Circuits and Signals: Biomedical Applications
Week 10

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Week 10 Agenda: Circuits and Sine/Cosine Waves

- Fourier Series and Inverse
- Fourier Transforms
- Digital Fourier Transforms
- Fast Fourier Transforms
- Examples
- Convolution
- Impulse Response and Transfer Function
- Some Fourier Transform Pairs

Fourier Series: Sine and Cosine

- Fourier Series

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

- Inverse

$$a_0 = \frac{1}{T}\int_{cycle} f(t)\, dt$$

$$a_n = \frac{2}{T}\int_{cycle} f(t)\cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T}\int_{cycle} f(t)\sin(n\omega_0 t)\, dt$$

Fourier Series: Exponential

$$v(t) = \sum_{n=-\infty}^{\infty} \left(\frac{\mathbf{V}_n}{2} e^{jn\omega_0 t} \right)$$

$$\mathbf{V}_n = \frac{1}{T} \int_{cycle} v(t)\, e^{-jn\omega_0 t} dt$$

Because (New Concept: Orthogonal Functions)

$$\frac{1}{T} \int_{cycle} e^{jn\omega_0 t} e^{-jn\omega_0 t} dt = 1$$

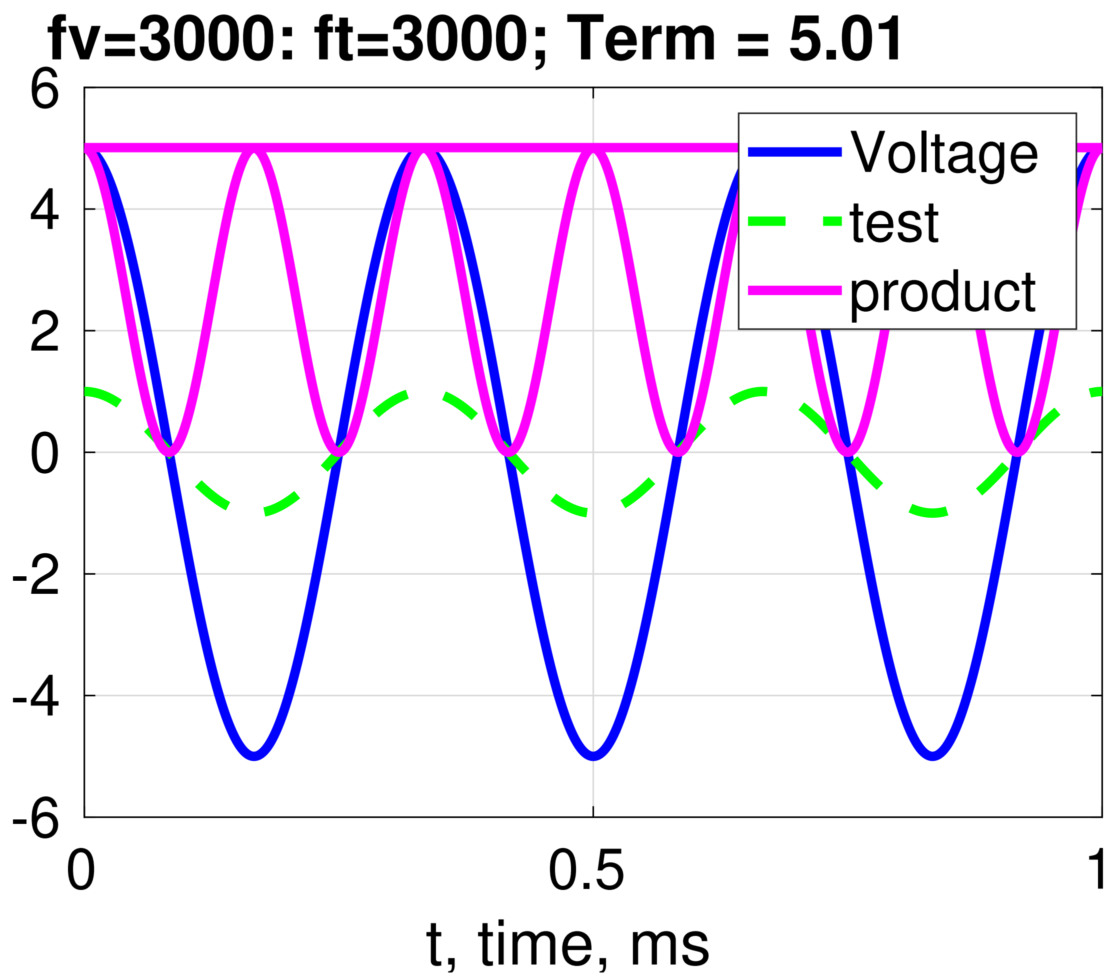
and

$$\frac{1}{T} \int_{cycle} e^{jm\omega_0 t} e^{-jn\omega_0 t} dt = 0$$

for $n \neq m$

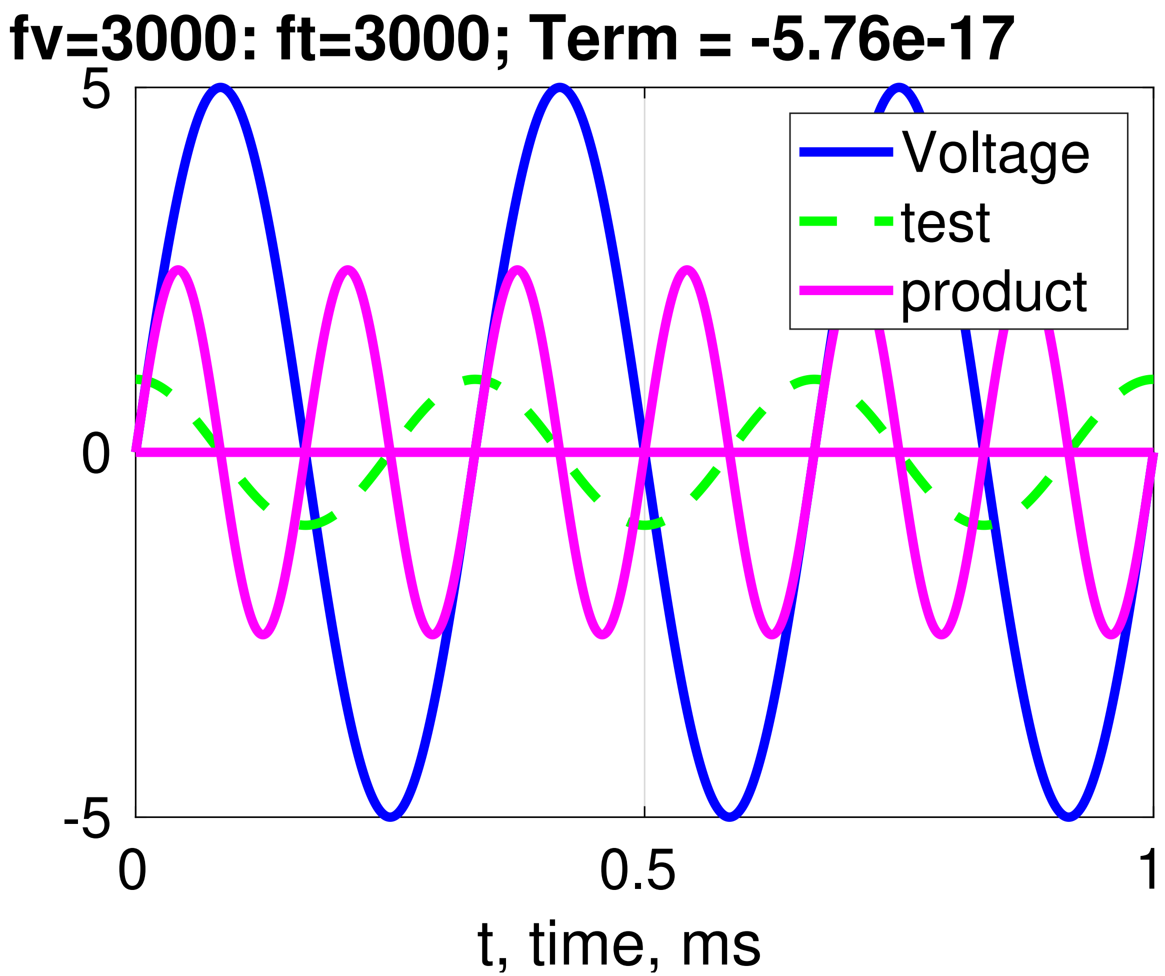
Computing the Coefficients (1)

Good Match: $a_n = \frac{2}{T}\int_{cycle} f\left(t\right)\cos\left(n\omega_0 t\right)dt$



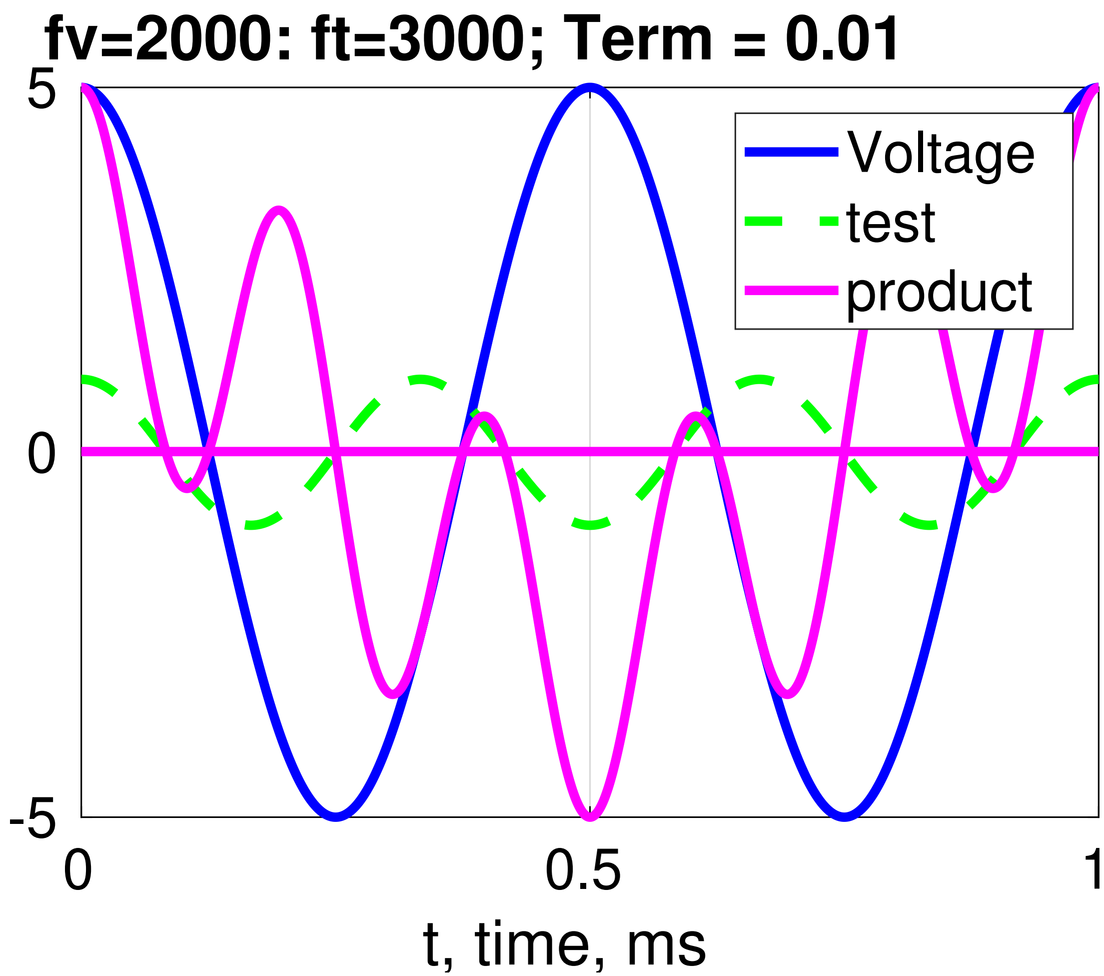
Computing the Coefficients (2)

Sine Signal with Cosine Test: $a_n = \frac{2}{T}\int_{cycle} f\left(t\right)\cos\left(n\omega_0 t\right)dt$



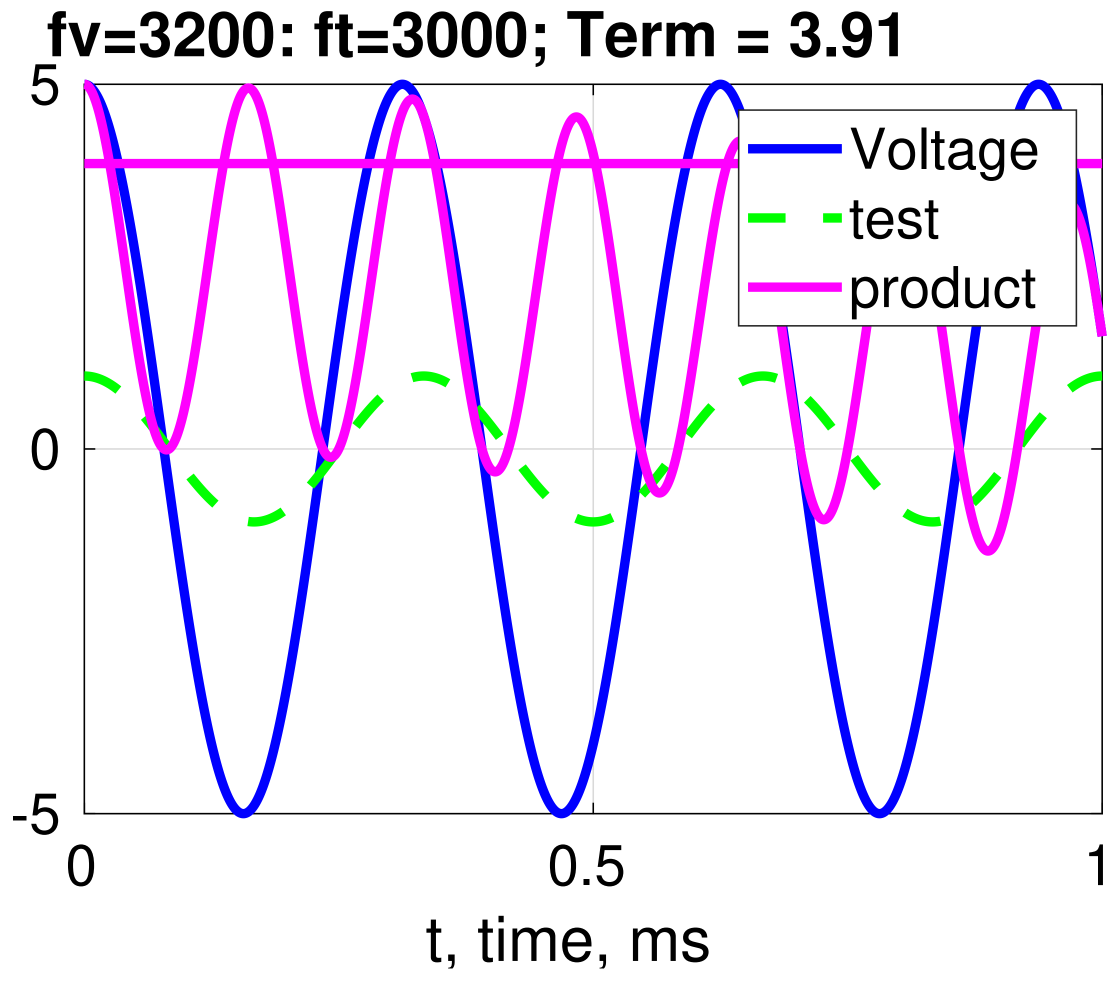
Computing the Coefficients (3)

Different Frequency: $a_n = \frac{2}{T}\int_{cycle} f\left(t\right)\cos\left(n\omega_0 t\right)dt$



Computing the Coefficients (4)

Non–Integer Multiple of Fundamental (Not allowed)



Fourier Series Properties

- Real Function of Time $\longleftrightarrow$ $F(-\omega) = F(+\omega)^*$
- Even Function of Time $\longleftrightarrow$ Just Cosines (F Real)
- Odd Function of Time $\longleftrightarrow$ Just Sines (F Imaginary)
- Half–Wave Symmetry $f(t) = -f(t - T/2)$ $\longleftrightarrow$ Terms for Even Harmonics Zero *e.g.* Square Wave

Fourier Transforms

- Let Period go to Infinity
- Sum Becomes an Integral
- Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

- Inverse Transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

- Bad News: The limits of integration are infinite.

Fourier Transform Properties (1)

- What is the Difference Between These Equations?

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

- Most of what's true for FT is true for IFT too.

- Periodic in Time $\longleftrightarrow$ Discrete in Frequency

- Discrete in Time $\longleftrightarrow$ Periodic in Frequency

Fourier Transform Properties (2)

- True for Fourier Series and Transforms

- Real Function of Time $\longleftrightarrow F(-\omega) = F(+\omega)^*$

- Even Function of Time $\longleftrightarrow$ Just Cosines (F Real)

- Odd Function of Time $\longleftrightarrow$ Just Sines (F Imaginary)

- Half–Wave Symmetry $f(t) = -f(t - T/2) \longleftrightarrow$ Terms for Even Harmonics Zero

Some Operational Fourier Transforms

- Time Shift

$$f(t+\tau) \longleftrightarrow F(\omega)\, e^{j\omega\tau}$$

- Derivative

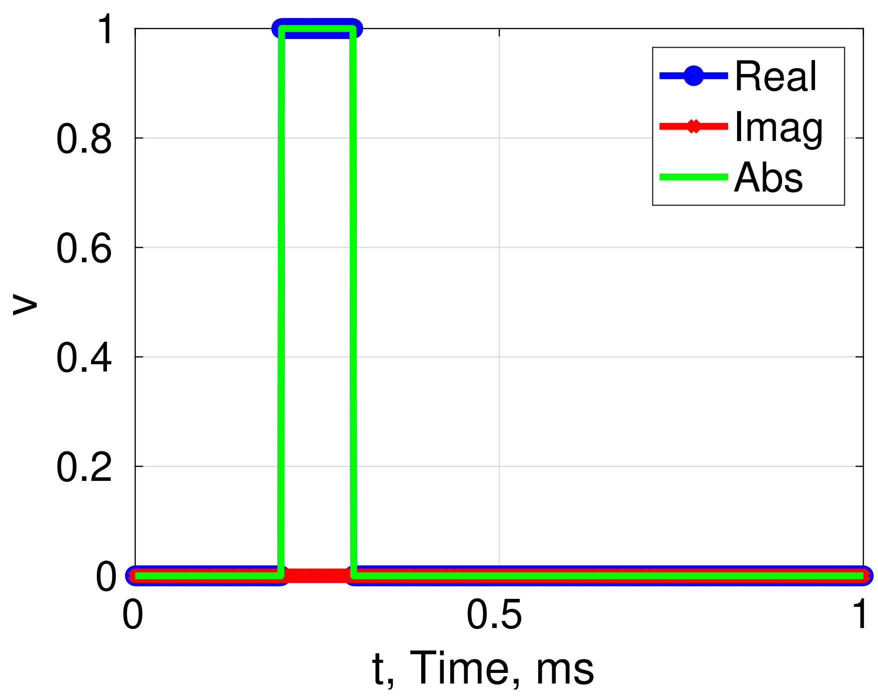
$$\frac{df(t)}{dt} \longleftrightarrow F(\omega)\, j\omega$$

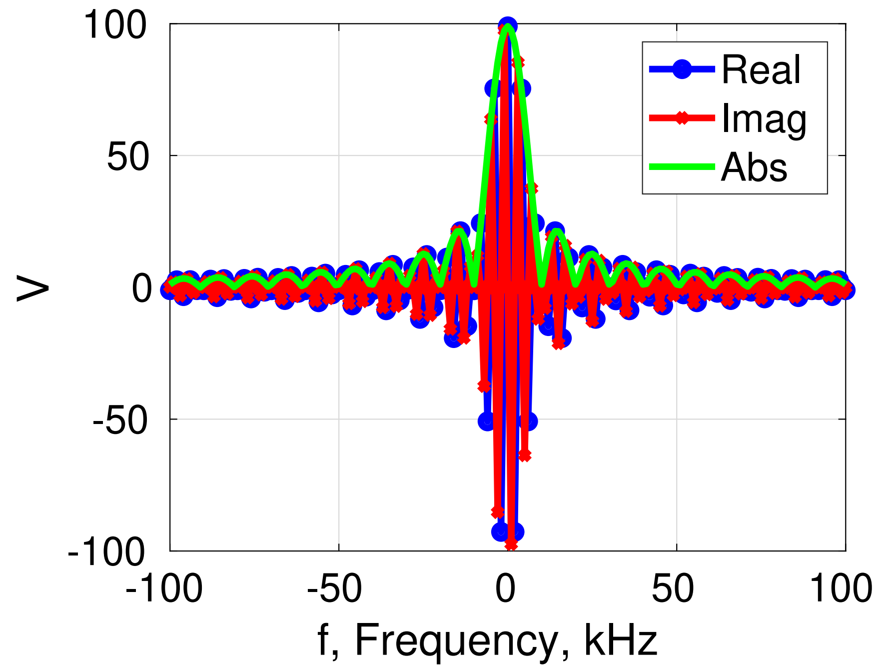
- Integral

$$\int f(t)\, dt \longleftrightarrow F(\omega)\, \frac{1}{j\omega}$$

Fourier Transform Pairs: Pulse

`rect` in Time: Width τ

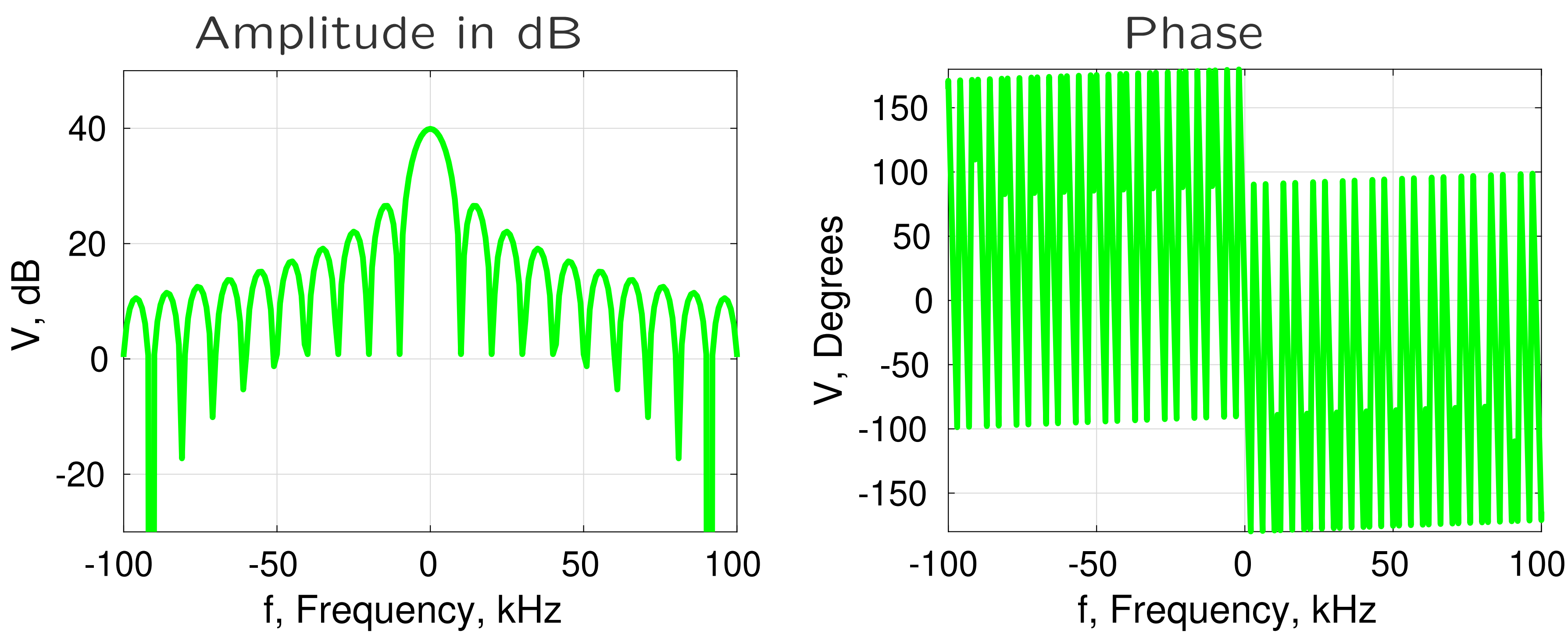




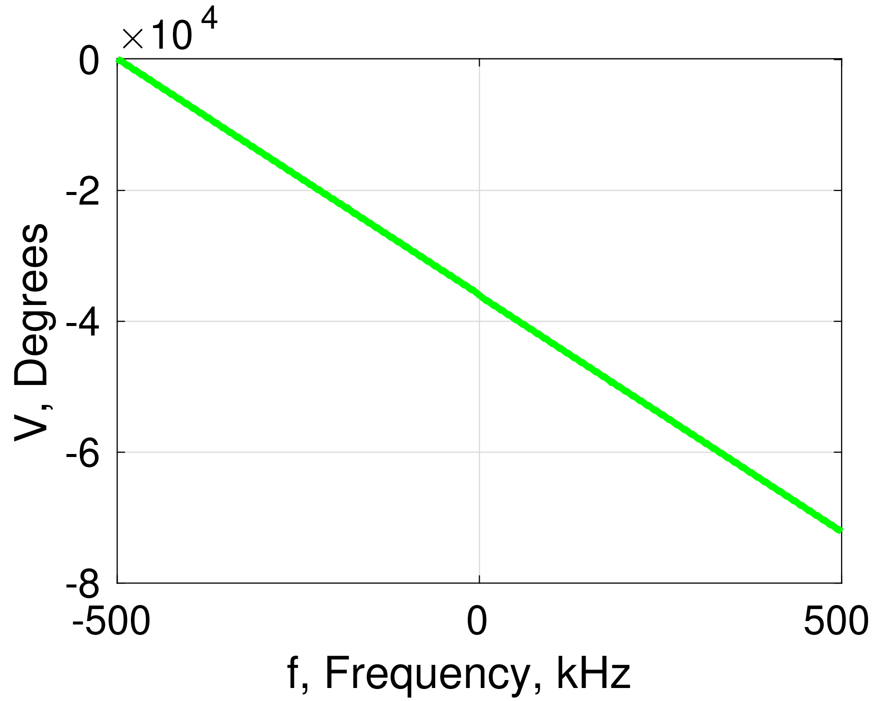
`sinc` in Frequency: $\frac{\sin \omega\tau}{\omega\tau}$

Width of transform decreases as width of pulse increases.

FT of Pulse


Amplitude in dB
40
20
0
-20
V, dB
-100
-50
0
50
100
f, Frequency, kHz
Phase
150
100
50
0
-50
-100
-150
V, Degrees
-100
-50
0
50
100
f, Frequency, kHz

Phase Unwrap



Fast Fourier Transform

- Discrete in Time, Periodic in Frequency
- Discrete in Frequency, Periodic in Time
- Evenly Spaced Times and Frequencies

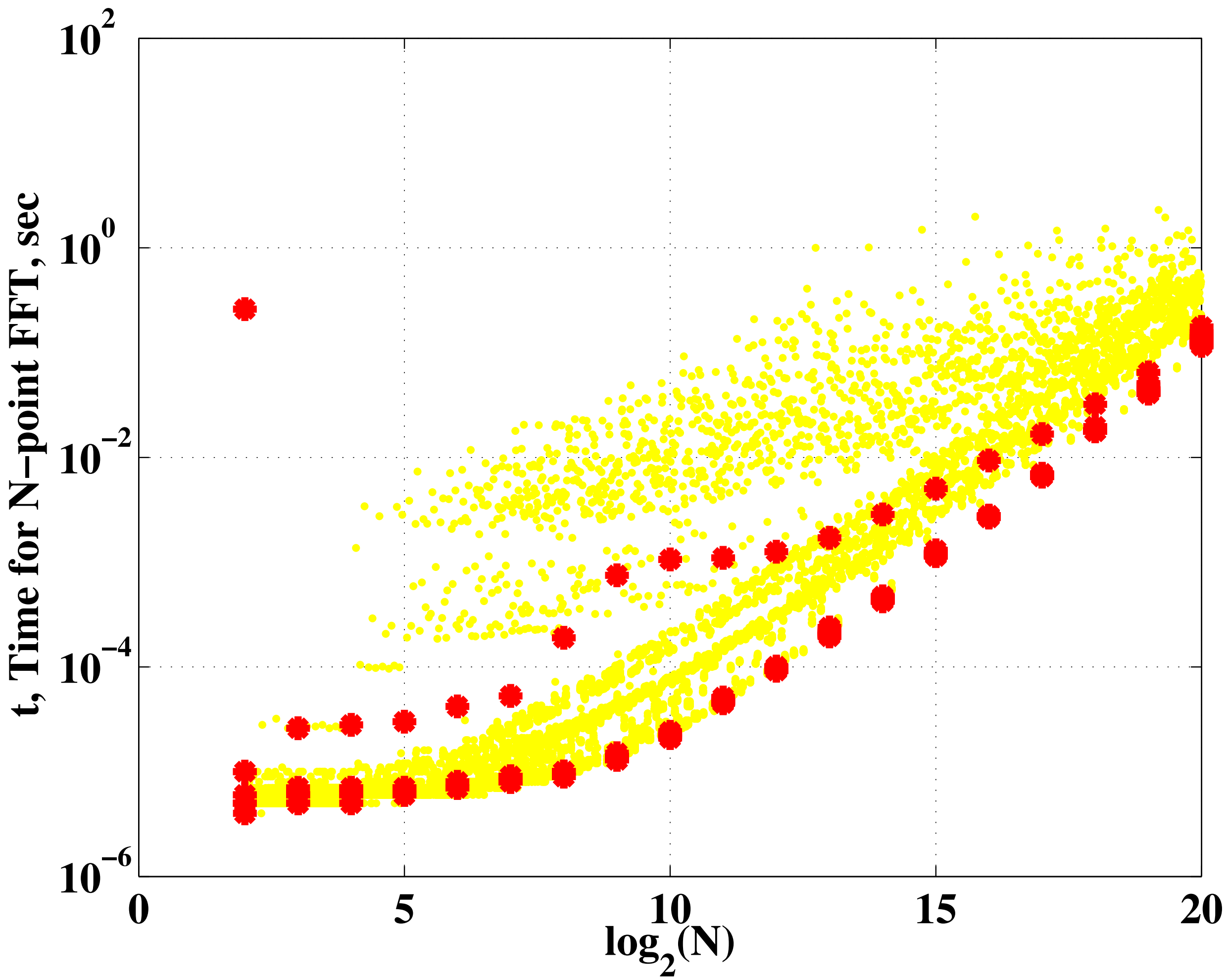
$$\cos\left(2\pi n f_{step} m t_{step}\right) = \cos\left(2\pi f_{step} t_{step} m n\right) = ?$$

$$f_{step} = 1/t_{max} \qquad t_{step} = t_{max}/N = 2 * f_{Nyquist}$$

$$f_{max} = N f_{step} = 2 * f_{Nyquist} - f_{step}$$

Fast Fourier Transform is Fast

Especially when $N = 2^{\texttt{integer}}$



Peculiarities of the FFT Algorithm

- Time and Frequency Axes

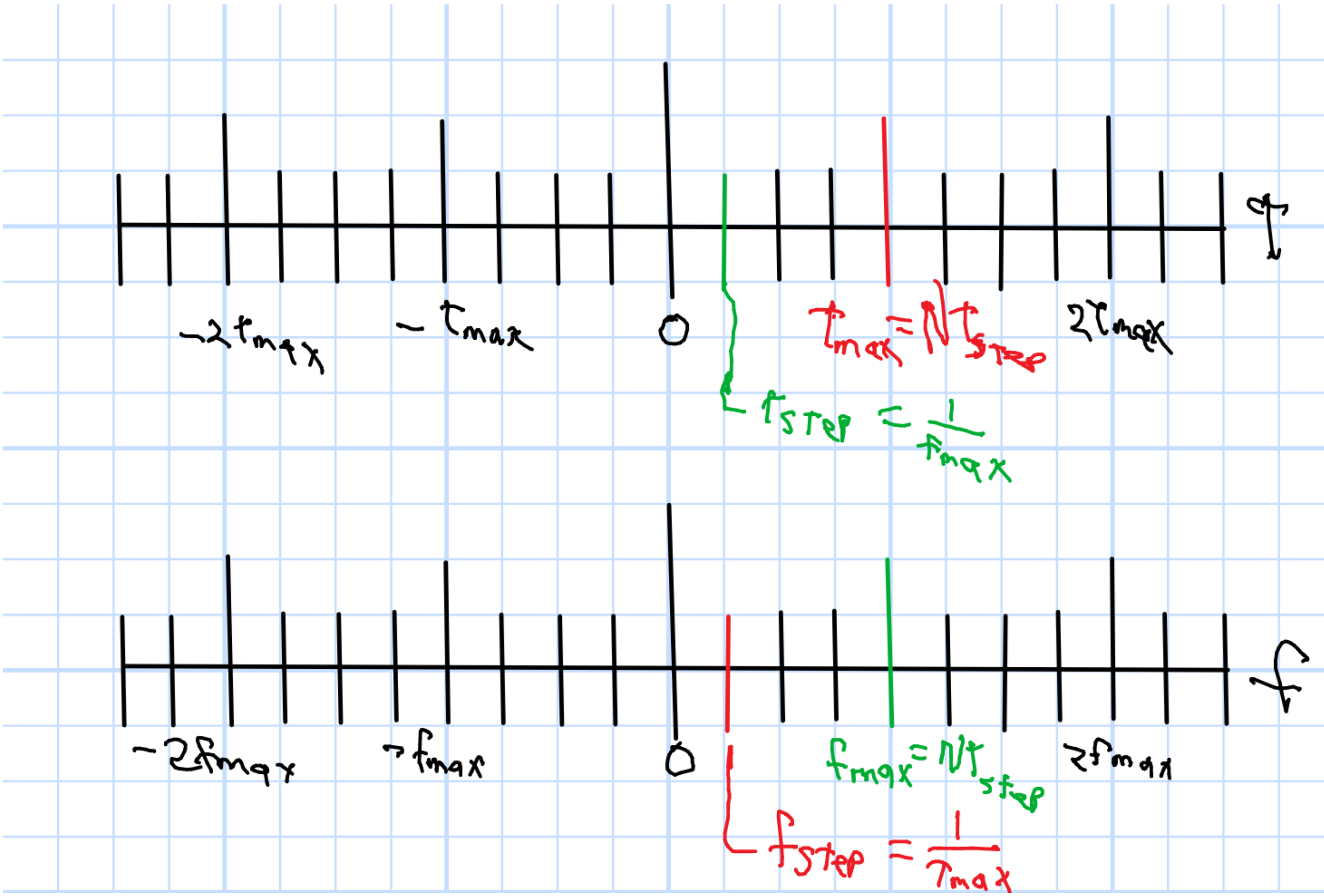
- Periodicity

- Shifted Negative Frequencies

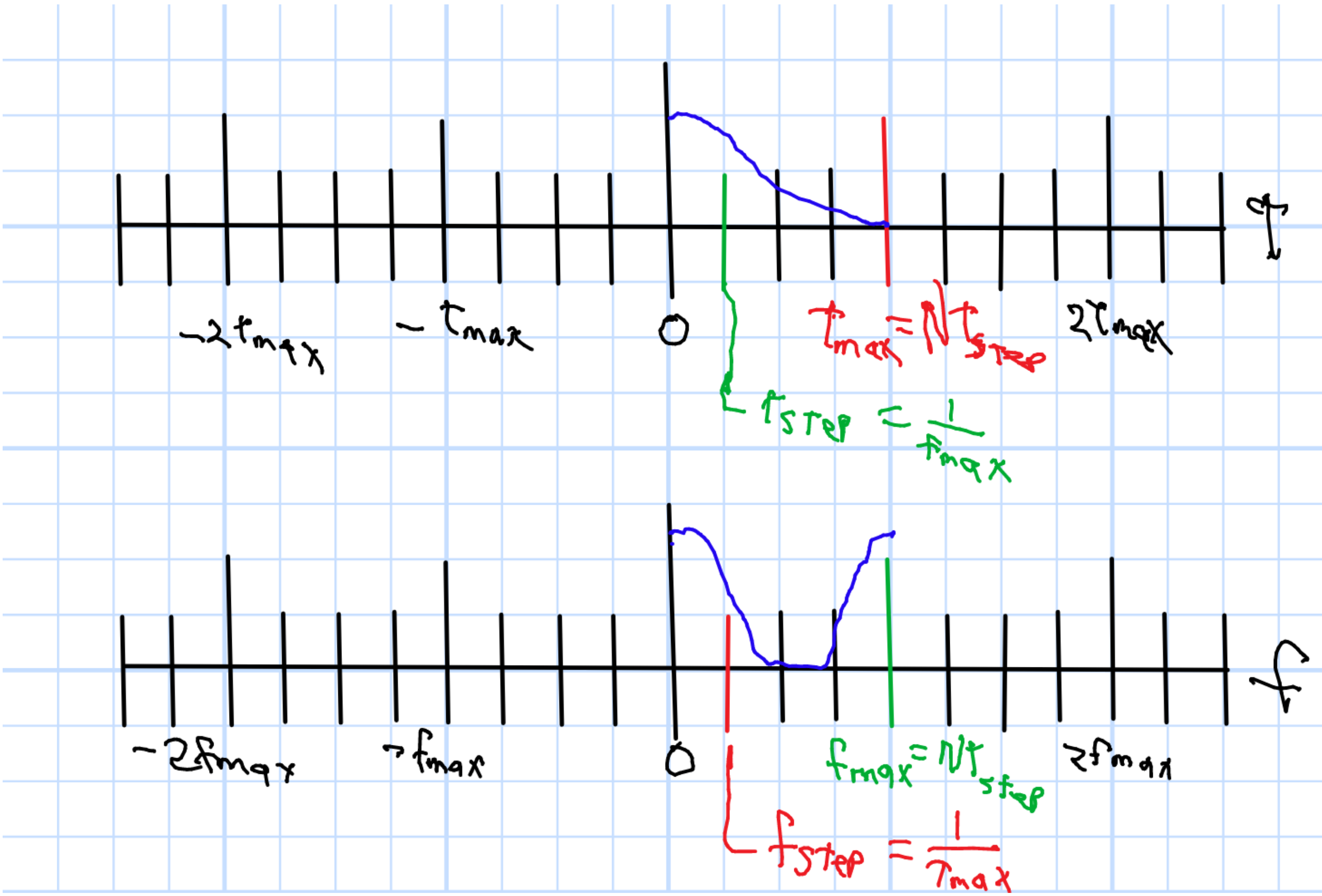
- See fft examples.

https://ece.northeastern.edu/courses/eece2150/2018fa/html/ftransform.html

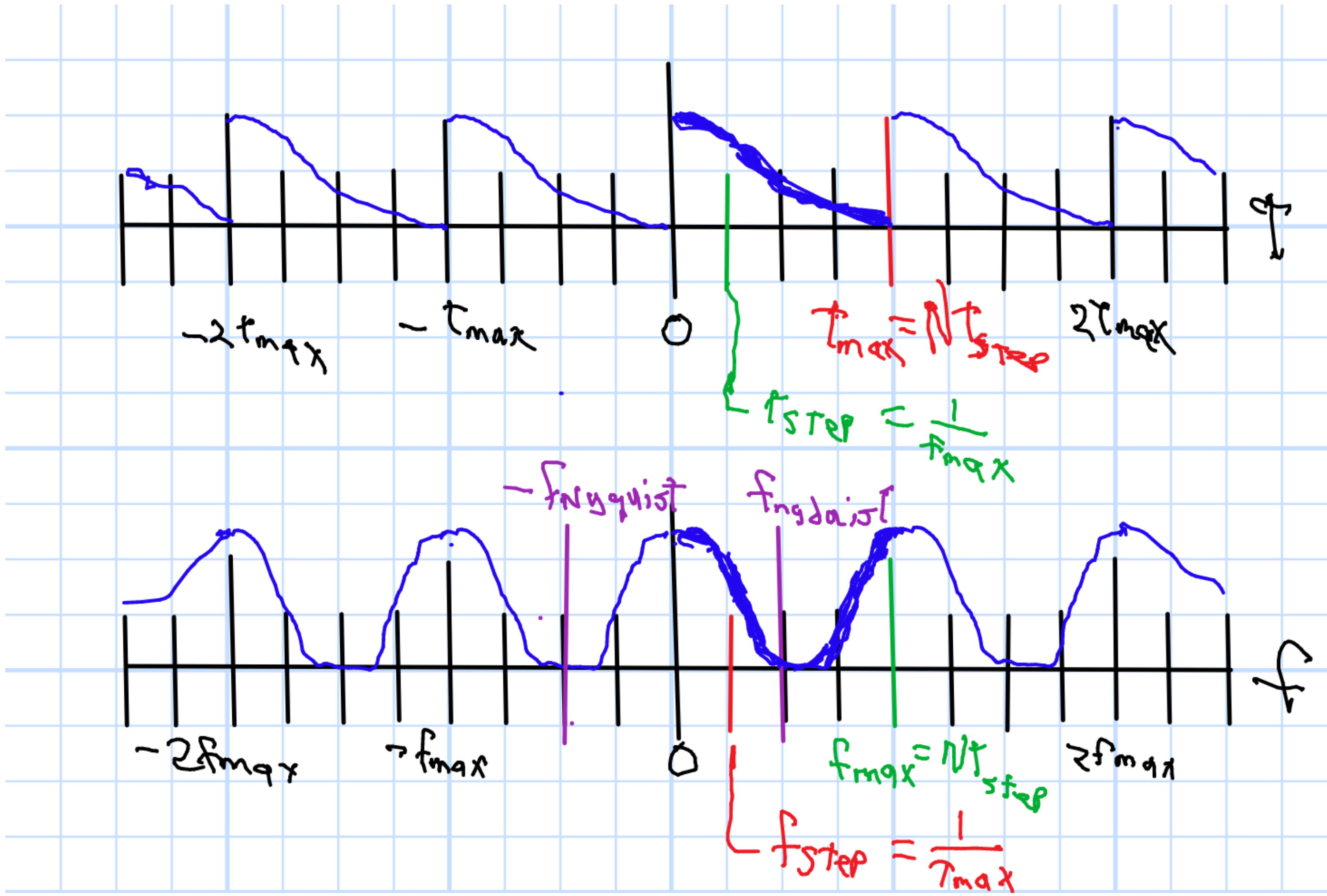
Time and Frequency Axes



FFT Algorithm Results

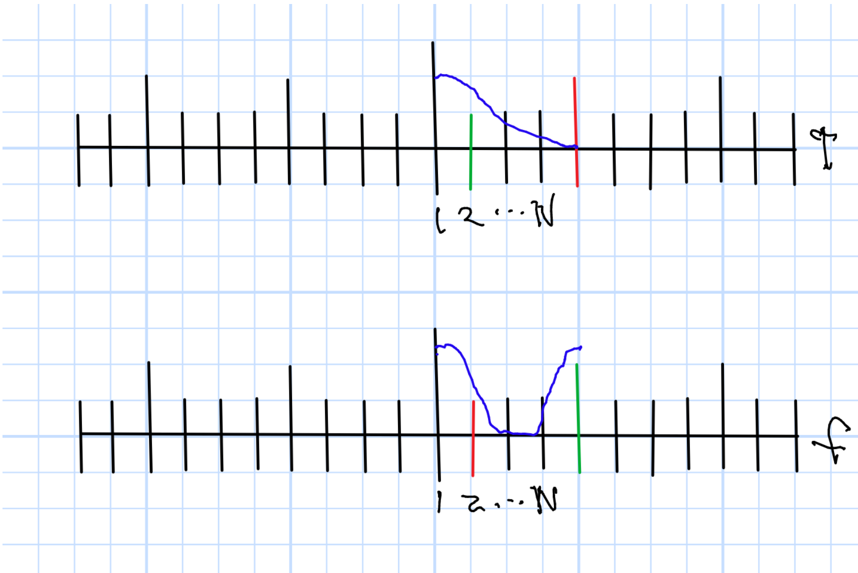


Applying Periodicity



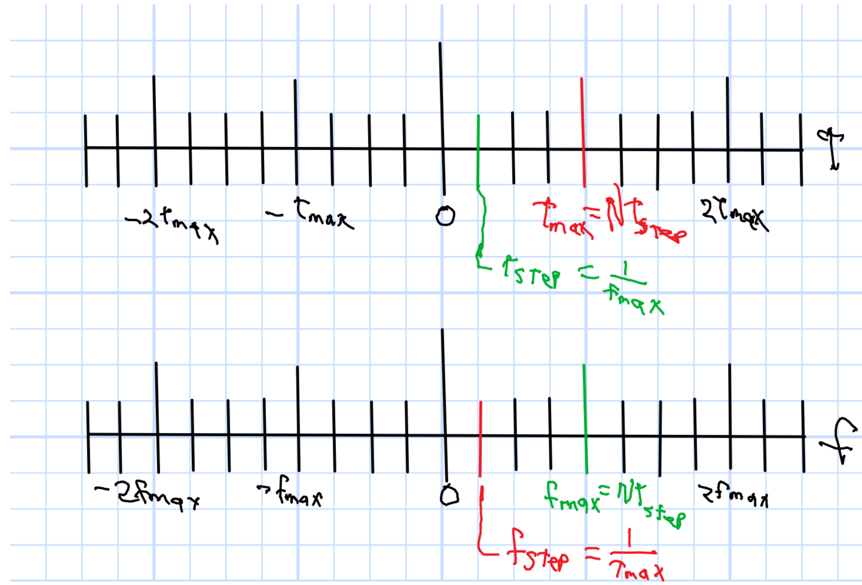
Matlab FFT Functions

Matlab Function `fft`



`V1=fft(v)`

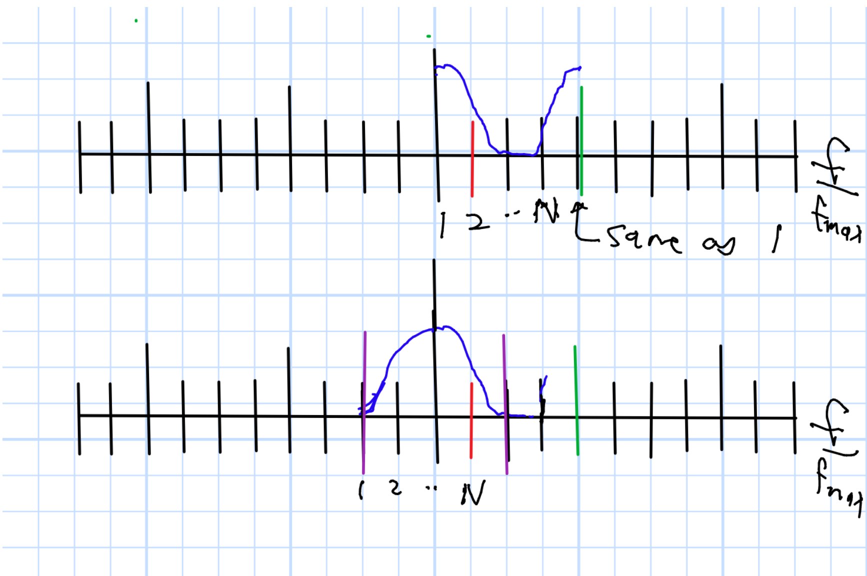
My `fftaxis.m` (See Website)



`faxis1=fftaxis(taxis)`

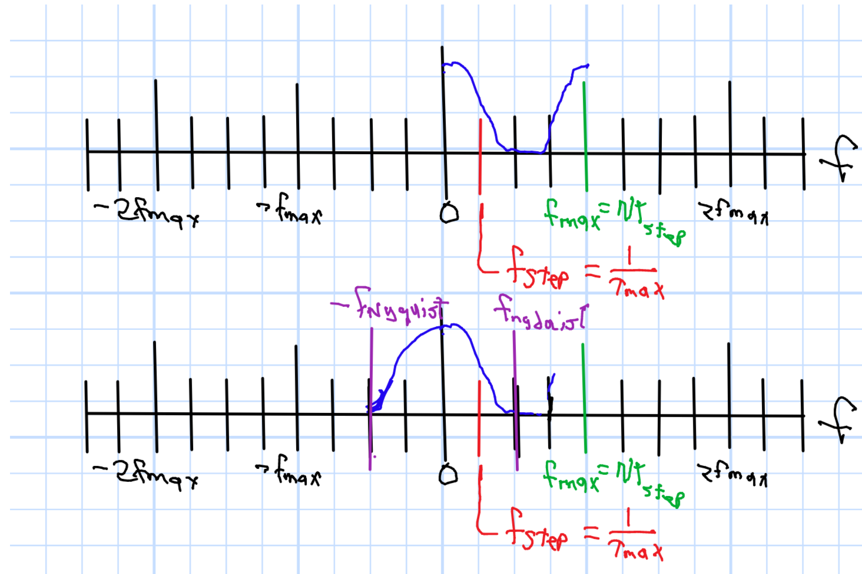
Matlab Shifting Functions

Matlab Function `fftshift`



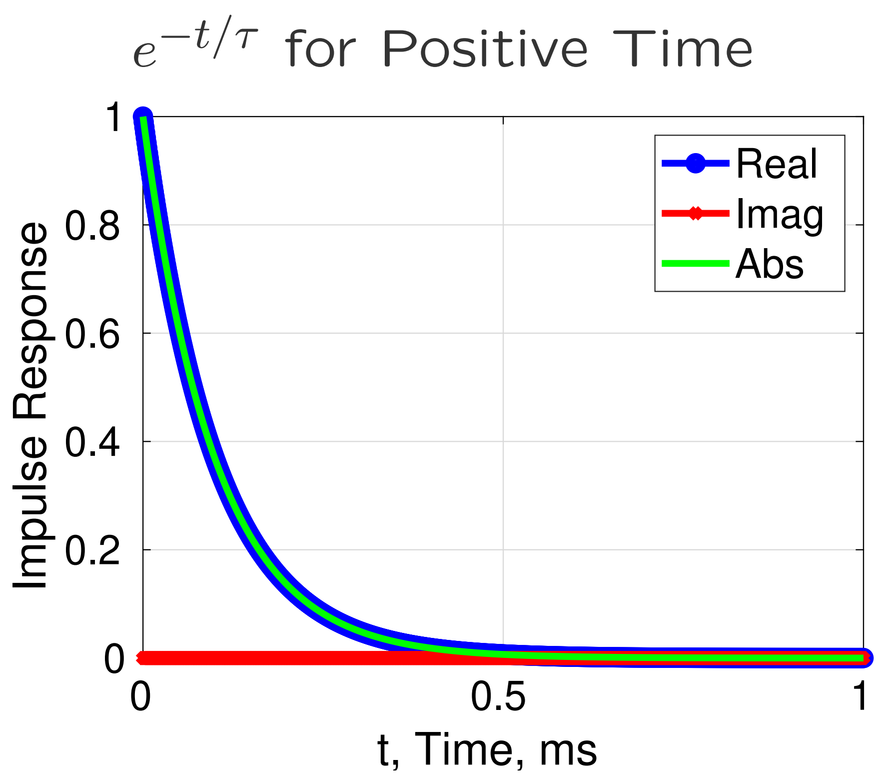
```
V=fftshift(V1)
```
or
```
V=fftshift(fft(v))
```

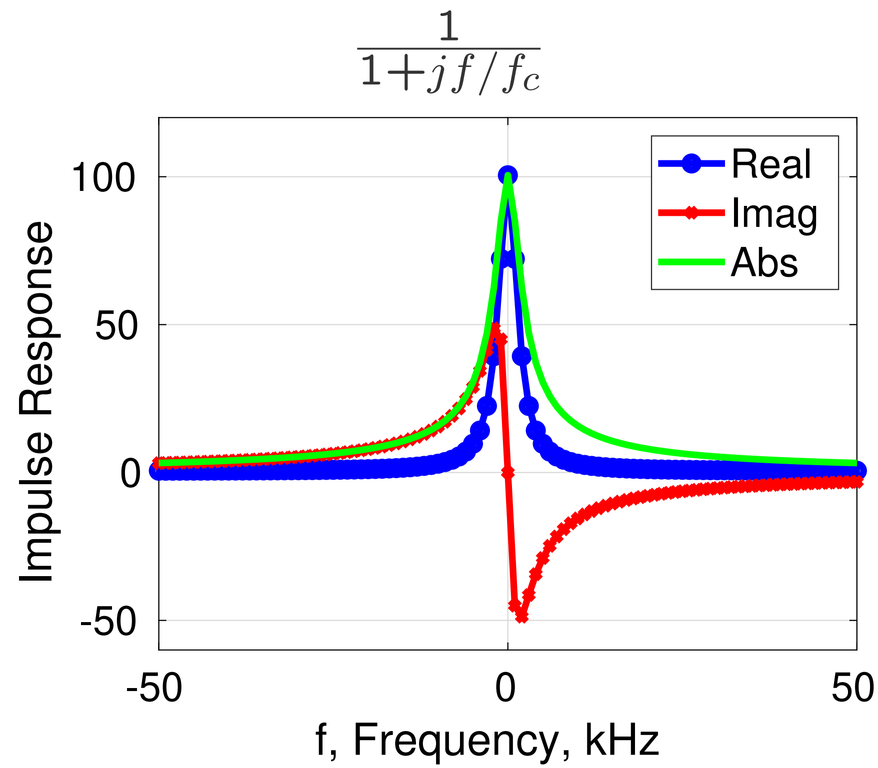
My `fftaxisshift.m` (See Website)



```
faxis=fftaxisshift(faxis1)
```
or
```
faxis=fftaxisshift(fftaxis(taxis))
```

An Important FT Pair





Lorentzian Function of Frequency with $\omega_c = \frac{1}{2\pi\tau}$

$$f_c = \frac{1}{2\pi\tau}$$

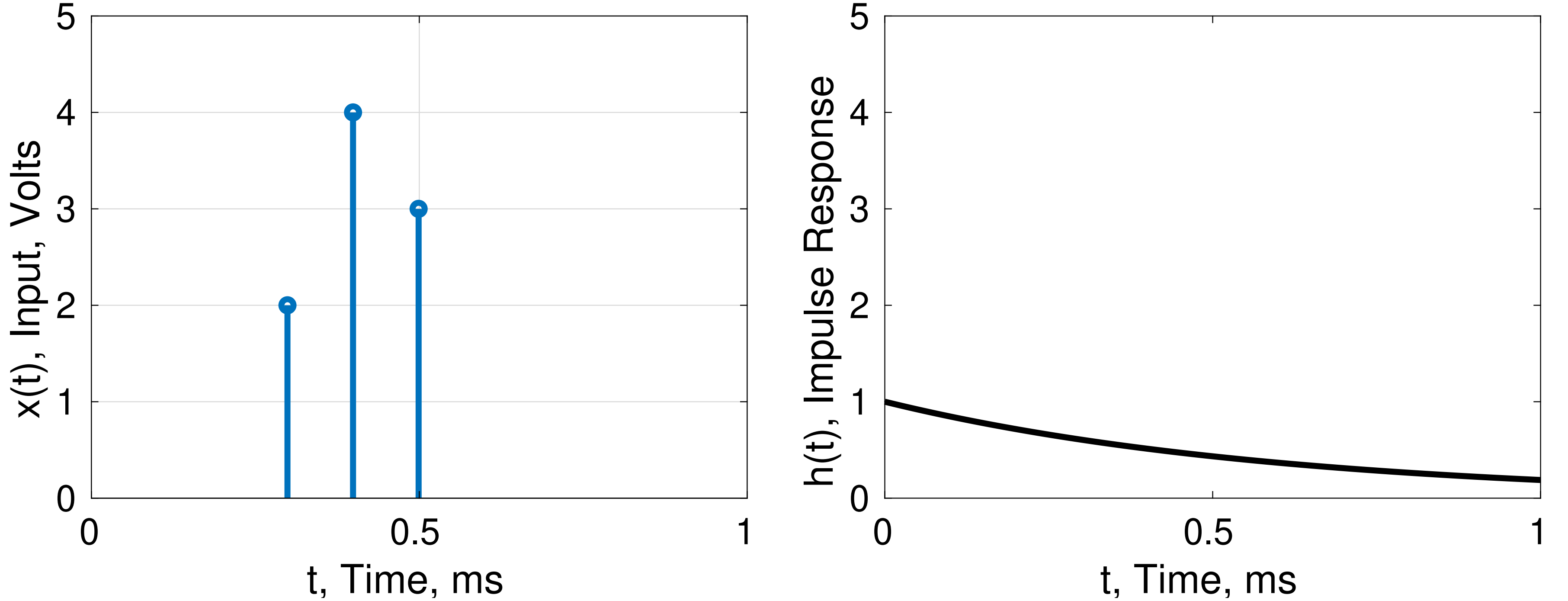
Remember; Wider in Time, Narrower in Frequency.

Convolution (1)

New Concept: Convolution

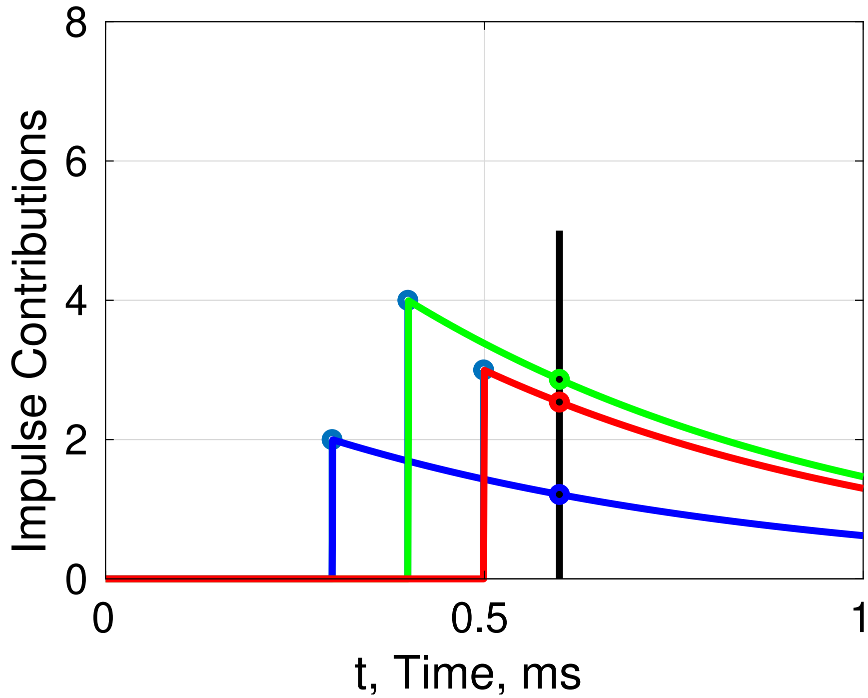
Remember Transient Solutions

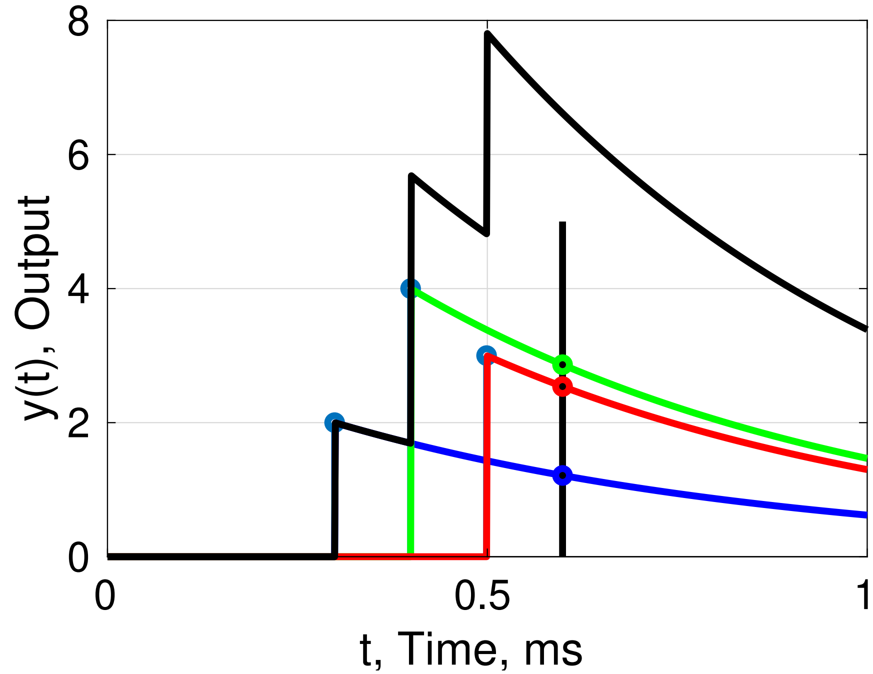
Consider Multiple Pulses as Input to a Circuit



Convolution (2)

Use Superposition to Add Transients



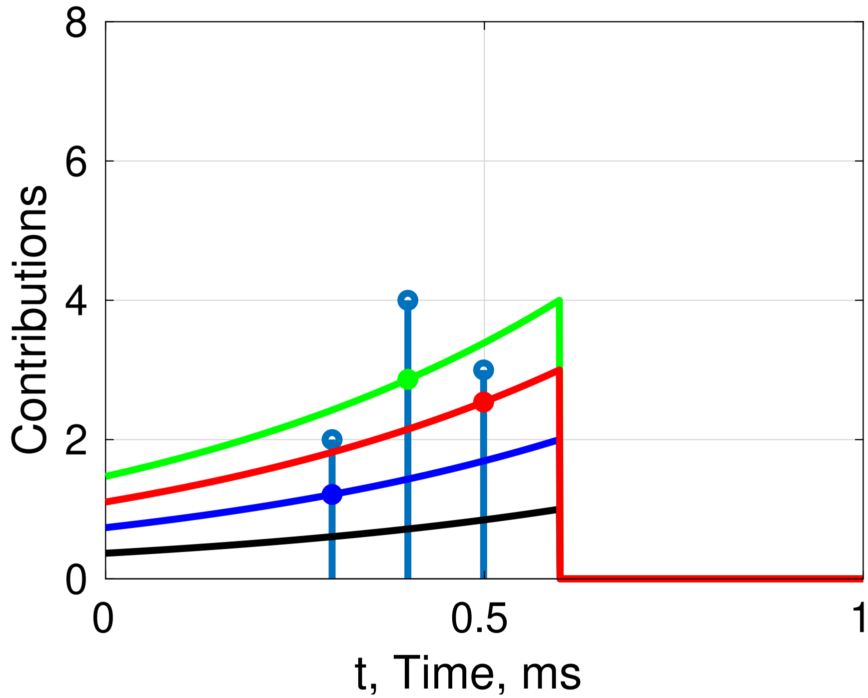


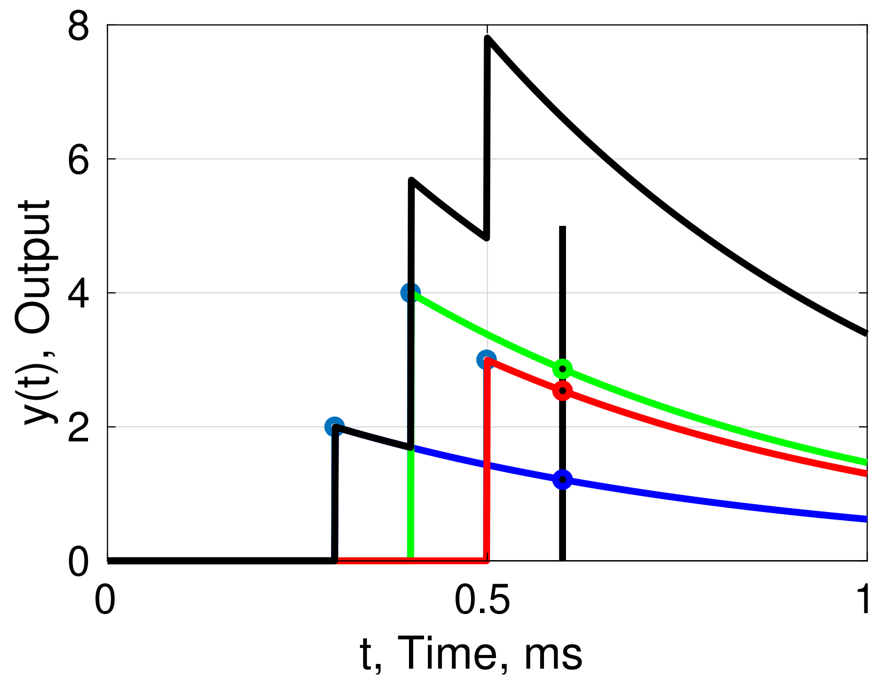
$$y(t) = x(t_1)h\,(t - t_1) + = x(t_2)h\,(t - t_2) + x(t_3)h\,(t - t_3)$$

In General, $y\,(t) = \int x\,(\tau)\,h\,(t - \tau)\,d\tau$

Convolution (3)

Use Superposition to Add Transients





Black Curve Moves Left to Right

$$y(t) = x(t_1)h\left(t - t_1\right) + x(t_2)h\left(t - t_2\right) + x(t_3)h\left(t - t_3\right)$$

In General, $y\left(t\right) = \int x\left(\tau\right) h\left(t - \tau\right) d\tau$

Convolution (4)

- Convolution Definition

$$y\left(t\right)=\int x\left(\tau\right)h\left(t-\tau\right)d\tau$$

- Symbol (one of many)

$$y\left(t\right)=x\left(t\right)\otimes h\left(t\right)$$

- Fourier Transform

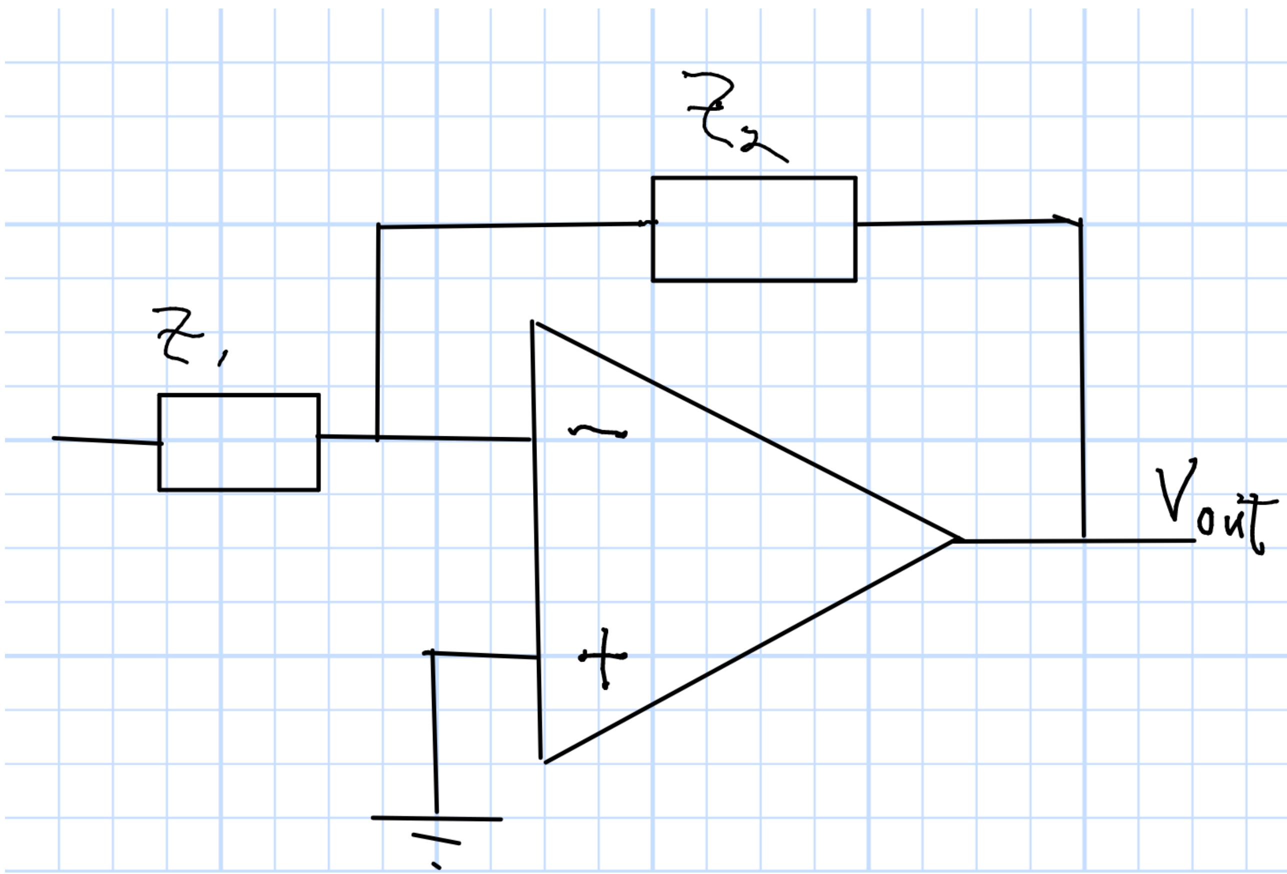
$$Y\left(\omega\right)=X\left(\omega\right)\times H\left(\omega\right)$$

- And of course

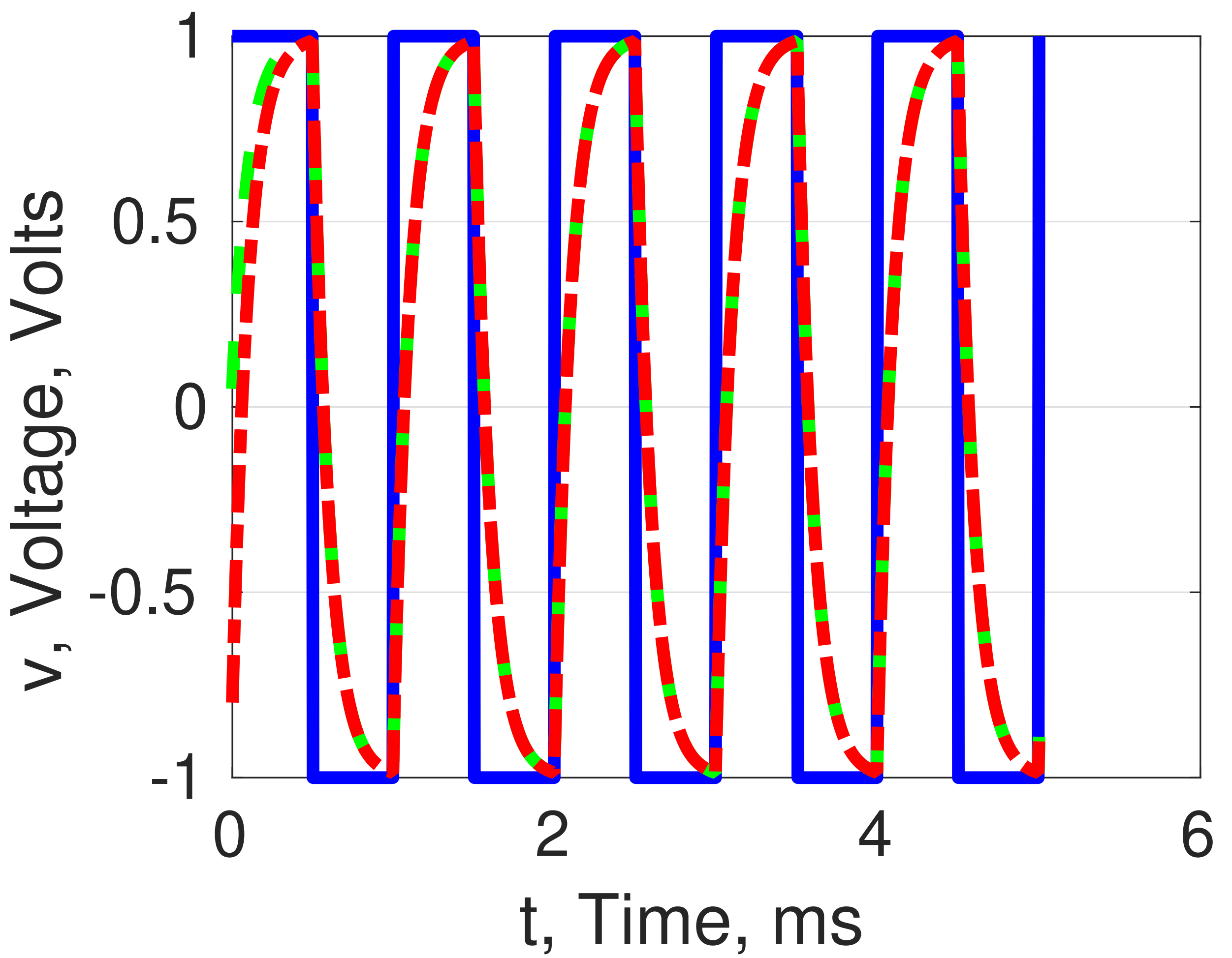
$$y\left(t\right)=x\left(t\right)\times h\left(t\right)\qquad Y\left(\omega\right)=X\left(\omega\right)\otimes H\left(\omega\right)$$

Impulse Response & Transfer Function (1)

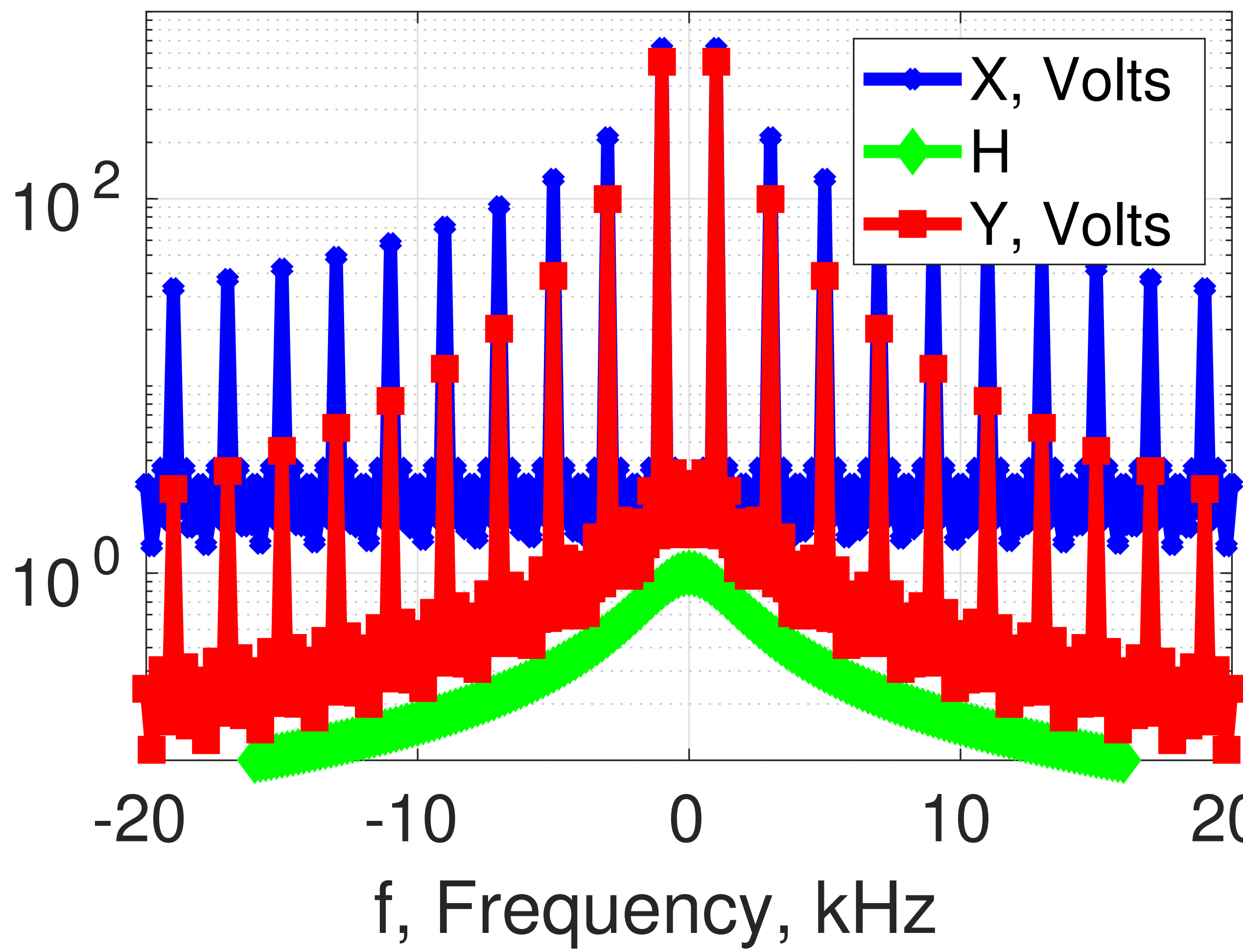
$y(t) = x(t) \otimes h(t) \qquad Y(\omega) = X(\omega) \times H(\omega)$



Impulse Response & Transfer Function (2)



Impulse Response & Transfer Function (3)



Some Fourier Transform Pairs

https://www.ethz.ch/content/dam/ethz/special-interest/baug/ibk/structural-mechanics-dam/education/identmeth/fourier.pdf

And many other sources.

For the Quiz (And More)

Time Domain	**Frequency Domain**
Real Function	Real Part Even, Imag. Odd
DC (Constant Signal)	Spike at $f = 0$
General Sinusoid	Spikes at $\pm f$.
Cosine	Pure Real (and Even)
Sine	Pure Imaginary (and Odd)
RECT	SINC
SINC	RECT
Convolution	Product
Product	Convolution
Exponential	Lorentzian