

**M. Parseval's Relations:**

$$\int_{-\infty}^{\infty} x_1(\lambda)X_2(\lambda) d\lambda = \int_{-\infty}^{\infty} X_1(\lambda)x_2(\lambda) d\lambda \quad (5.62)$$

$$\int_{-\infty}^{\infty} x_1(t)x_2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega)X_2(-\omega) d\omega \quad (5.63)$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \quad (5.64)$$

Equation (5.64) is called *Parseval's identity* (or *Parseval's theorem*) for the Fourier transform. Note that the quantity on the left-hand side of Eq. (5.64) is the normalized energy content  $E$  of  $x(t)$  [Eq. (1.14)]. Parseval's identity says that this energy content  $E$  can be computed by integrating  $|X(\omega)|^2$  over all frequencies  $\omega$ . For this reason,  $|X(\omega)|^2$  is often referred to as the *energy-density spectrum* of  $x(t)$ , and Eq. (5.64) is also known as the *energy theorem*.

Table 5-1 contains a summary of the properties of the Fourier transform presented in this section. Some common signals and their Fourier transforms are given in Table 5-2.

**TABLE 5-1 Properties of the Fourier Transform**

PROPERTY	SIGNAL	FOURIER TRANSFORM
	$x(t)$	$X(\omega)$
	$x_1(t)$	$X_1(\omega)$
	$x_2(t)$	$X_2(\omega)$
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(\omega)$
Frequency shifting	$e^{j\omega_0 t} x(t)$	$X(\omega - \omega_0)$
Time scaling	$x(at)$	$\frac{1}{ a } X\left(\frac{\omega}{a}\right)$
Time reversal	$x(-t)$	$X(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time differentiation	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Frequency differentiation	$(-jt)x(t)$	$\frac{dX(\omega)}{d\omega}$
Integration	$\int_{-\infty}^t x(\tau) d\tau$	$\pi X(0) \delta(\omega) + \frac{1}{j\omega} X(\omega)$
Convolution	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$
Real signal	$x(t) = x_e(t) + x_o(t)$	$X(\omega) = A(\omega) + jB(\omega)$ $X(-\omega) = X^*(\omega)$
Even component	$x_e(t)$	$\text{Re}\{X(\omega)\} = A(\omega)$
Odd component	$x_o(t)$	$j \text{Im}\{X(\omega)\} = jB(\omega)$
Parseval's relations	$\int_{-\infty}^{\infty} x_1(\lambda)X_2(\lambda) d\lambda = \int_{-\infty}^{\infty} X_1(\lambda)x_2(\lambda) d\lambda$ $\int_{-\infty}^{\infty} x_1(t)x_2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega)X_2(-\omega) d\omega$ $\int_{-\infty}^{\infty}  x(t) ^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty}  X(\omega) ^2 d\omega$	

TABLE 5.2 Common Fourier Transforms Pairs

$x(t)$	$X(\omega)$
$\delta(t)$	1
$\delta(t - t_0)$	$e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin \omega_0 t$	$-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$u(-t)$	$\pi\delta(\omega) - \frac{1}{j\omega}$
$e^{-at}u(t), a > 0$	$\frac{1}{j\omega + a}$
$t e^{-at}u(t), a > 0$	$\frac{1}{(j\omega + a)^2}$
$e^{-a t }, a > 0$	$\frac{2a}{a^2 + \omega^2}$
$\frac{1}{a^2 + t^2}$	$e^{-a \omega }$
$e^{-at^2}, a > 0$	$\sqrt{\frac{\pi}{a}} e^{-\omega^2/4a}$
$p_a(t) = \begin{cases} 1 &  t  < a \\ 0 &  t  > a \end{cases}$	$2a \frac{\sin \omega a}{\omega a}$
$\frac{\sin at}{\pi t}$	$p_a(\omega) = \begin{cases} 1 &  \omega  < a \\ 0 &  \omega  > a \end{cases}$
$\text{sgn } t$	$\frac{2}{j\omega}$
$\sum_{k=-\infty}^{\infty} \delta(t - kT)$	$\omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0), \omega_0 = \frac{2\pi}{T}$

## 5.5 The Frequency Response of Continuous-Time LTI Systems

### A. Frequency Response:

In Sec. 2.2 we showed that the output  $y(t)$  of a continuous-time LTI system equals the convolution of the input  $x(t)$  with the impulse response  $h(t)$ ; that is,

$$y(t) = x(t) * h(t) \quad (5.65)$$

Applying the convolution property (5.58), we obtain

$$Y(\omega) = X(\omega)H(\omega) \quad (5.66)$$

where  $Y(\omega)$ ,  $X(\omega)$ , and  $H(\omega)$  are the Fourier transforms of  $y(t)$ ,  $x(t)$ , and  $h(t)$ , respectively. From Eq. (5.66) we have

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} \quad (5.67)$$