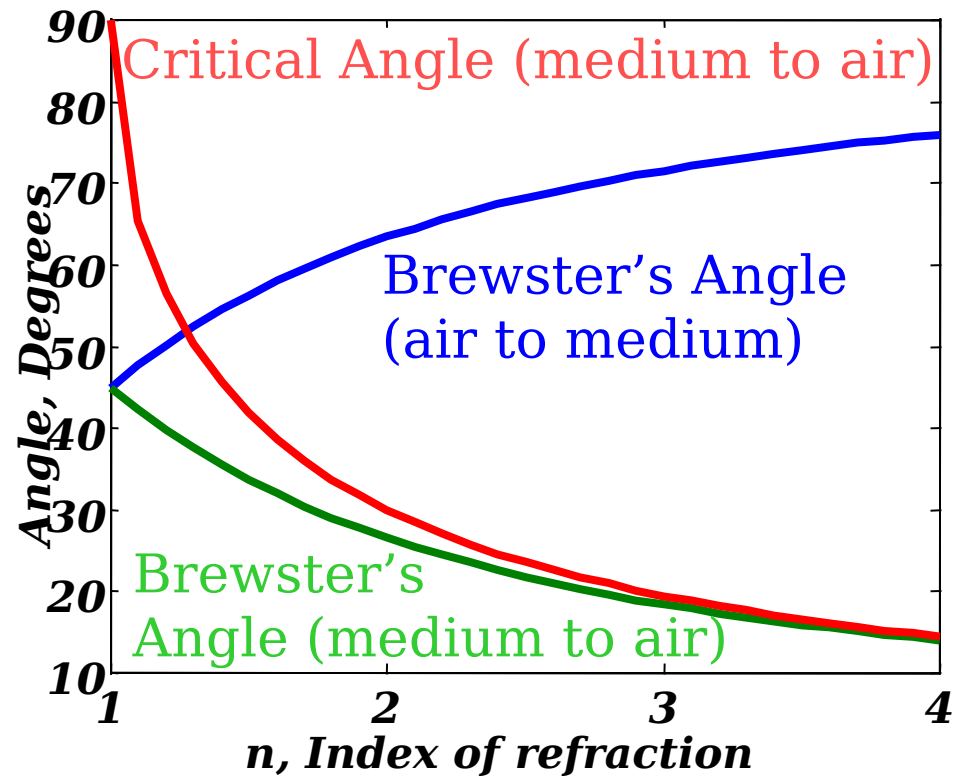
## Critical Angle

Reflectivity goes to 1

Arg of  $\sqrt{\cdot} = 0$

$$\sin \theta_c = \frac{n_2}{n_1}$$

4061ppt5.tex



# Summary of Reflectivities

Sign Reversal

P-Polarization

Artifact of Sign Choice

Normal Incidence

$$\rho_s = \frac{1 - n}{1 + n} \quad \rho_p = \frac{n - 1}{1 + n}$$

| $n_2 > n_1$              |           |          | $n_2 < n_1$              |                 |                 |
|--------------------------|-----------|----------|--------------------------|-----------------|-----------------|
| $\theta$                 | $\rho_s$  | $\rho_p$ | $\theta$                 | $\rho_s$        | $\rho_p$        |
| 0                        | $-\rho_0$ | $\rho_0$ | 0                        | $\rho_0$        | $-\rho_0$       |
| $0 < \theta < \theta_B$  | —         | +        | $0 < \theta < \theta_B$  | +               | —               |
| $\theta_B$               | —         | 0        | $\theta_B$               | +               | 0               |
| $\theta_B < \theta < 90$ | —         | —        | $\theta_B < \theta < 90$ | +               | +               |
| 90                       | —1        | —1       | $\theta_C$               | 1               | 1               |
|                          |           |          | $\theta_C < \theta < 90$ | $e^{j\delta_s}$ | $e^{j\delta_p}$ |
|                          |           |          | 90                       | —1              | —1              |

4061ppt5.tex-43

4061ppt5.tex-42

# Power Coefficients

[Poynting Vector]  $I = \frac{|E|^2}{Z}$   $I = \frac{dP}{dA'}$

$$dA' = dA \cos \theta \quad I = \frac{dP}{\cos \theta dA}$$

Power per Projected Area

$$I_r = I_i \rho \rho^* \quad I_t = I_i \tau \tau^* \frac{Z_1}{Z_2}$$

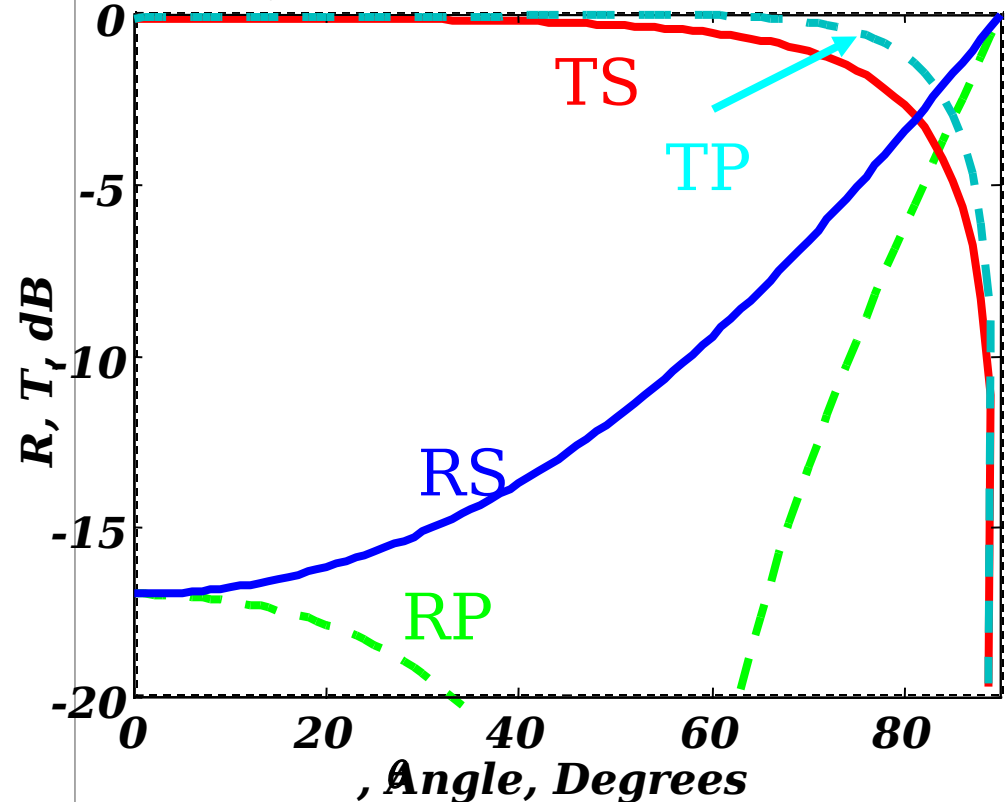
Power Equations

$$P_r = P_i \rho \rho^* \quad \frac{P_t}{\cos \theta_t} = \frac{P_i}{\cos \theta_i} \tau \tau^* \frac{Z_1}{Z_2}$$

$$R = \rho \rho^* \quad T = \frac{\cos \theta_t}{\cos \theta_i} \tau \tau^* \frac{Z_1}{Z_2}$$

4061ppt5.tex-44

**Fresnel Reflection: Air to Water,  $n=1.33$**



# Total Internal Reflection

$$\mathbf{E}_i = E_i \hat{\mathbf{x}} \exp[-jk_1 (\sin \theta_i z + \cos \theta_i y)]$$

$$\mathbf{E}_r = E_r \hat{\mathbf{x}} \exp[-jk_1 (\sin \theta_r z - \cos \theta_r y)]$$

$$\mathbf{E}_t = E_t \hat{\mathbf{x}} \exp[-jk_2 (\sin \theta_t z + \cos \theta_t y)]$$

Snell's Law

$$k_1 \sin \theta_i = k_2 \sin \theta_t$$

$$\cos \theta_t = \pm j \sqrt{\frac{k_1^2}{k_2^2} \sin^2 \theta_i - 1}$$

$$\mathbf{E}_t = E_t \hat{\mathbf{x}} \times$$

$$\exp \left[ -jk_2 (\sin \theta_t z) - \sqrt{k_1^2 \sin^2 \theta_i - k_2^2} y \right]$$

4061ppt5.tex-45

$$\rho_s = \frac{\cos \theta_i - j \sqrt{\sin^2 \theta_i - \frac{n_2^2}{n_1^2}}}{\cos \theta_i + j \sqrt{\sin^2 \theta_i - \frac{n_2^2}{n_1^2}}} =$$

$$\exp \left[ -j2 \sqrt{\sin^2 \theta_i - \frac{n_2^2}{n_1^2}} / \cos \theta_i \right]$$

$$|\rho_s|^2 = 1$$

Application: Fresnel Rhomb, etc.

4061ppt5.tex-46

Topics for Later  
in the Course

11270-05-20

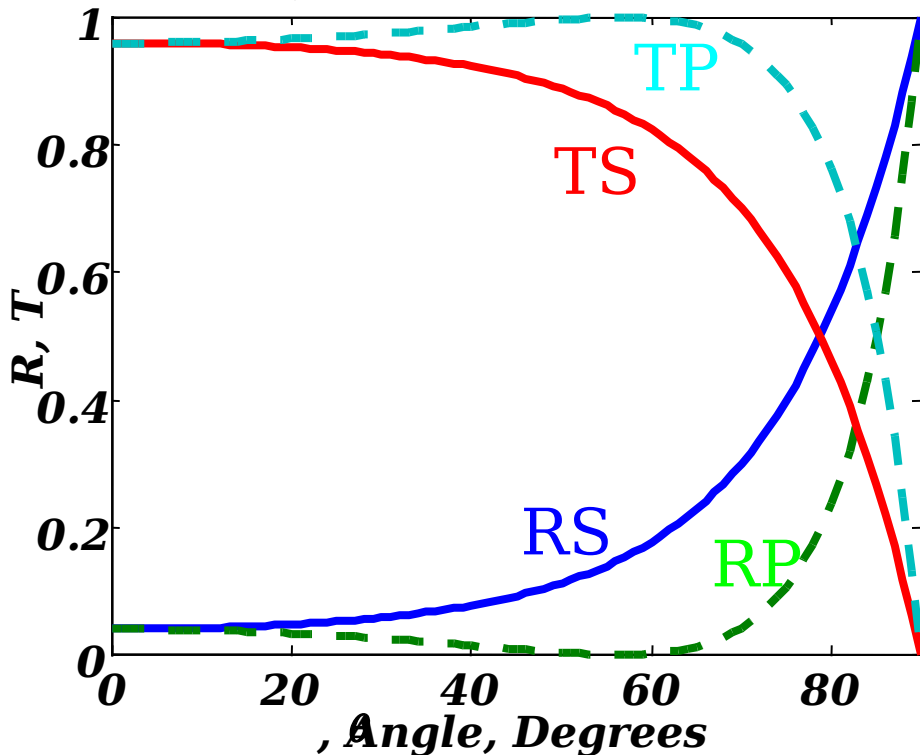
# Fresnel Reflection

## Examples (1)

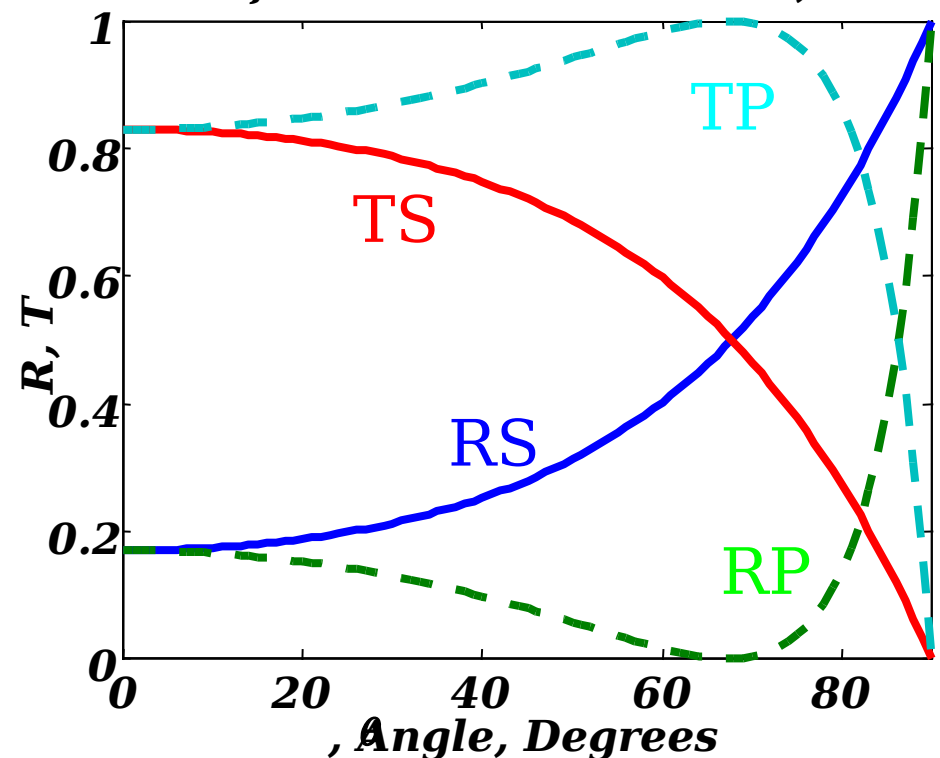
Air to Glass

Air to ZnSe (IR)

*Fresnel Reflection: Air to Glass,  $n=1.5$*



*Fresnel Reflection: Air to ZnSe in IR,  $n=2.4$*



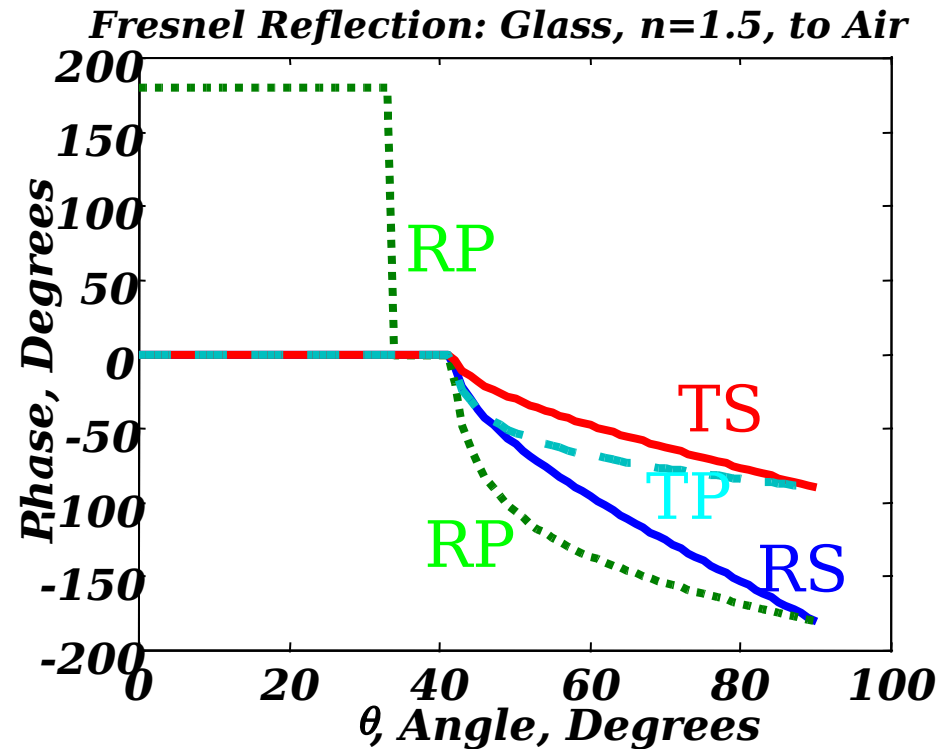
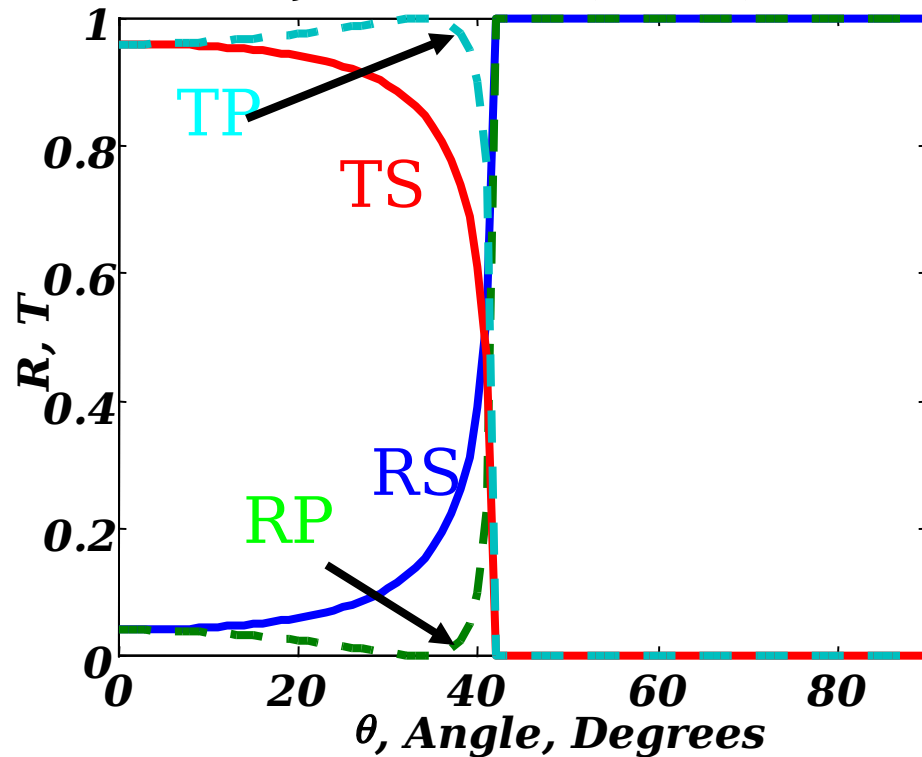
# Fresnel Reflection

## Examples (2)

Glass to Air Amplitude

Glass to Air Phase

*Fresnel Reflection: Glass,  $n=1.5$ , to Air*



# Complex Index

## Plane Waves

$$\mathbf{E}_i = E_i \hat{\mathbf{x}} \exp[-jk_1 (\sin \theta_i z + \cos \theta_i y)]$$

$$\mathbf{E}_r = E_r \hat{\mathbf{x}} \exp[-jk_1 (\sin \theta_r z - \cos \theta_r y)]$$

$$\mathbf{E}_t = E_t \hat{\mathbf{x}} \exp[-jk_2 (\sin \theta_t z + \cos \theta_t y)]$$

## “Phase” Matching Condition

$$\mathbf{k}_i \bullet \mathbf{r} = \mathbf{k}_r \bullet \mathbf{r} = \mathbf{k}_t \bullet \mathbf{r}$$

Matches Amplitude and Phase

## Direction

$$(\mathbf{k}'_i + j\mathbf{k}''_i) \bullet \mathbf{r} = (\mathbf{k}'_t + j\mathbf{k}''_t) \bullet \mathbf{r}$$

1st Medium Lossless:  $\mathbf{k}''_i = 0$

$\mathbf{k}''_t \bullet \mathbf{r} = 0$ ;  $\mathbf{k}''_t$  normal to surface

4061ppt5.tex-47

## Relation to Wavevector

$$\mathbf{k} \bullet \mathbf{k} = (n' + jn'')^2 \left(\frac{\omega}{c}\right)^2$$

Not...

$$k = (n' + jn'') \frac{\omega}{c}$$

## Real and Imaginary Parts

$$(k')^2 + (k'')^2 + 2jk'k'' \cos \alpha = [(n')^2 - (n'')^2 + 2n'n''] \left(\frac{\omega}{c}\right)^2$$

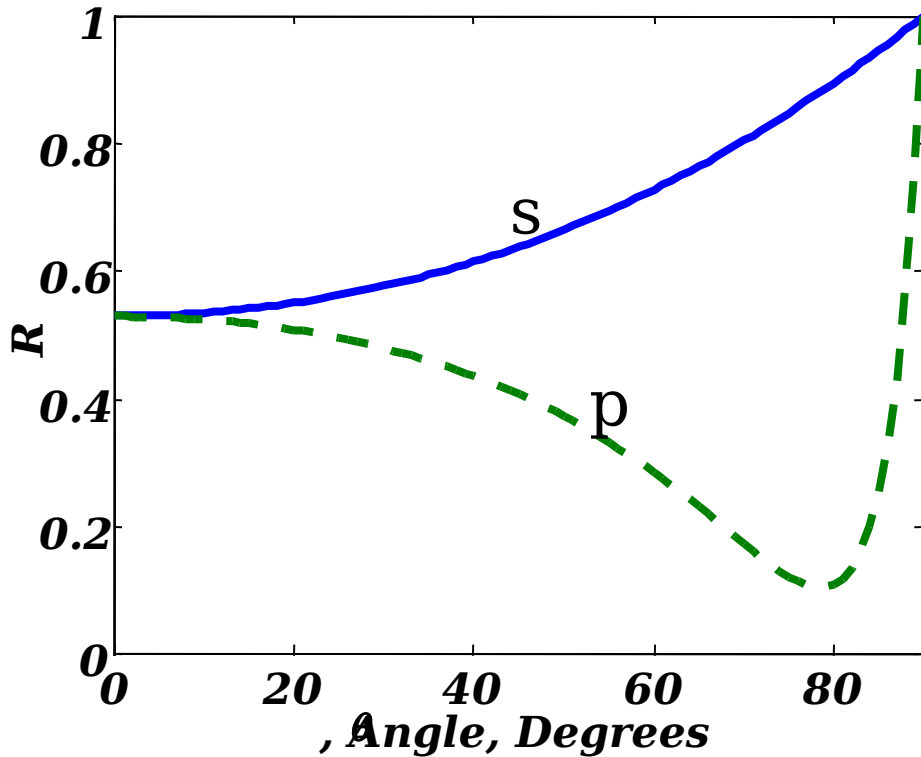
## Fresnel Coefficients

$$\rho_s = \frac{E_{sr}}{E_{si}} = \frac{\cos \theta_i - \sqrt{\frac{\epsilon'_2 + j\epsilon''_2}{\epsilon_1} - \sin^2 \theta_i}}{\cos \theta_i + \sqrt{\frac{\epsilon'_2 + j\epsilon''_2}{\epsilon_1} - \sin^2 \theta_i}}$$

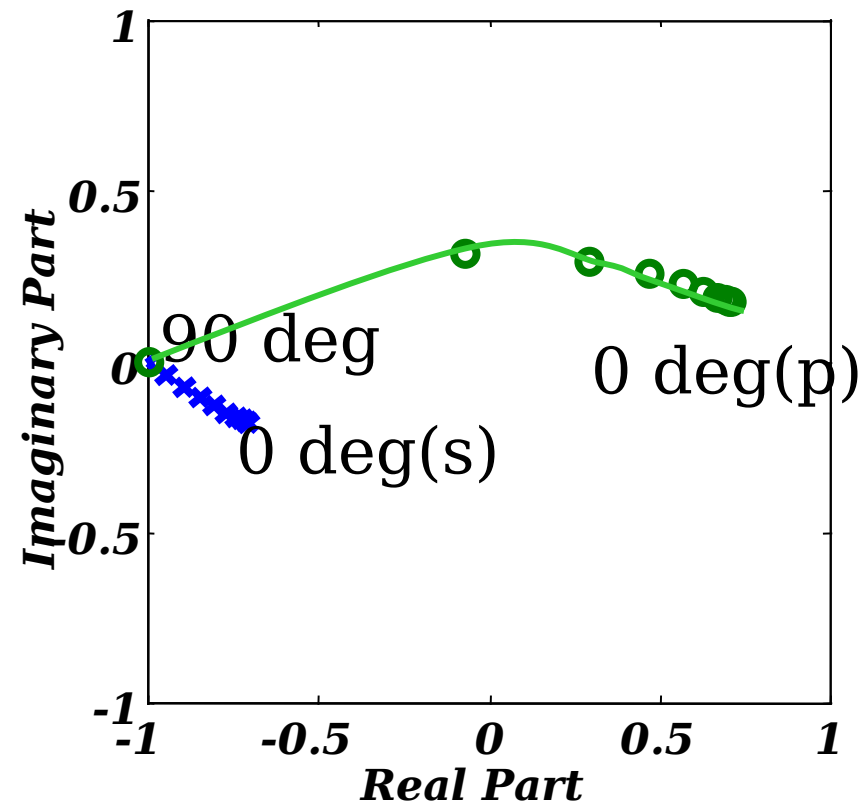
4061ppt5.tex-48

# Fresnel Reflection for a Metal

*Fresnel Reflection: Air to Metal,  $n=4+3i$*



*Fresnel Reflection: Air to Metal,  $n=4+3i$*



# Device Applications

Linear Polarizers

Half-Wave Plates

Quarter-Wave Plates

Example: T/R Switch

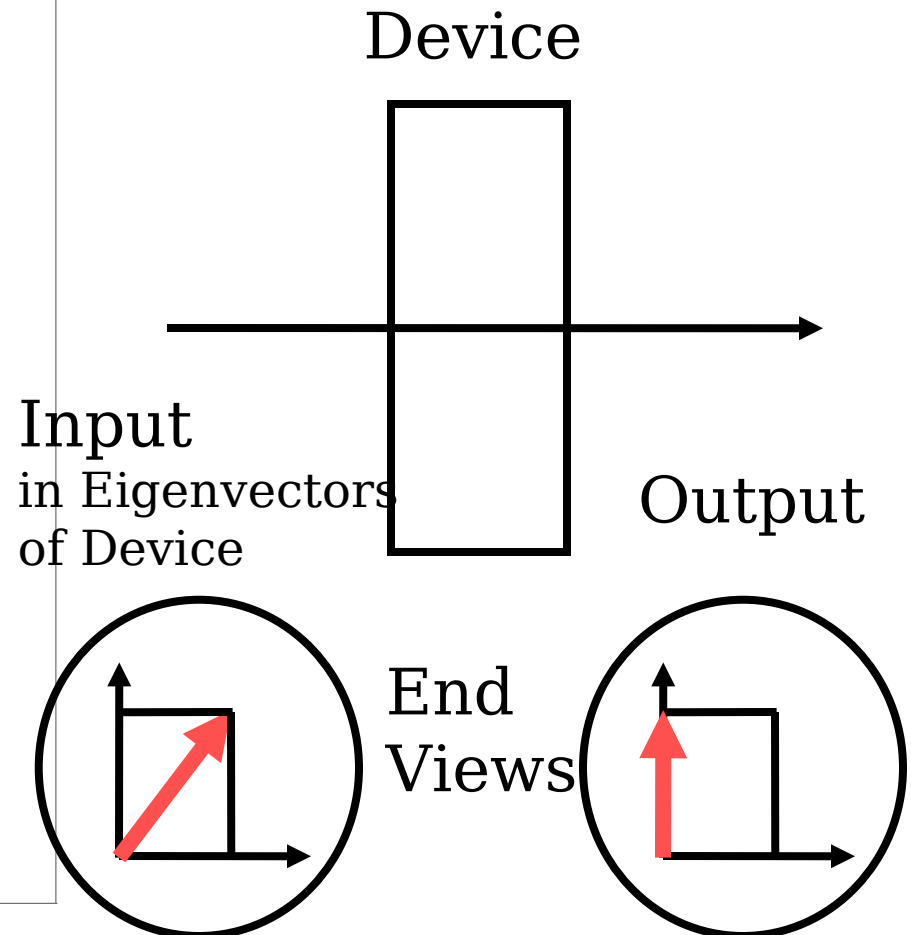
E/O Modulators

Phase Modulation

Intra-Cavity FM Modulation

AM Modulation

4061ppt5.tex-6



# Brewster Plates

Dielectric Interface

Incident Field

$$\mathbf{E}_{in} = E_p \hat{\mathbf{E}}_p + E_s \hat{\mathbf{E}}_s$$

Output

$$\mathbf{E}_{out} = E_p \hat{\mathbf{E}}_p + (\sqrt{1 - R_s})^2 E_s \hat{\mathbf{E}}_s$$

Example

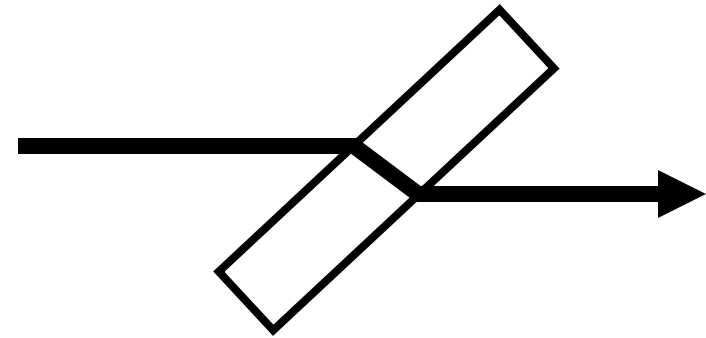
Ge at 10.6  $\mu\text{m}$

$$n = 4 \quad \theta_b = 76 \text{ deg.} \quad R_b = 0.78$$

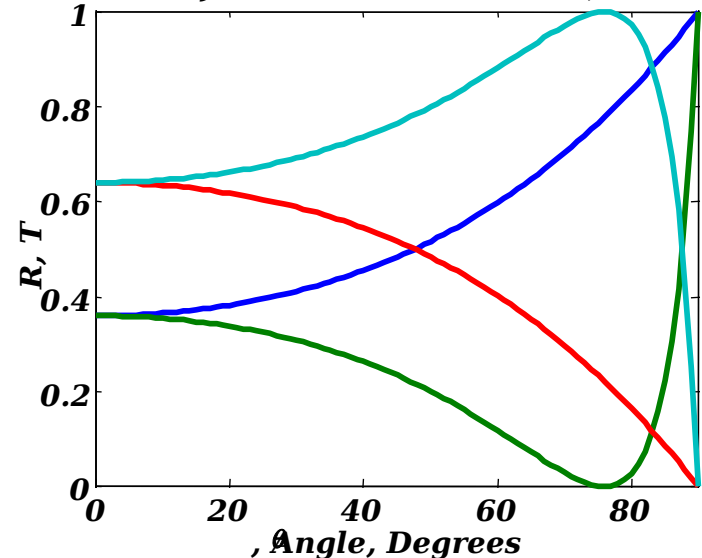
$$\mathbf{E}_{out} = E_p \hat{\mathbf{E}}_p + (\sqrt{0.22})^2 E_s \hat{\mathbf{E}}_s \quad (2 \text{ Surfaces})$$

Leakage = 0.048 (Proof Later)

4061ppt5.tex-7



*Fresnel Reflection: Air to Ge in IR,  $n=4$*



# Tent Polarizers

Two Brewster Plates

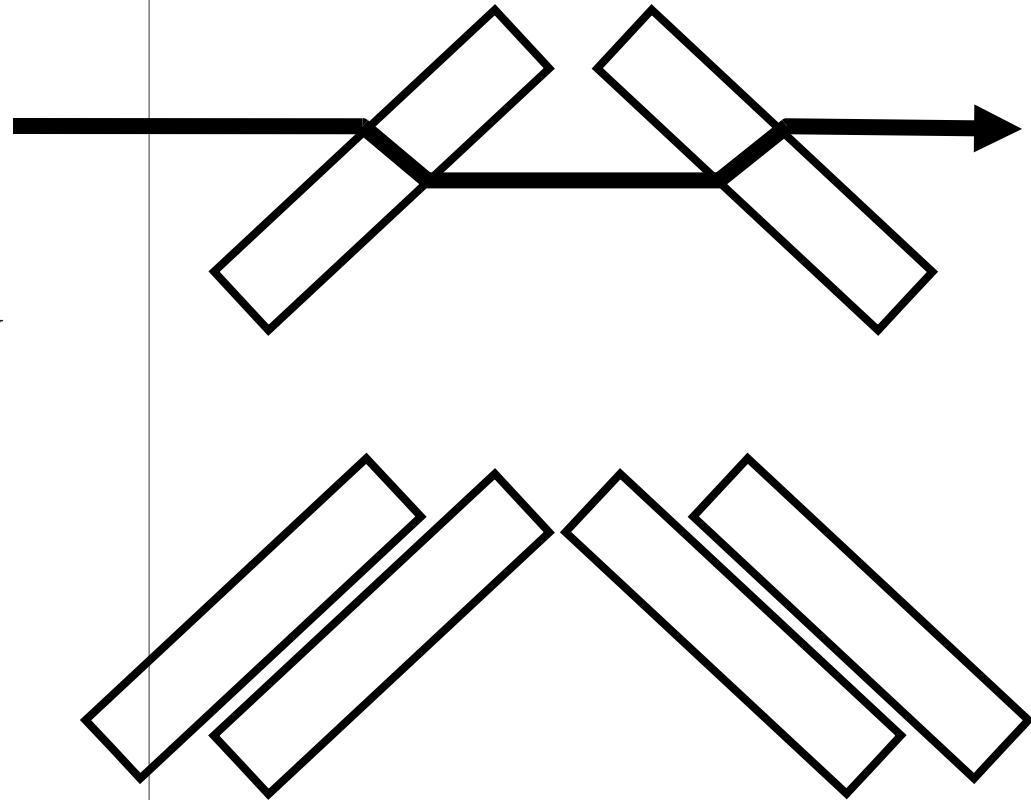
Multiple Tents

Transmission (N Plates)

$$T_s = [1 - \sin^2(2\theta_B - 90 \text{ deg})]^{2N}$$

Works well for High Index:

$$\tan \theta_B = n_2/n_1$$



4061ppt5.tex-8

# Wire Grid Polarizers

## The Principle

Wires Are Conductors: No  $E_x$

## Advantages

Compactness

## Problems

Poor Beam Quality

Limited Power Handling

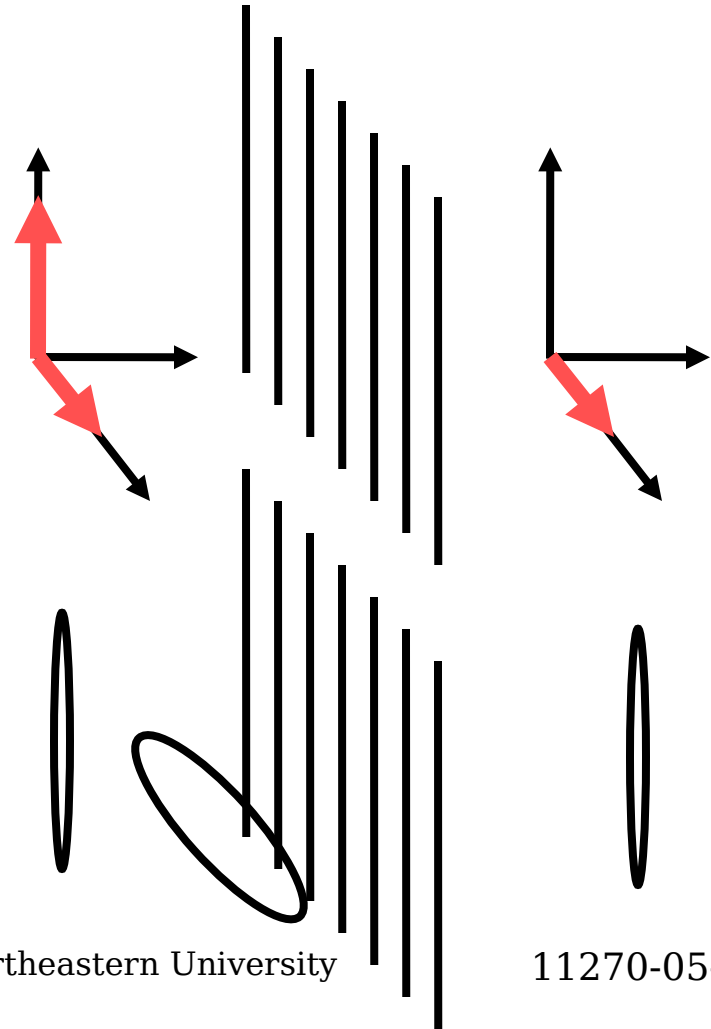
## Warning!

They Don't Work the Way Most People Think

4061ppt5.tex-13

July 2003

Chuck DiMarzio, Northeastern University



11270-05-28

# Polaroid H Sheets

## Composition

Polyvinyl Alcohol and Iodine

## Fabrication

Stretch PVA Sheet

Dip in Iodine

Chains of Iodine Act Like Wire Grid

## Examples:

HN-38, HN-32, HN-22

N = "Neutral," 38 = "38% Transmission"

50% Loss to Polarization, 8% to Fresnel Refl.

4061ppt5.tex-10 (H42 would be "perfect")

## Advantages

Large Size

Low Cost

Low Scattering

## Problems

Limited Power-Handling

Limited Performance

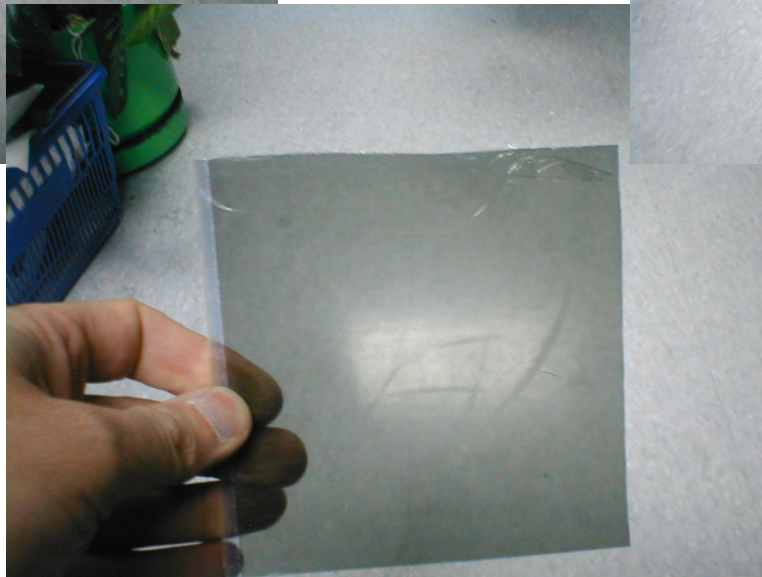
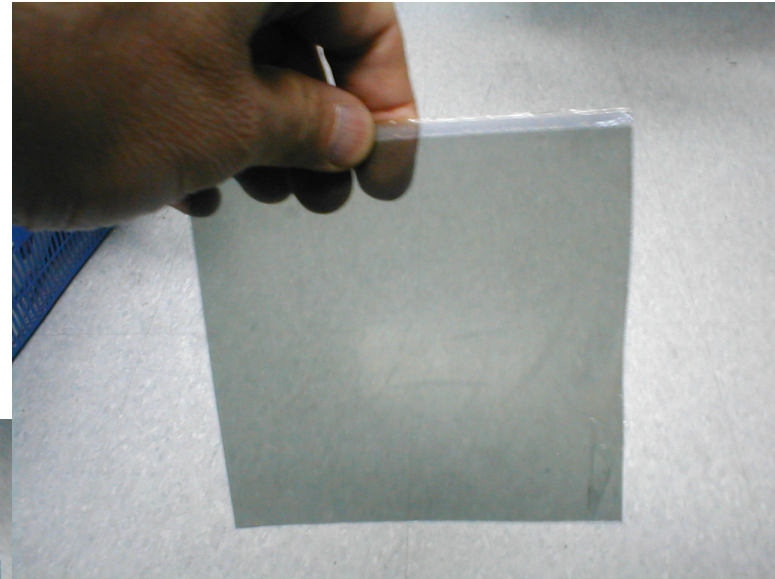
Insertion Loss

Extinction

Wavelength Dependence

4061ppt5.tex-12

# S Reflection > P Reflection?



- How Do Polarized Sunglasses Work?
- What is the Underlying Assumption?

# Birefringence Model

## The Model

Mass and Springs

## The Results

Index Dep. on Field Direction

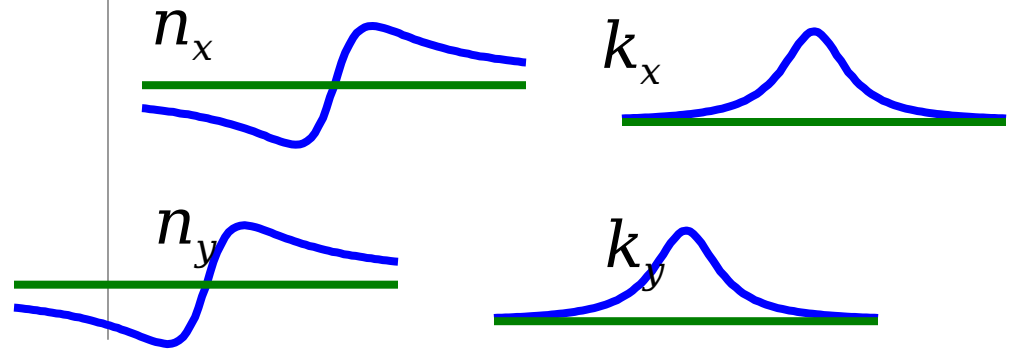
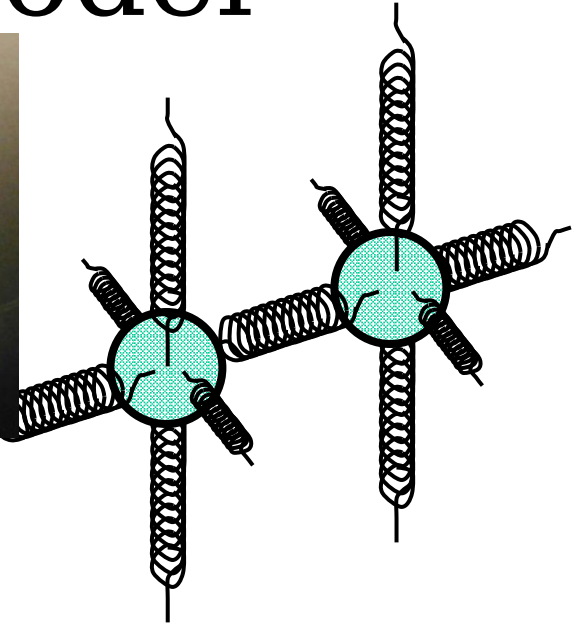
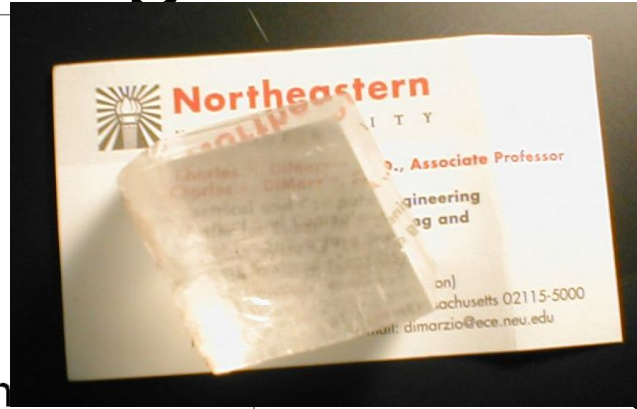
Anisotropic Speed and

Absorption Resonances

## Applications

Polarizer at  $\omega_1$  or  $\omega_2$

Other Freqs: Waveplate



4061ppt5.tex-9

# Waveplates (1)

Input

$$E_{in} = (E_{xi}\hat{\mathbf{E}}_x + E_{yi}\hat{\mathbf{E}}_y) e^{i(\omega t - kz)}$$

Anywhere Inside

$$E(z) = E_{xi}\hat{\mathbf{E}}_x e^{i(\omega t - k_x z)} + E_{yi}\hat{\mathbf{E}}_y e^{i(\omega t - k_y z)}$$

Specifically at  $z = T$

$$E(T) = \left( E_{xi}\hat{\mathbf{E}}_x e^{-i2\pi n_x T/\lambda} + E_{yi}\hat{\mathbf{E}}_y e^{-i2\pi n_y T/\lambda} \right) e^{i\omega t}$$

Using

$$k_x = \frac{2\pi n_x}{\lambda} \quad k_y = \frac{2\pi n_y}{\lambda}$$

4061ppt5.tex-14

Output

$$\mathbf{E}_{out} = (E_{xo}\hat{\mathbf{E}}_x + E_{yo}\hat{\mathbf{E}}_y)$$

Magnitudes

$$|E_{xo}| = |E_{xi}| \quad |E_{yo}| = |E_{yi}|$$

Phases

$$\phi_{xo} = \phi_{xi} - \frac{2\pi(n_x - 1)T}{\lambda}$$

$$\phi_{yo} = \phi_{yi} - \frac{2\pi(n_y - 1)T}{\lambda}$$

Phase Difference

$$(\phi_{yo} - \phi_{xo}) = (\phi_{yi} - \phi_{xi}) + \frac{2\pi T}{\lambda} (n_y - n_x)$$

4061ppt5.tex-15

# Waveplates (2)

## Examples

### Quarter Wave Plate

Quartz at 589.3 nm

$$2\pi (n_y - n_x) = 2\pi (1.5534 - 1.5443) = 0.057$$

Quarter-Wave Condition

$$\frac{2\pi T \Delta n}{\lambda} = \frac{\pi}{2}$$

$$T = \frac{\lambda}{4\Delta n} \quad T = \frac{\pi/2}{0.057} \lambda = 27.56\lambda$$

(16.24  $\mu\text{m}$ )

5/4 Waveplate

$$T = 137.79\lambda \quad (81 \mu\text{m})$$

## Watch Out For Dispersion

$$\frac{d\phi}{d\lambda} = -\frac{2\pi T \Delta n}{\lambda^2} = \frac{\phi}{\lambda}$$

$$d\lambda = 100\text{nm}, \lambda = 800\text{nm}$$

$$\phi = 45 \text{ deg} \rightarrow d\phi = 6 \text{ deg}$$

$$\phi = 225 \text{ deg} \rightarrow d\phi = 30 \text{ deg}$$

# Optical Activity

## Mathematical Description

Different Speeds for...

Left-Hand Circular Polarization

Right-Hand Circular Polarization

## The Mechanism

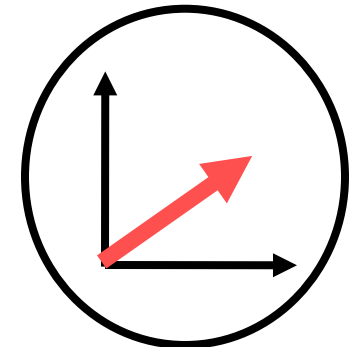
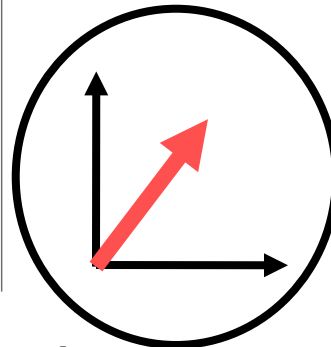
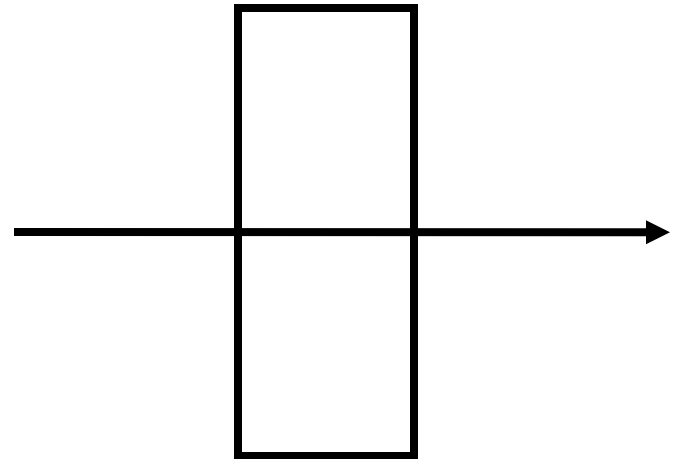
Helical Path for Electrons

Crystal Structure

Molecular Structures, eg. Benzene

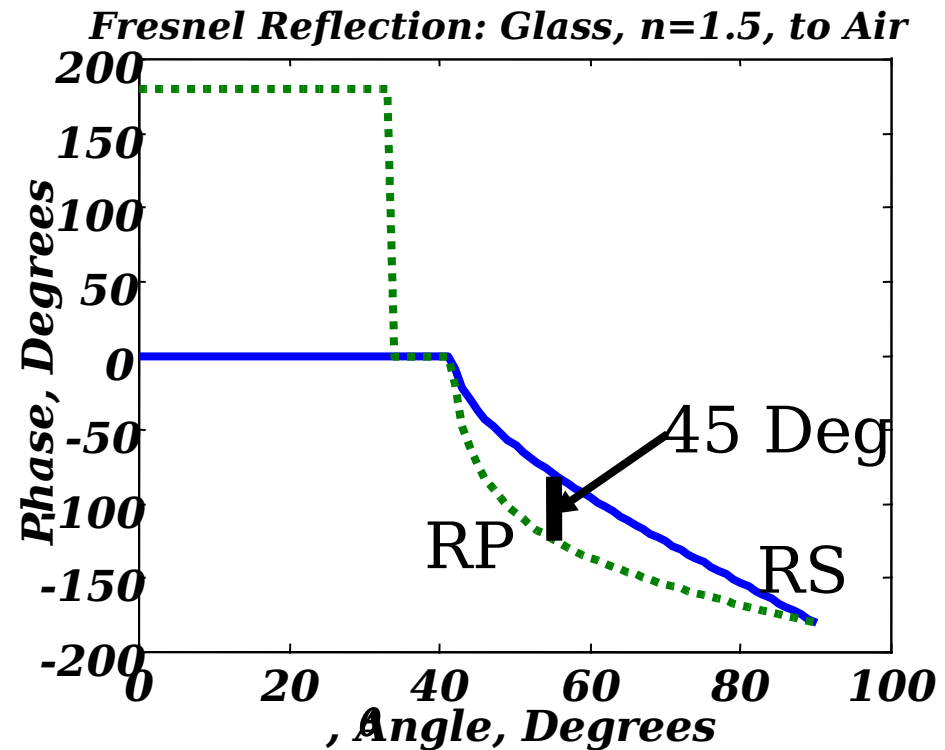
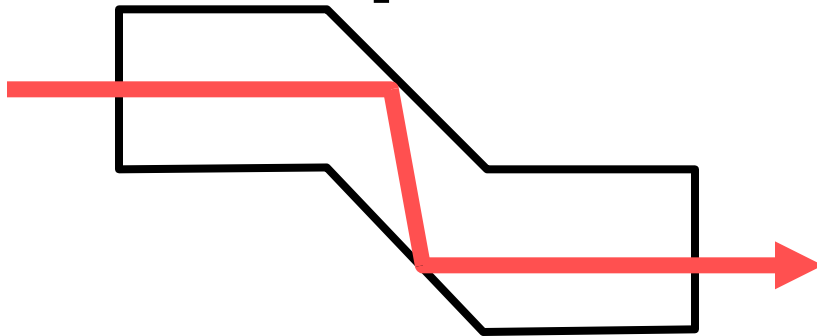
Faraday Effect, Introduced by Mag. Field

Device



# Fresnel Rhomb

- Functions as QWP
  - True Phase Shift rather than Time Delay
  - Broadband: Limited only by material Dispersion



# Jones Matrices

## Notation

$$\mathbf{E} = |E\rangle = \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$
$$\langle E| = \begin{pmatrix} E_x^* & E_y^* \end{pmatrix}$$

## Linear Basis Vectors

$$\hat{\mathbf{E}}_x = |E_x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \hat{\mathbf{E}}_y = |E_y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

## Basic Operations

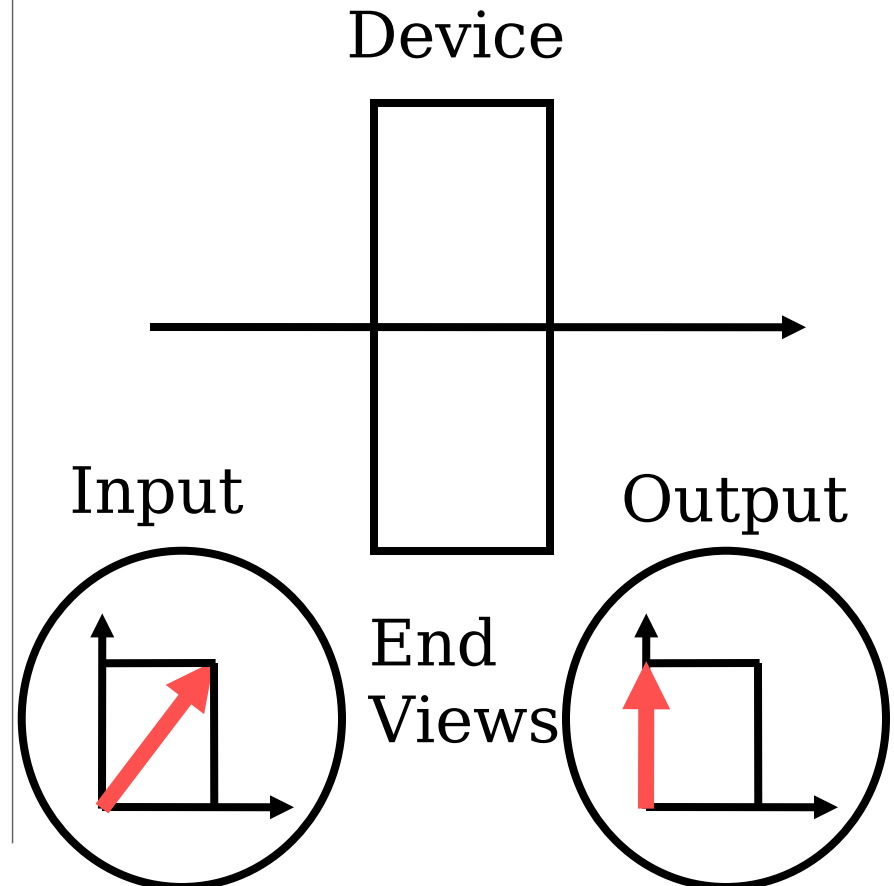
$$|E_{out}\rangle = \mathcal{J} |E_{in}\rangle$$

## Jones Matrix

Component, or Coordinate Transform

4061ppt5.tex-19

## Example: Polarizer



# Rotation of Coordinates

Equation

$$|E'\rangle = \mathcal{R}_c |E\rangle$$

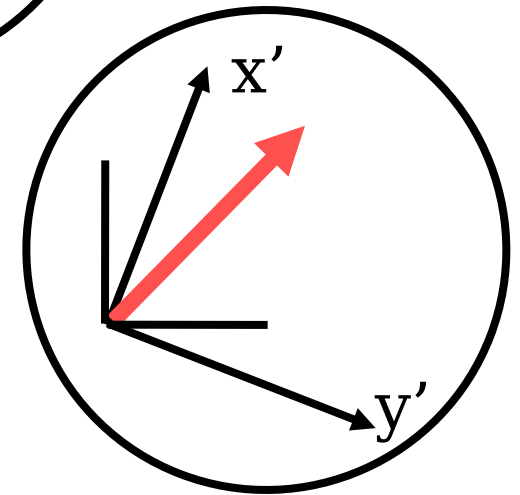
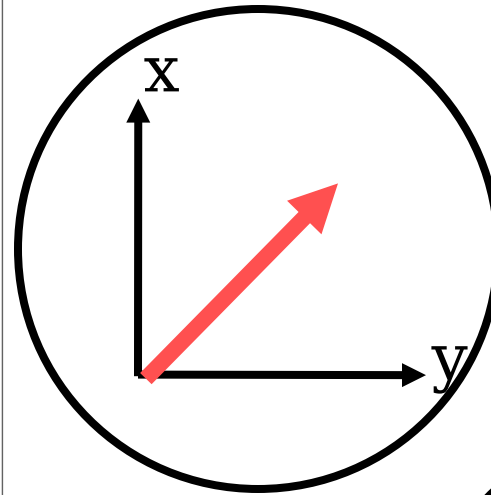
$$\mathcal{R}_c = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

E in Old Coordinates

E' in New Coordinates

Examples

$$\mathcal{R}_{c90} |E_x\rangle = -|E_y\rangle \quad \mathcal{R}_{c90} |E_y\rangle = |E_x\rangle$$



# Alternative Basis Sets

## Requirements

2 Complex Numbers

2 Magnitudes and 2 Phases

Coefficients Multiplying 2 Basis Vectors

## Vector Space

Basis Vectors Orthogonal

$$\langle E_1 | E_2 \rangle = 0$$

Basis Vectors Normalized

$$\langle E_1 | E_1 \rangle = 1$$

4061ppt5.tex-25

## Any Two Orthonormal Vectors

Example: Circular Polarization

Transformation

$$\hat{\mathbf{E}}_L = \frac{\hat{\mathbf{E}}_x + j\hat{\mathbf{E}}_y}{\sqrt{2}}$$

$$\hat{\mathbf{E}}_R = \frac{\hat{\mathbf{E}}_x - j\hat{\mathbf{E}}_y}{\sqrt{2}}$$

Transformation Matrix

$$\mathcal{T} = \begin{pmatrix} 1 & j \\ 1 & -j \end{pmatrix}$$

Inverse Transformation

$$\hat{\mathbf{E}}_x = \frac{\hat{\mathbf{E}}_R + j\hat{\mathbf{E}}_L}{\sqrt{2}} \quad \hat{\mathbf{E}}_y = \frac{\hat{\mathbf{E}}_R - j\hat{\mathbf{E}}_L}{\sqrt{2}}$$

4061ppt5.tex-26

# Other Transforms

Old Basis: Linear X, Y

New Basis: LHC, RHC

$$\begin{pmatrix} E_L \\ E_R \end{pmatrix} = \begin{pmatrix} \langle E_L | E_X \rangle & \langle E_L | E_Y \rangle \\ \langle E_R | E_X \rangle & \langle E_R | E_Y \rangle \end{pmatrix} \begin{pmatrix} E_X \\ E_Y \end{pmatrix}$$

$$\begin{pmatrix} E_L \\ E_R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} E_X \\ E_Y \end{pmatrix}$$

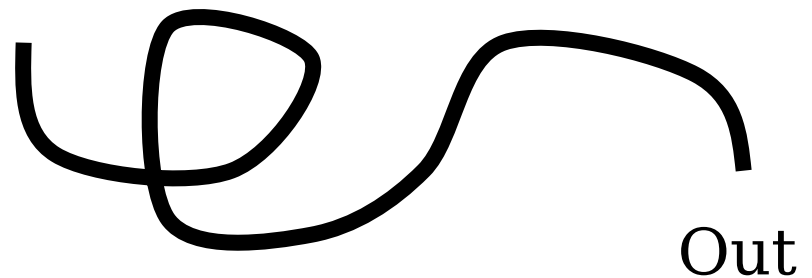
Unitary Matrix

$$\langle E_{out} | E_{out} \rangle = \langle E_{in} | J^\dagger J | E_{in} \rangle$$

$$J^\dagger J = I$$

4061ppt5.tex-21

Transform to  
Eigenstates  
of a Fiber may be  
complicated, but  
useful



# Some Jones Matrices

## X, Y Polarizers

$$\mathcal{P}_X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \mathcal{P}_Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

## Quarter-Wave Plate

Fast on Y-Axis

$$\mathcal{W}_{\lambda/4} = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} = e^{i\pi/4} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

## Rotator

Rotation of Coordinates  
(No Polarization Change)

$$\mathcal{R}_\theta = \mathcal{R}_{c(-\theta)} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Physical Rotation of Polarization

## X-Polarizer Jones Matrix

$$\mathcal{P}_X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad |E_{in}\rangle = \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$$|E_{out}\rangle = \mathcal{P}_X |E_{in}\rangle =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_x \\ 0 \end{pmatrix}$$

Conjugate:  $\langle E_{out}| = \langle E_{in}| \mathcal{P}_X^\dagger =$

$$\begin{pmatrix} E_x^* & E_y^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} E_x^* & 0 \end{pmatrix}$$

$$\mathcal{P}^\dagger = (\mathcal{P}^*)^T$$

Pwr:  $\langle E_{out}|E_{out}\rangle = \langle E_{in}| \mathcal{P}_X^\dagger \mathcal{P}_X |E_{in}\rangle$

Malus Law,  $\cos^2 \theta$

# Components at Arbitrary Angles

## Rotated Coordinates

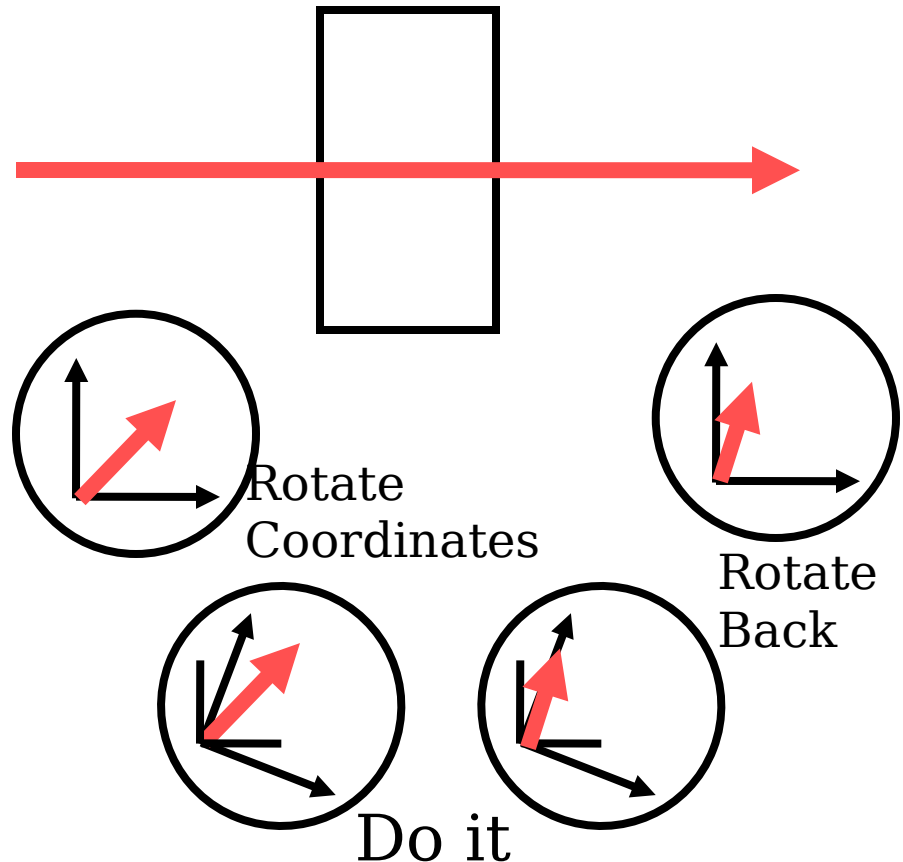
$\mathcal{M}'$  is Matrix of  
Element in  $x', y'$  Coordinates  
 $\mathcal{M}$  is Matrix in  
 $x, y$  Coordinates

Rotate  $|E'_{in}\rangle = \mathcal{R}_{-\theta}|E_{in}\rangle$   
Do it  $|E'_{out}\rangle = \mathcal{M}'|E'_{in}\rangle = \mathcal{M}'\underline{\mathcal{R}_{\theta}}|E_{in}\rangle$   
Return  $|E_{out}\rangle = \mathcal{R}_{+\theta}\mathcal{M}'\underline{\mathcal{R}_{\theta}}|E_{in}\rangle$

## Overall Result

$$\mathcal{M} = \mathcal{R}_{+\theta}\mathcal{M}'\mathcal{R}_{-\theta}$$

4061ppt5.tex-24





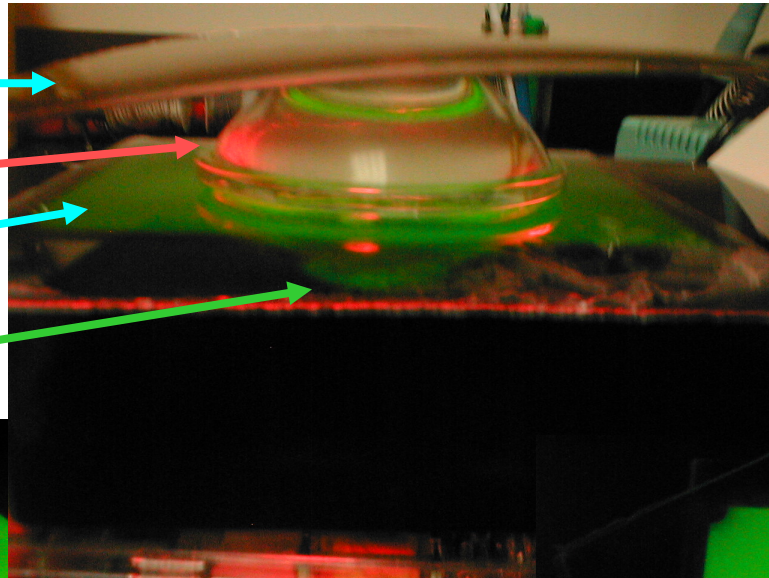
# Maltese Cross (1)

Polarizer

Fast Lens

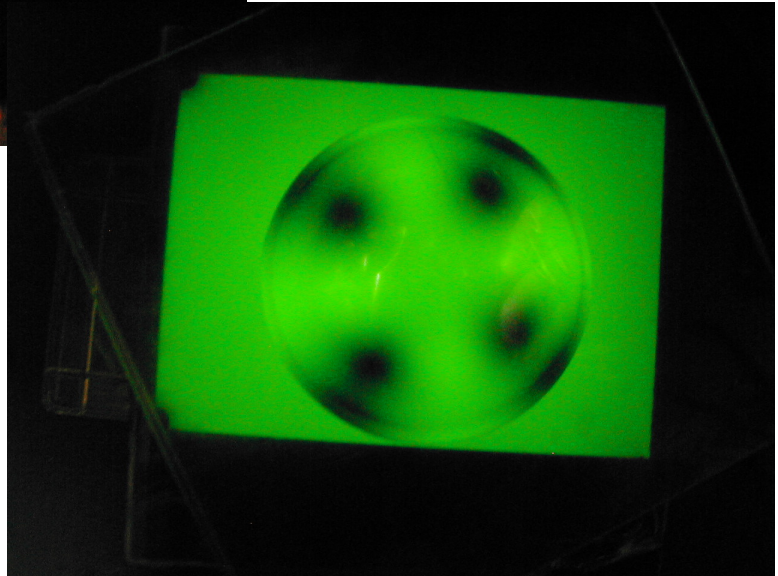
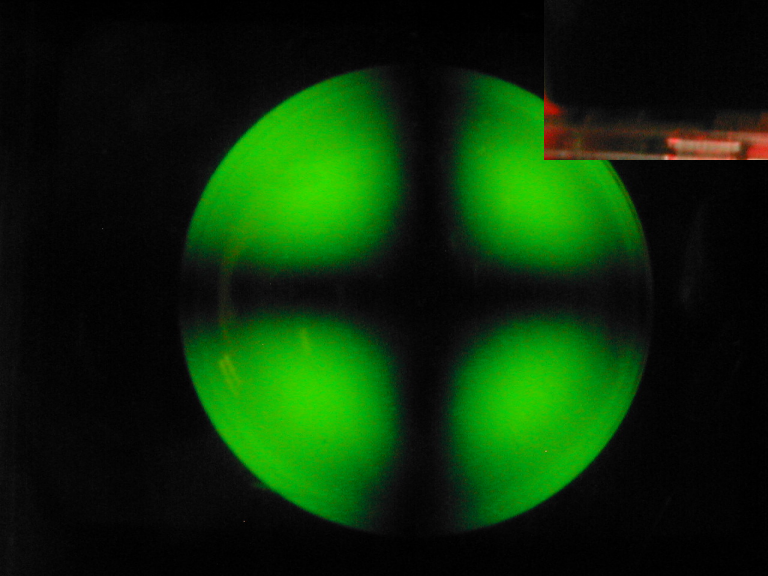
Polarizer

Green Light



What is the Orientation of the Polarizers in Each Photo Below?

Why must the lens be fast to show this effect?



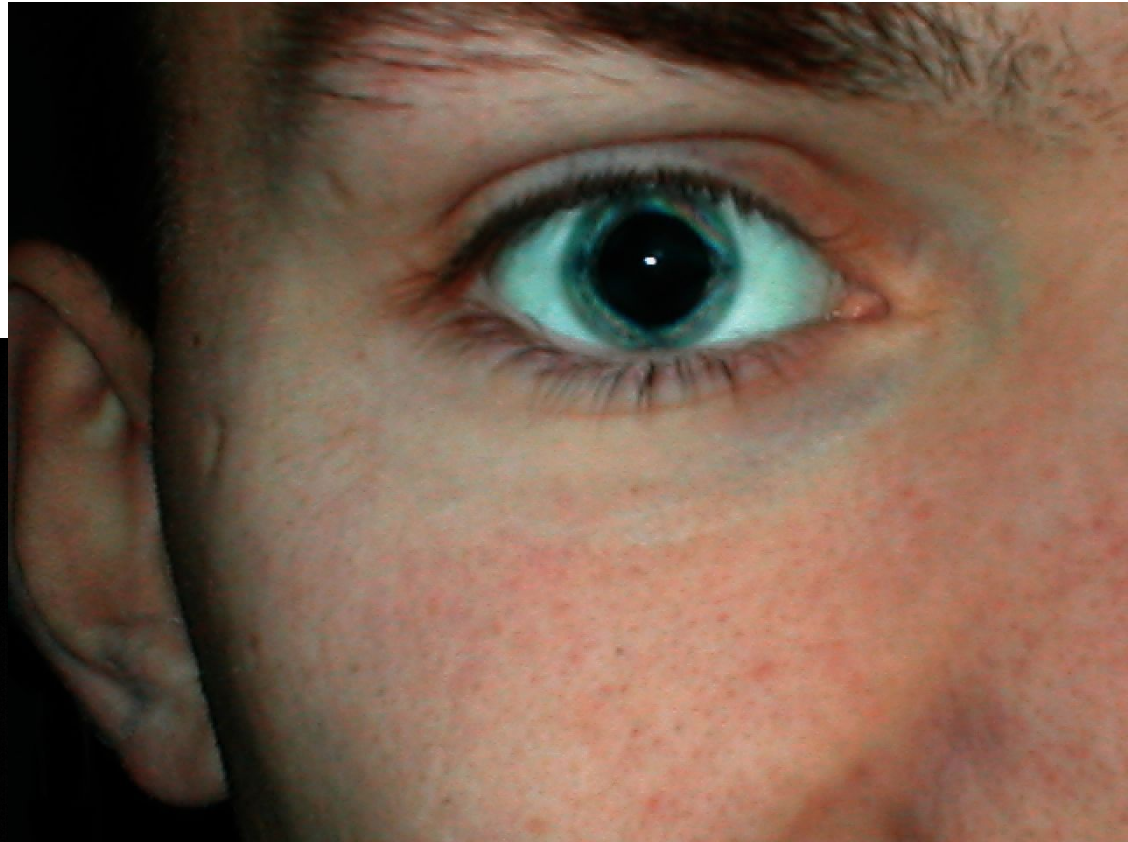


# Maltese Cross (2)

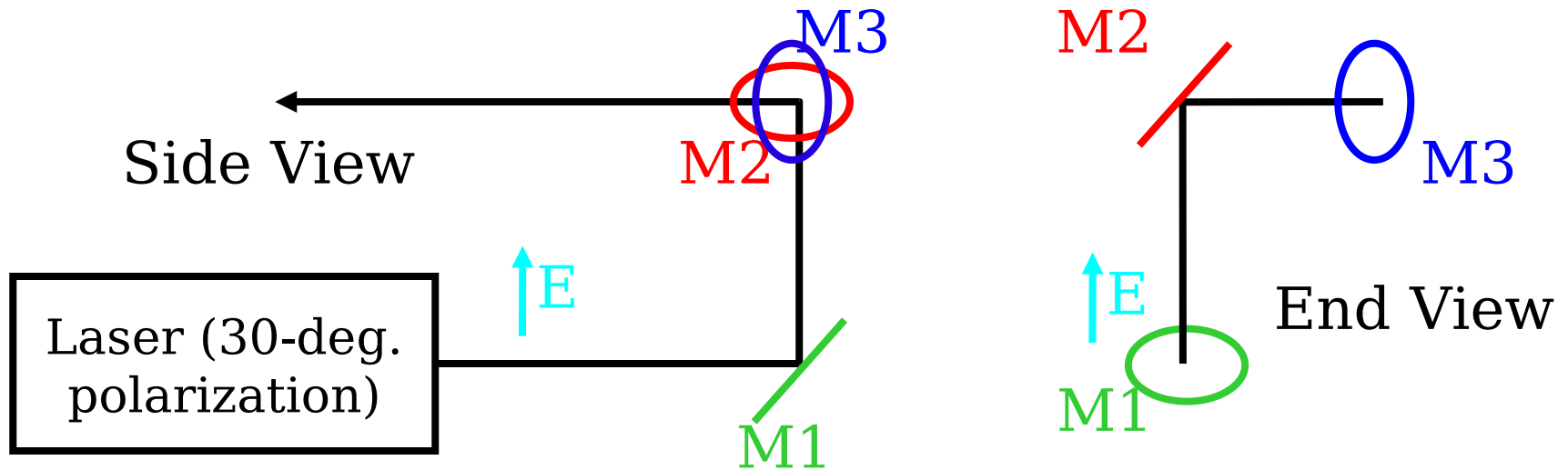


Photo at right  
taken with vertical  
polarizer over flash  
and horizontal

polarizer  
camera.



# Rotation Example



- Beam Path:
  - “North” to M1, Up to M2, “West” to M3, “South”
- What is the Polarization after M3?
  - Suppose Mirrors Are Metal,  $n=4+3i$ , at 45 Degrees

# Partial Polarization

Coherent Light is Polarized

$$|E\rangle = \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

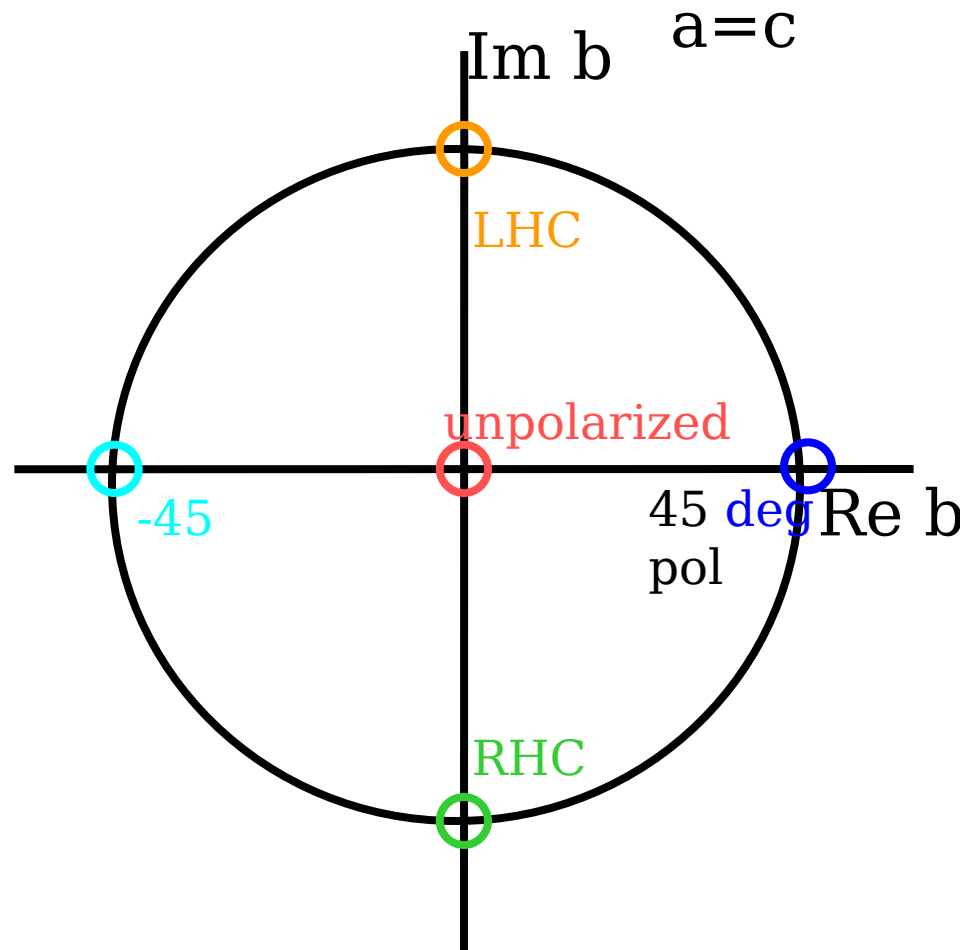
Coherency Matrix

Outer Product

$$|E\rangle\langle E| = \begin{pmatrix} E_x E_x^* & E_x E_y^* \\ E_y E_x^* & E_y E_y^* \end{pmatrix}$$

Ensemble Average

$$\overline{|E\rangle\langle E|} = \begin{pmatrix} \overline{E_x E_x^*} & \overline{E_x E_y^*} \\ \overline{E_y E_x^*} & \overline{E_y E_y^*} \end{pmatrix}$$



# Coherency Matrices (1)

## Definition

$$\overline{|E\rangle\langle E|} = \begin{pmatrix} \overline{E_x E_x^*} & \overline{E_x E_y^*} \\ \overline{E_y E_x^*} & \overline{E_y E_y^*} \end{pmatrix} =$$

$$C = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$$

## Physical Interpretation

$$a = \overline{E_x E_x^*} = \text{Power in x-polarization}$$

$$c = \overline{E_y E_y^*} = \text{Power in y-polarization}$$

$$b = \overline{E_x E_y^*} = \text{Cross-Correlation}$$

## Polarizing Elements

Given Jones Matrix

$$|E\rangle_{out} = J |E\rangle_{in}$$

Conjugate Equation

$$\langle E|_{out} = \langle E|_{in} J^\dagger$$

Outer Product

$$|E\rangle_{out} \langle E|_{out} = J |E\rangle_{in} \langle E|_{in} J^\dagger$$

$$C_{out} = \overline{J C_{in} J^\dagger}$$

Constant J

$$C_{out} = J C_{in} J^\dagger$$

# Coherency Matrices (2)

## Example

Input: Random Polarization

$$C_{in} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Element: X-Polarizer

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

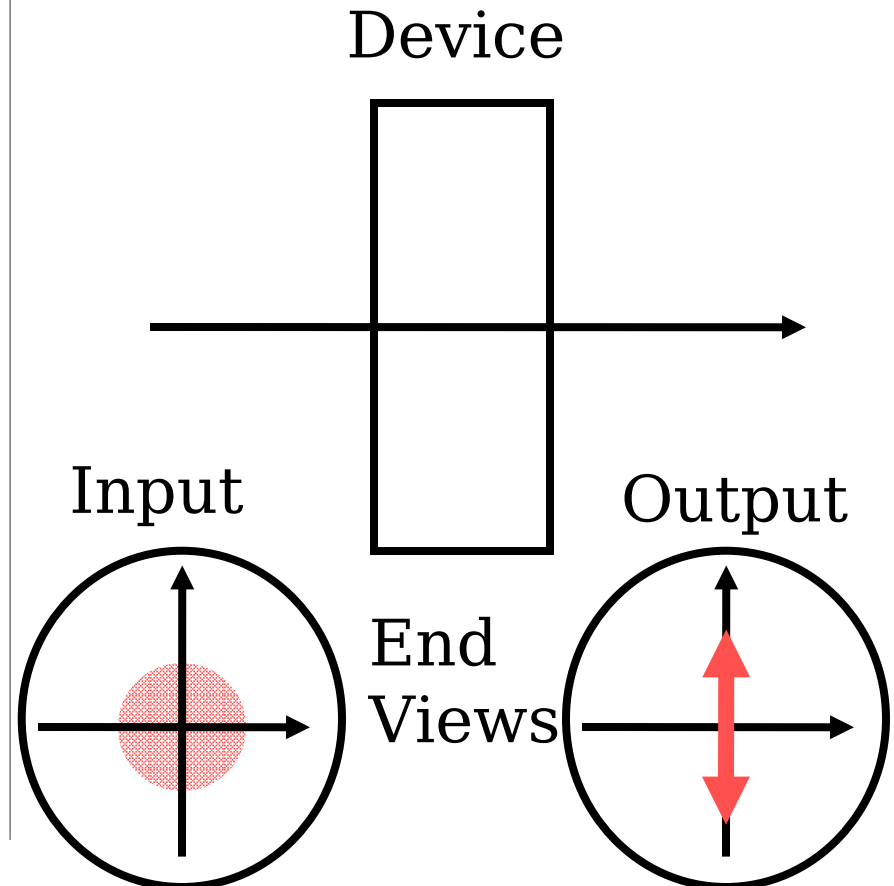
Output

$$C_{out} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

4061ppt5.tex-30

July 2003

Chuck DiMarzio, Northeastern University



11270-05-47

# Stokes Parameters

For Partial Polarization  
Statistical Description  
Coherency Matrices  
Alternative Form  
Equations

$$\mathbf{c} = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$$

|             |               |                            |
|-------------|---------------|----------------------------|
| $S_0 = I =$ | $a + c$       | Total Power                |
| $S_1 = M =$ | $a - c$       | Pref. for X vs. Y          |
| $S_2 = C =$ | $b + b^*$     | $45^\circ$ vs. $-45^\circ$ |
| $S_3 = S =$ | $(b - b^*)/j$ | RHC vs. LHC                |

4061ppt5.tex-31

Degree of Polarization

$$V = \frac{\sqrt{M^2 + C^2 + S^2}}{I} \leq 1$$

Stokes Vector

$$\mathbf{C} = \begin{pmatrix} I \\ M \\ C \\ S \end{pmatrix}$$

Examples

$$\mathbf{S}_x = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{S}_y = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{S}_{45} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

4061ppt5.tex-32

# Stokes Vectors, Mueller

## Matrices

$$\mathbf{S}_{RHC} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{S}_{LHC} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{S}_{random} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

## Mueller Matrices

Transform Stokes Vectors

Have Real Components

Examples

| x-polarizer                                                                                                  | y-polarizer                                                                                                    |
|--------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------|
| $\frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ | $\frac{1}{2} \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ |

Depolarizer

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

# A Depolarizer? What's That?

Depolarization

Optical Fiber

Two Modes, Different Speeds

$$\int |E_{out}|^2 d\lambda =$$

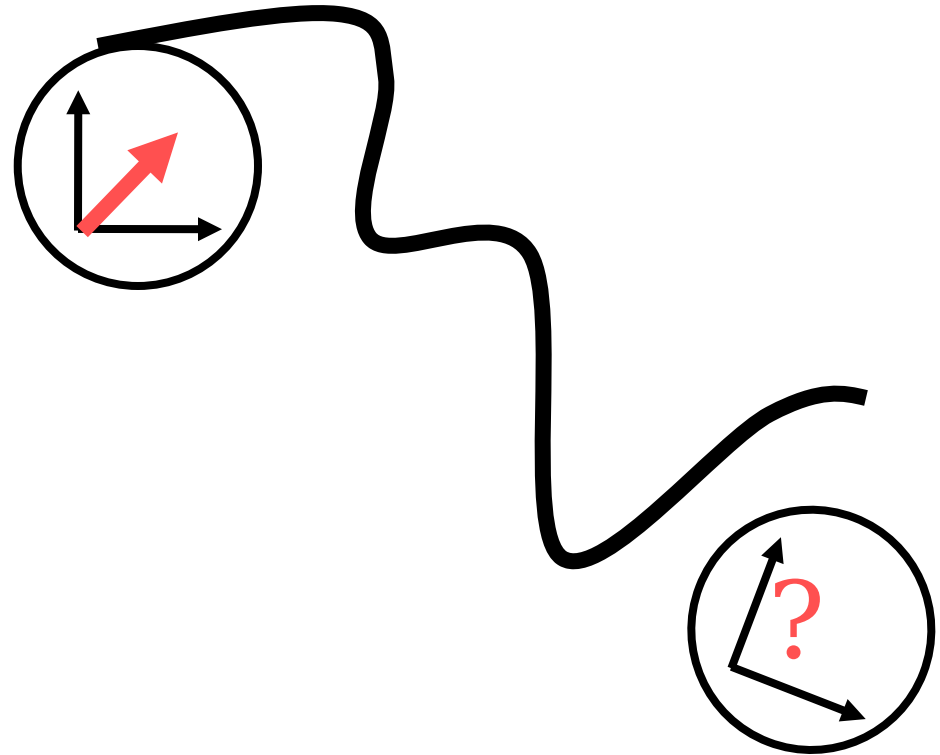
$$\int \mathcal{T}^\dagger \begin{pmatrix} e^{in_1\ell/\lambda} & 0 \\ 0 & e^{in_2\ell/\lambda} \end{pmatrix} \mathcal{T} |E_{in}|^2 d\lambda$$

$\mathcal{T}$  Transforms to Eigenvectors

Large Bandwidth

Many Cycles in  $e^{i\Delta n\ell/\lambda}$ , Randomizes Phases

4061ppt5.tex-35



# The Poincaré Sphere

- Graphical Representation of Stokes Parameters
- DOP is Radius

$$V_s = \begin{bmatrix} I \\ M \\ S \\ C \end{bmatrix}$$

