Circuits and Signals: Biomedical Applications
Week 4

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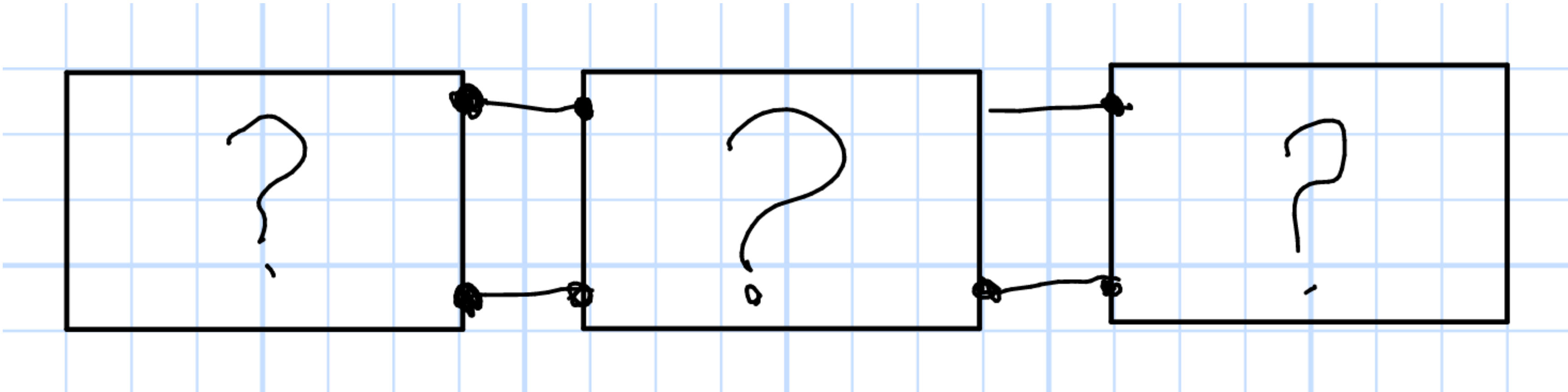
Week 4 Agenda

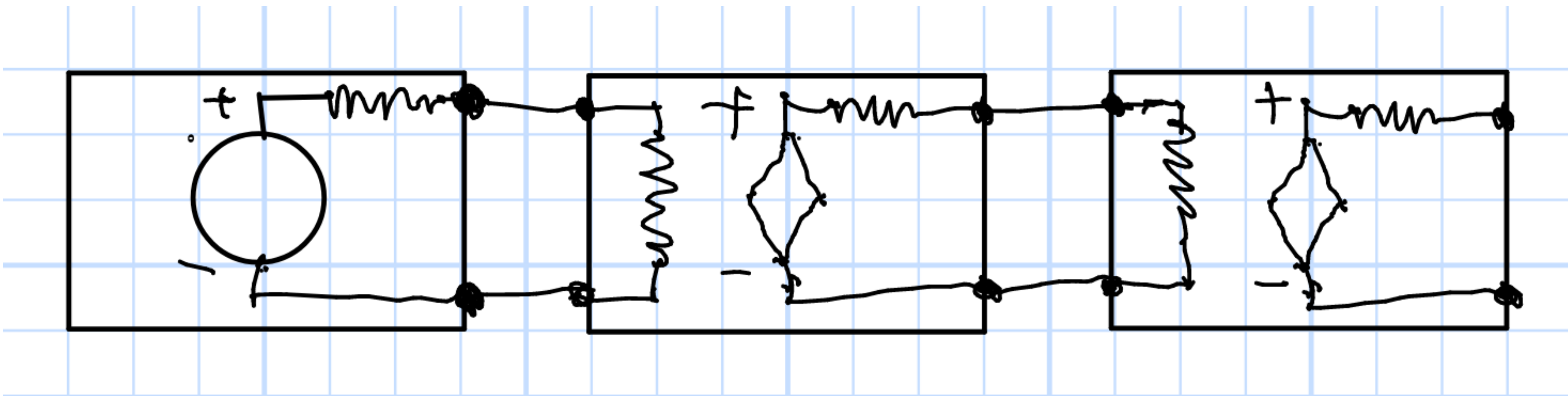
- Source Transformation
  - Thévenin Equivalent Circuit
  - Norton Equivalent Circuit
- Superposition
- Class Problems
  - Determining Equivalent Circuits
  - Using Equivalent Circuits
- Wheatstone Bridge

Why Equivalent Circuits

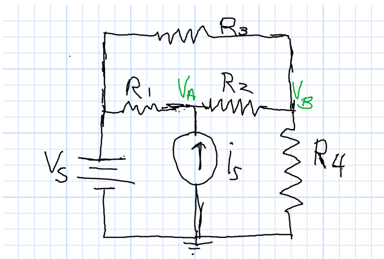
Each box contains a complicated Circuit. . .



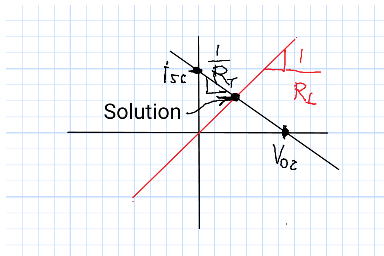
. . . but we don't really care what is in them.



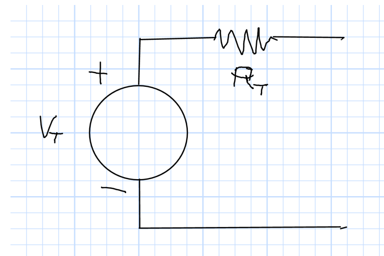
Equivalent Circuits



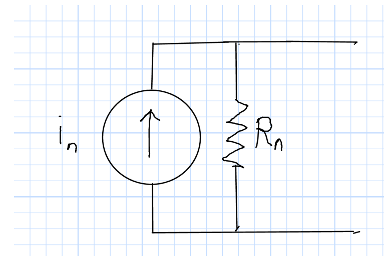
A Circuit



I-V PLots



Thévenin



Norton

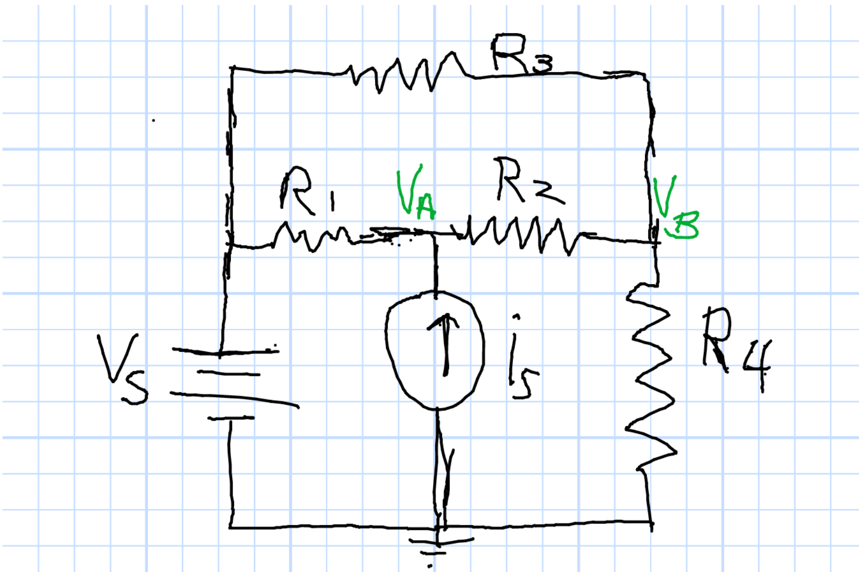
R_4 is the Load: Analyze everything else.

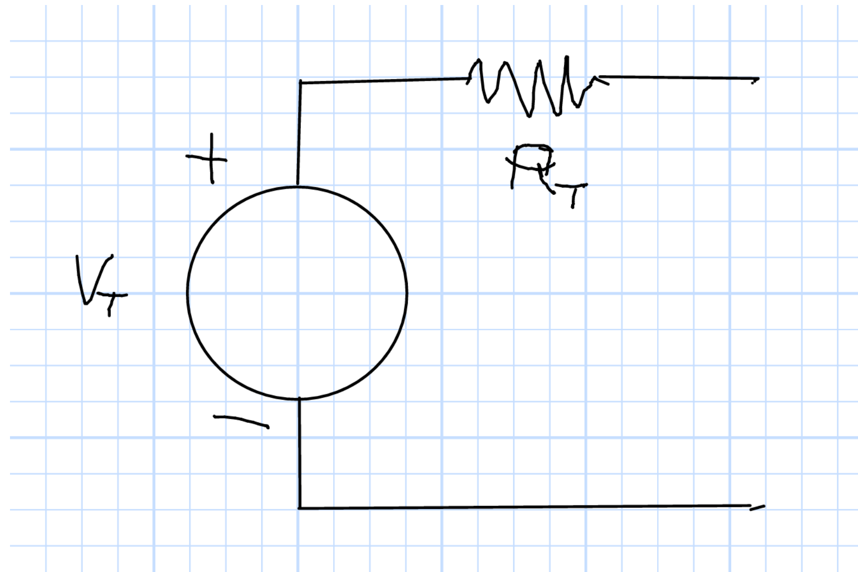
For a linear circuit, i vs. v is a straight line. Need two of v_{oc}, i_{sc}, R_T

- Calculation Options: Use any one.
  - Calculate V_{oc} and i_{sc}
  - Calculate V_{oc} and R_T
  - Calculate R_T and i_{sc}
- Equivalent Circuits
  - Thévenin: Voltage Source, V_{oc}, with Series R_T
  - Norton: Current Source, i_{sc}, with Parallel $R_N = R_T$

Thévenin Equivalent Circuit

$R_1 = 1\text{k}\Omega$, $R_2 = 1\text{k}\Omega$, $R_3 = 5\text{k}\Omega$, $v_s = 12\text{V}$, $i_s = 6\text{mA}$

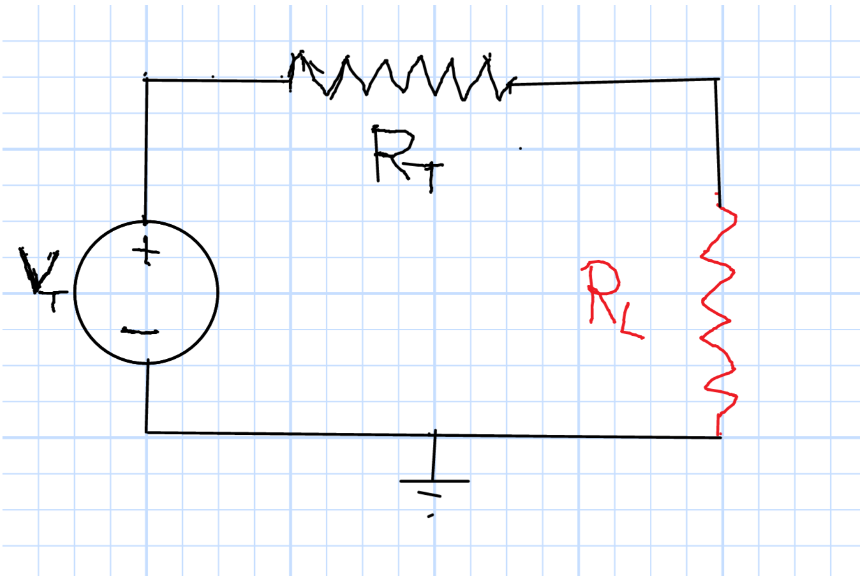




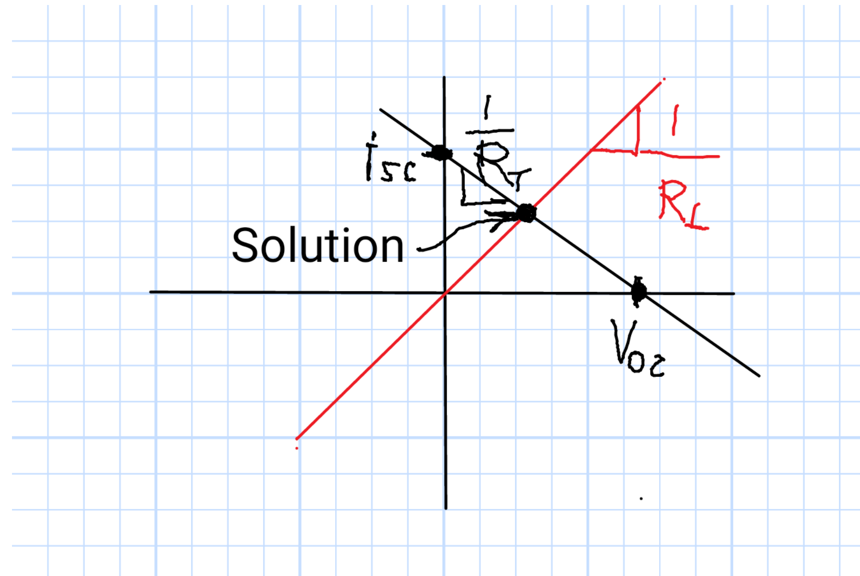
R_4 is the Load: Draw a circuit for everything else.
Linear Circuits: i *vs.* v is a straight line.
We need two points to determine the line.
$i = 0$: Open Circuit Voltage. $v = 0$: Short–Circuit Current

Load Resistor: $i = \frac{v}{R_4}$ Everything Else: $i = \frac{v_T - v}{R_T}$

Thévenin Concept



Circuit Equation



Load Equation

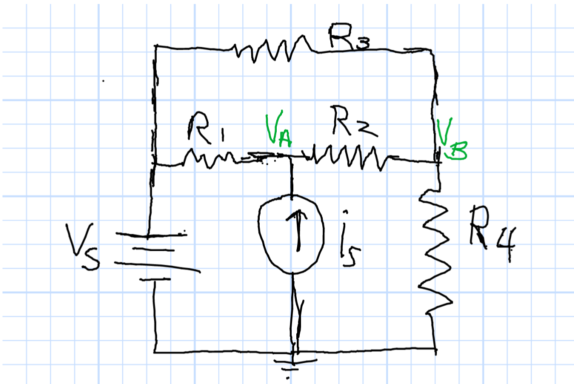
$$i = \frac{v_T - v}{R_T}$$

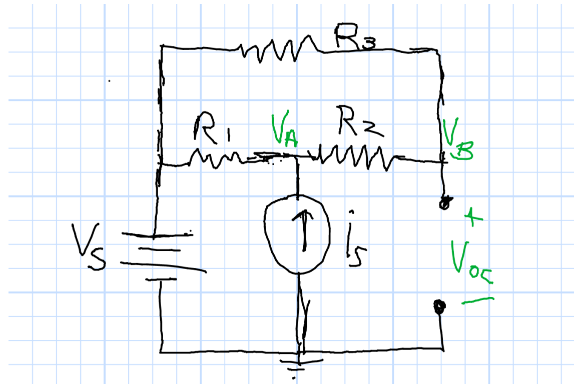
Load Line

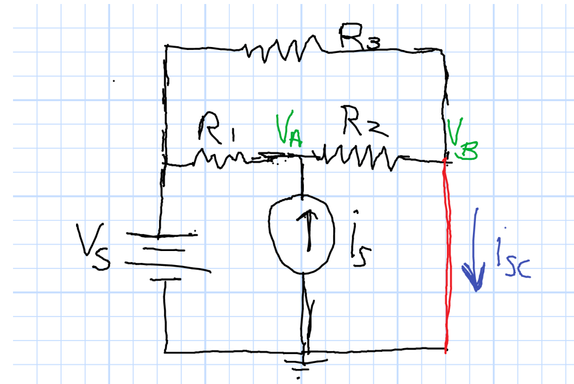
$$i = \frac{v}{R}$$

Solution is at the intersection.

Thévenin Calculation: v_{oc}, i_{sc}





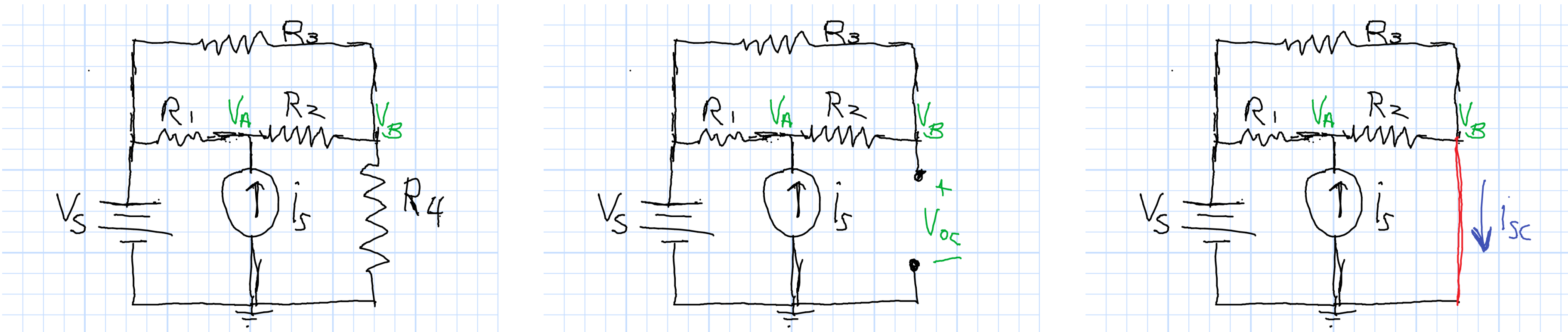


- Calculate Open–Circuit Voltage, v_{oc}.

- Calculate Short–Circuit Current, i_{sc}.

- Find Equivalent Circuit, $v_T = v_{oc}$ and $R_T = \frac{v_{oc}}{i_{sc}}$.

Thévenin Calculation

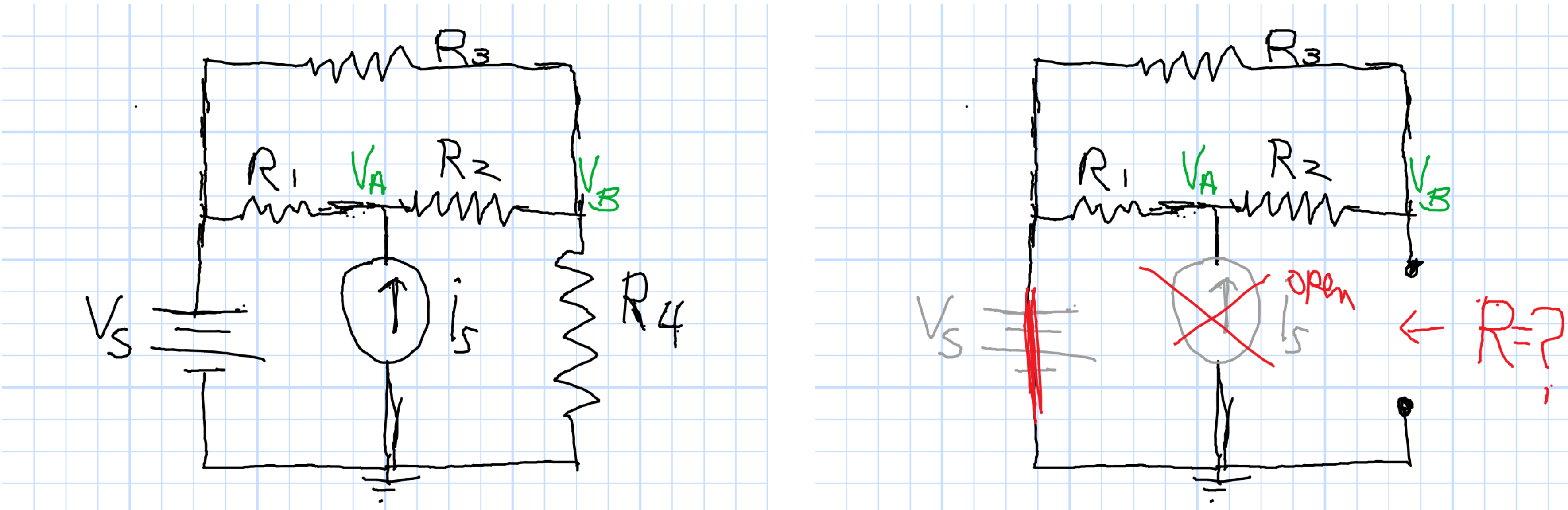


- OC: Draw with one equivalent resistor: $v_A = v_s + i_s\left[R_1 \parallel (R_2 + R_3)\right]$,
  Voltage Divider: $v_{oc} = v_s + (v_A - v_s)\frac{R_3}{R_3+R_2} = 16.28\text{V}$.

- SC: $\frac{v_s - v_A}{R_1} + i_s - \frac{v_A}{R_2} = 0$, $\left(\frac{1}{R_1} + \frac{1}{R_2}\right) v_A = \frac{v_s}{R_1} + i_s$
  $v_A = 9\text{V}$, $i_{sc} = \frac{v_s}{R_3} + \frac{v_A}{R_2} = 11.4\text{mA}$

- Equivalent Circuit, $v_T = 16.28\text{V}$ and $R_T = \frac{v_{oc}}{i_{sc}} = 1430\text{Ohms}$.

Alternative: Find R_T



Zero the Sources:

Voltage Sources Shorted ($v = 0$)

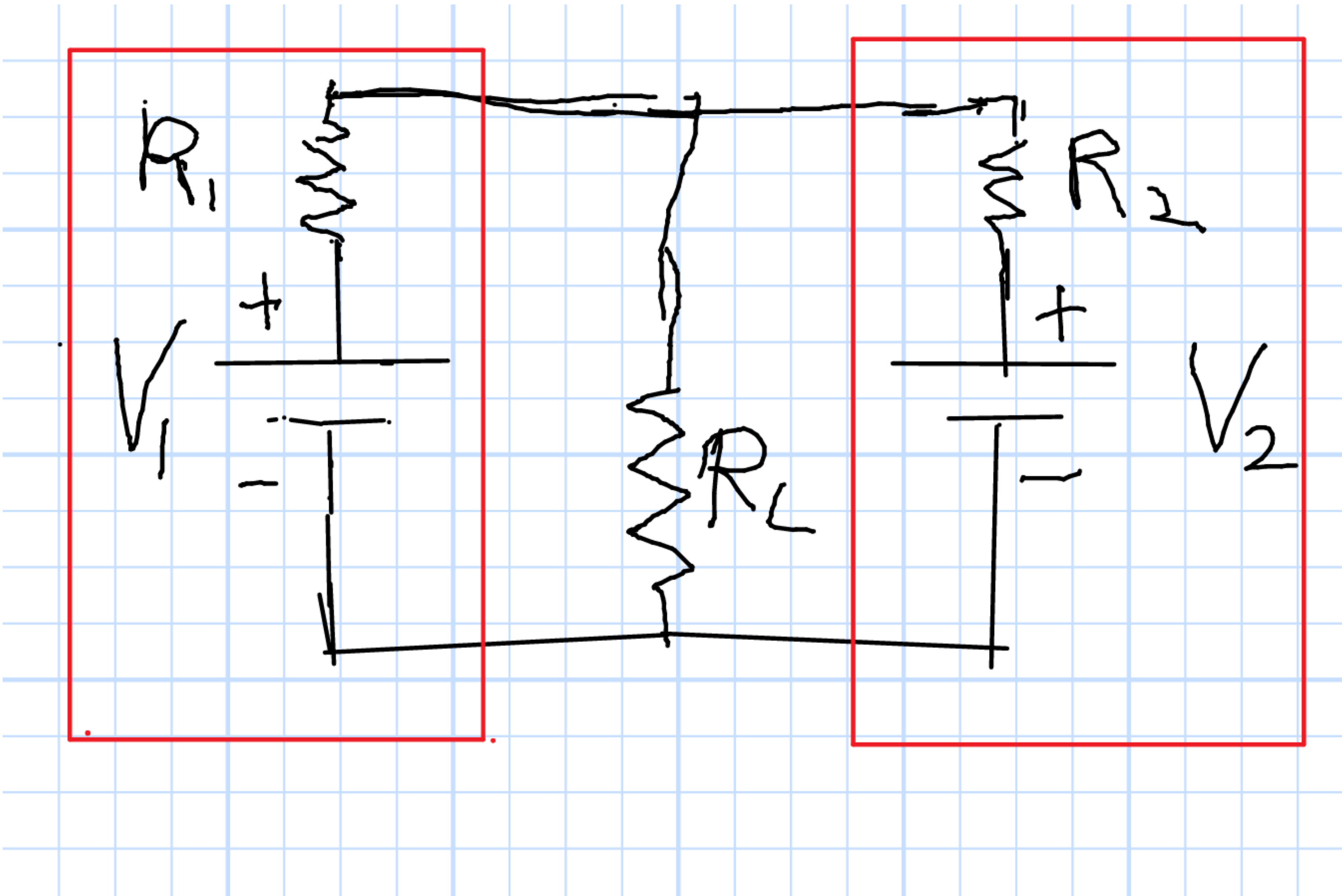
Current Sources Opened ($i = 0$)

Use Series and Parallel Combinations

Do This if v_{oc} or i_{sc} Looks Messy.

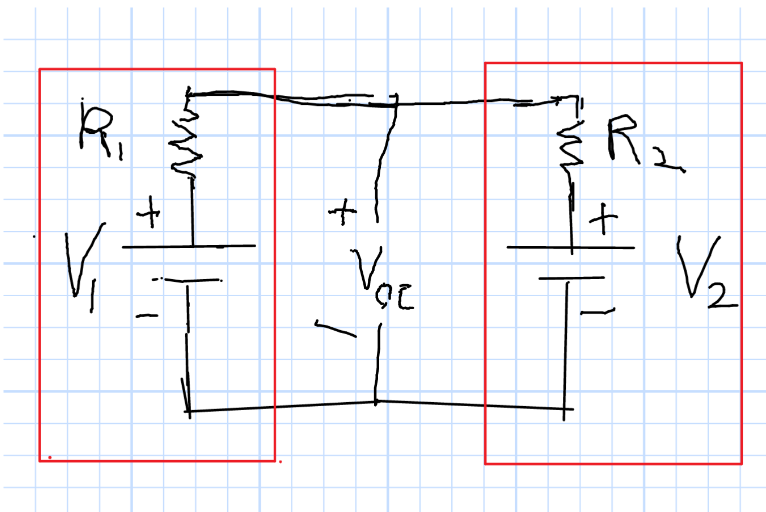
In This Example, $R_T = (R_1 + R_2) \parallel R_3 = 1430$Ohms

Thévenin Example: Parallel Batteries



Parallel Batteries Solution

$v_1 = 12\text{V},\ R_1 = 1\Omega,\ v_2 = 11.9\text{V},\ R_2 = 3\Omega$



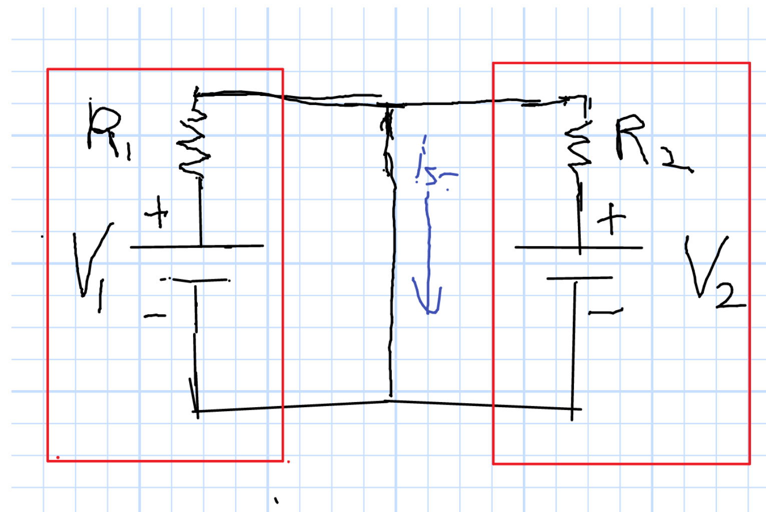
Voltage Divider

$$v_{oc} = v_1 - (v_1 - v_2)\frac{R_1}{R_1+R_2}$$

$$v_1\frac{R_2}{R_1+R_2} + v_2\frac{R_1}{R_1+R_2}$$

$$= \frac{v_1R_2+v_2R_1}{R_1+R_2}$$

$$= 11.97\text{V}$$



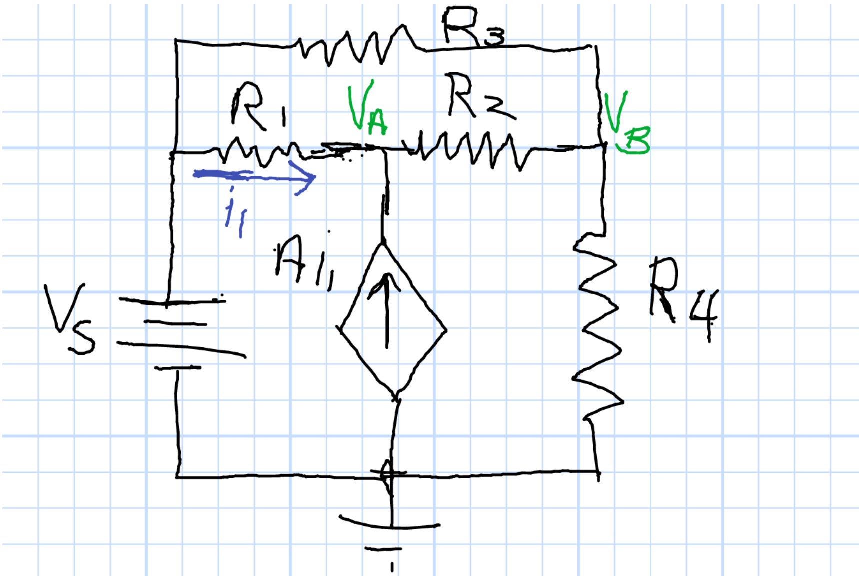
KCL

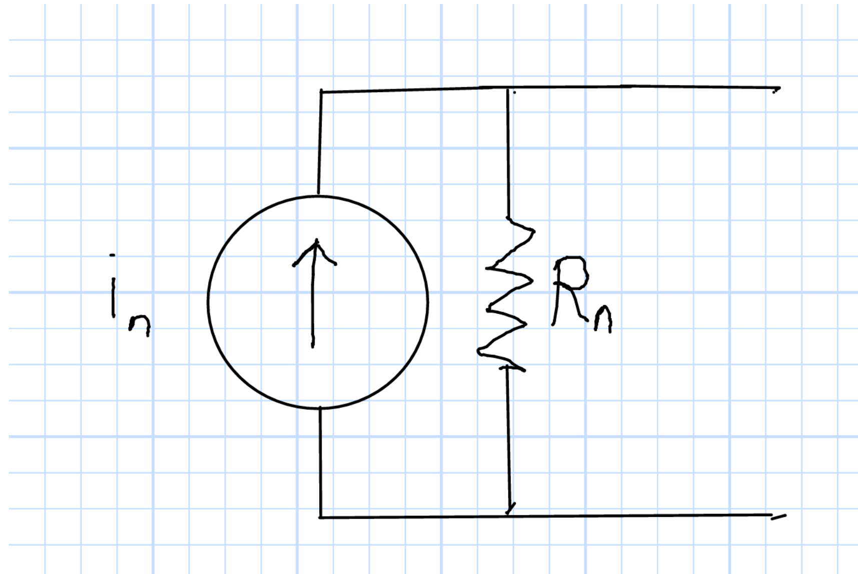
$$i_{sc} = \frac{v_1}{R_1} + \frac{v_2}{R_2} = \frac{v_1R_2+v_2R_1}{R_1R_2}$$

Thévenin Resistor

$$R_T = \frac{v_{oc}}{i_{sc}} = 0.75\Omega$$

Norton Equivalent Circuit





R_4 is the Load: Draw a circuit for everything else.

Linear Circuits: i *vs.* v is a straight line.

We need two points to determine the line.

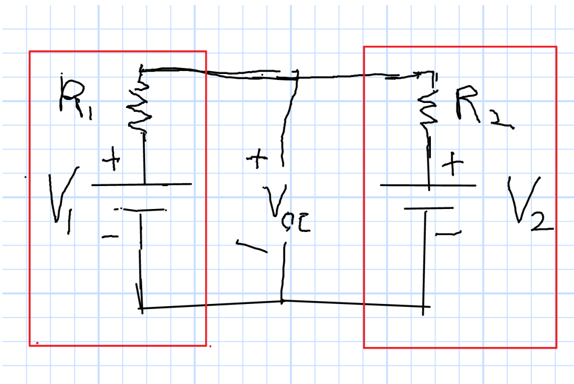
$i = 0$: $V_{oc} = 16.28$V. $v = 0$: $i_{sc} = 11.4$ma

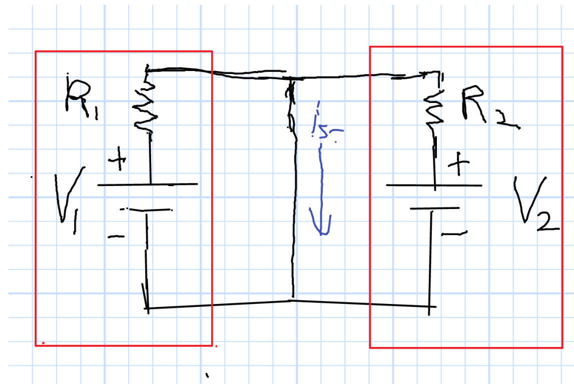
Load Resistor: $i = \frac{v}{R_4}$

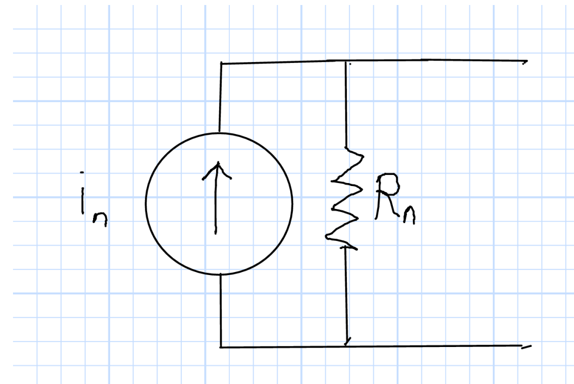
Solution: Current Divider

Everything Else: $i = i_N - \frac{v}{R_N}$

Norton Example







From Thévenin Equivalent

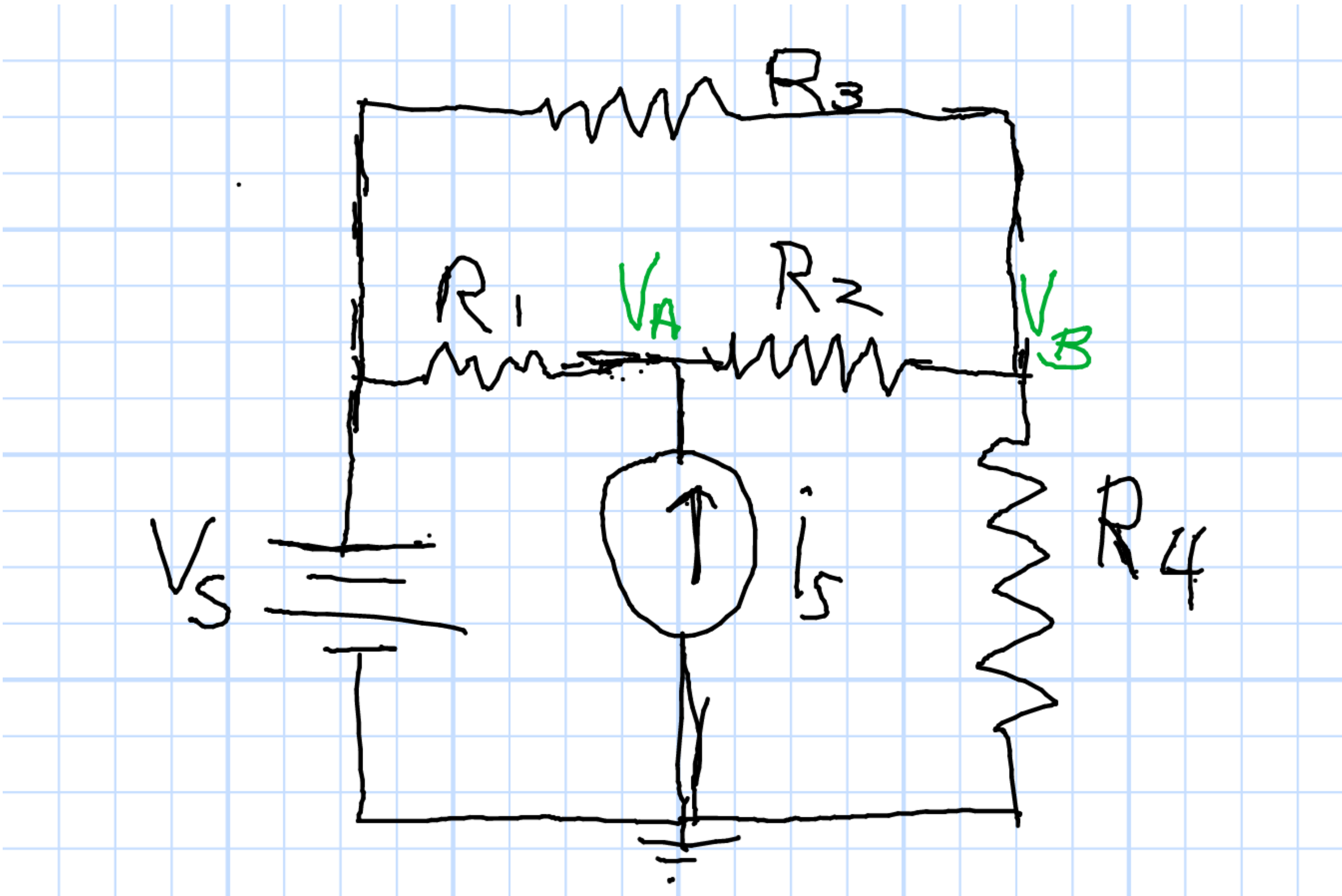
$$v_{oc} = 11.97\text{V} \qquad R_T = 0.75\Omega$$

$$i_{sc} = \frac{v_1}{R_1} + \frac{v_2}{R_2} = \frac{v_1 R_2 + v_2 R_1}{R_1 R_2} = 15.96\text{A}$$

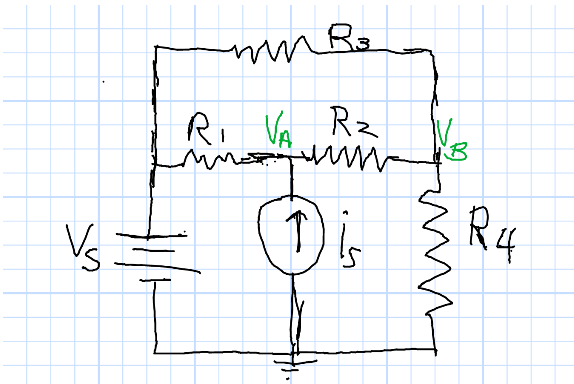
Norton Equivalent

$$i_N = i_{sc} = 15.96\text{A} \qquad R_N = R_T = 0.75\Omega$$

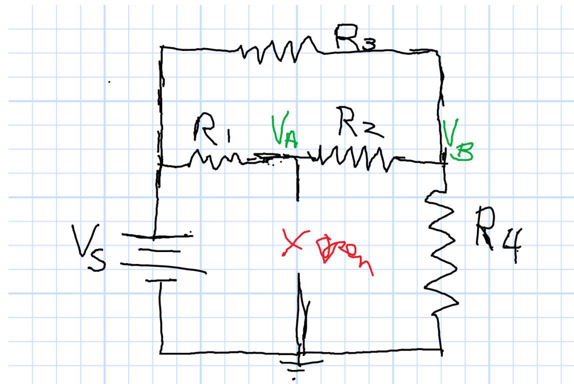
Superposition Example (1)



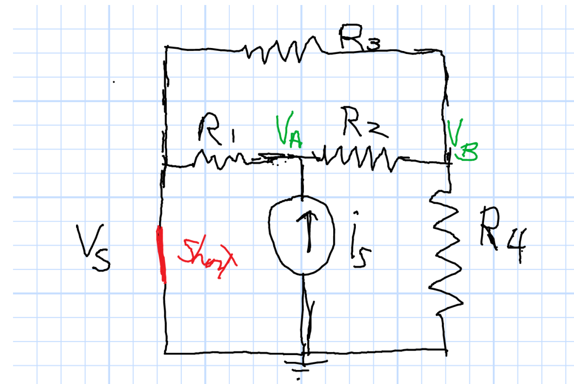
Superposition Example (2)



Original



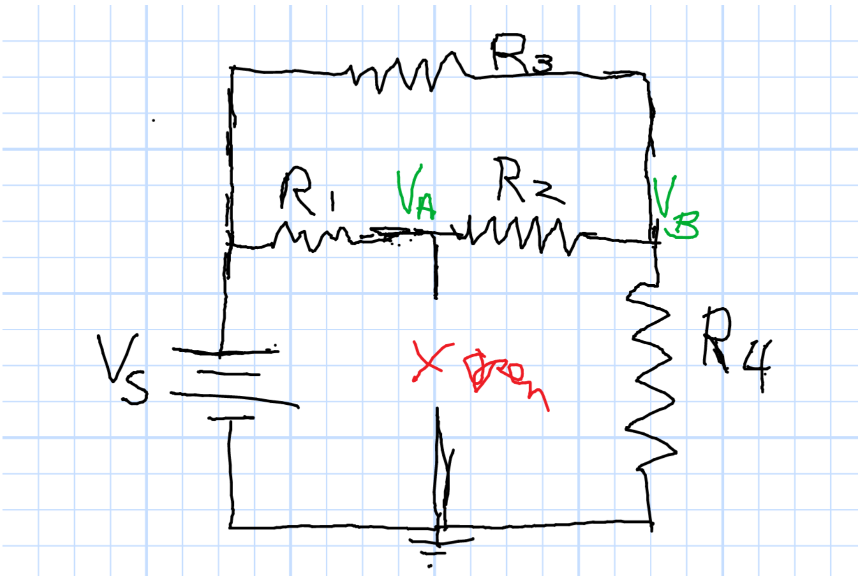
Zero Current Source Open



Zero Voltage Source Short

- Zero the Sources One at a Time
- Solve the Circuit for All Unknowns in Each Case
- Sum the Results

Superposition Solution



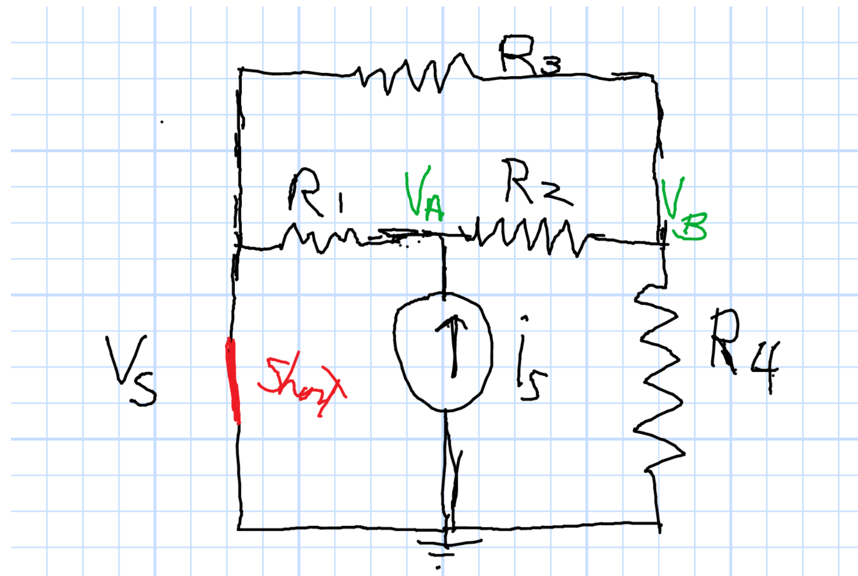
Series–Parallel Solution

$$i = v_s / \left[(R_1 + R_2) \parallel R_3 \right] + R_4$$

$$v_B = i R_4$$

Voltage Divider

$$v_A = v_s + (v_B - v_s) \frac{R_1}{R_1 + R_2}$$



Series–Parallel Solution

$$v_A = i_s \left\{ R_1 \parallel \left[R_2 + (R_3 \parallel R_4) \right] \right\}$$

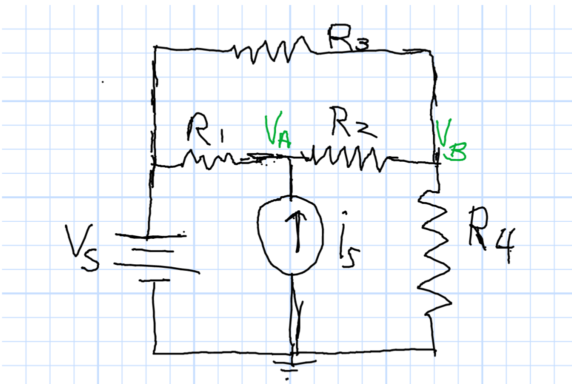
Current Divider (i_2 to the right)

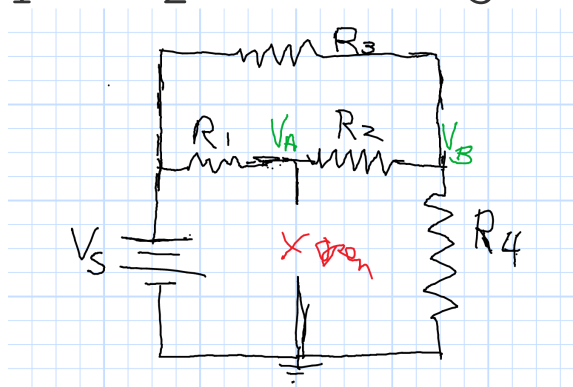
$$i_2 = i_s \frac{R_1}{R_1 + R_2 + (R_3 \parallel R_4)}$$

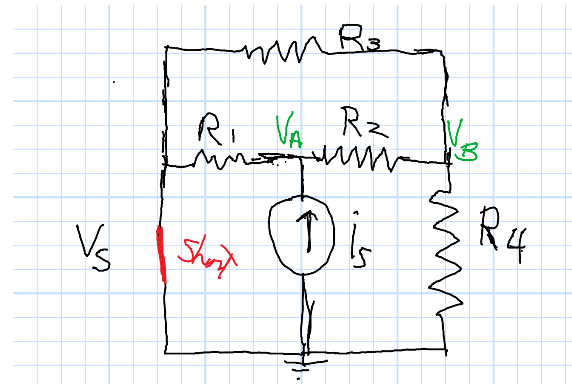
$$v_B = i_2 \left(R_3 \parallel R_4 \right)$$

Conclusion

$v_s = 12\text{V}, i_s = 6\text{mA},\ R_1 = R_2 = 1k\Omega,\ R_3 = 5k\Omega, R_4 = 200\Omega$







v_{total} = v_1 + v_2

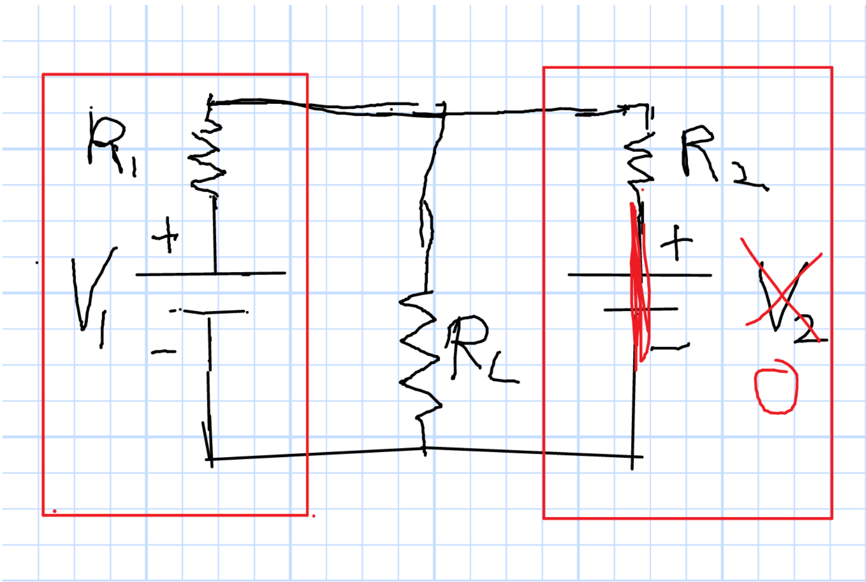
v_1	v_2
$v_A = 6.74\text{V}$	$v_A = 3.26\text{V}$
$v_B = 1.47\text{V}$	$v_B = 0.53\text{V}$

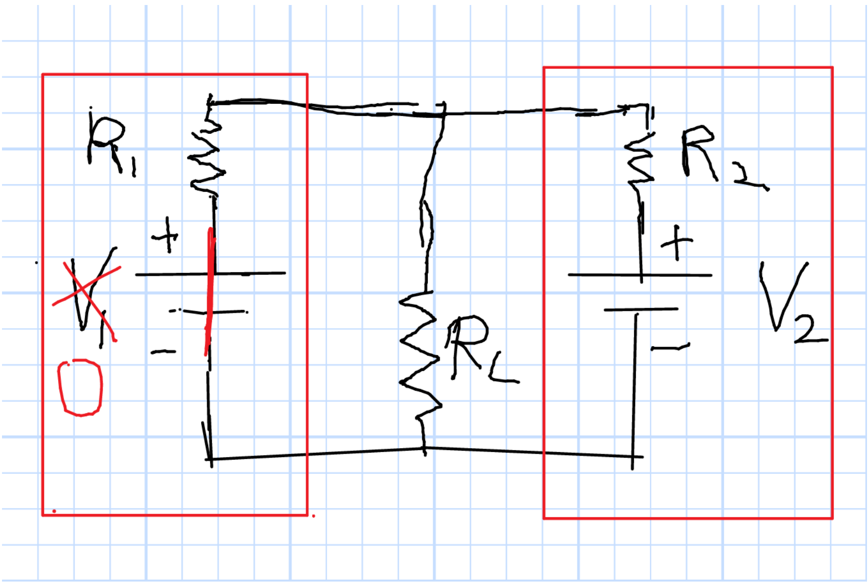
$$v_A = 10.0\text{V} \qquad v_B = 2.0\text{V}$$

Same Result We Had Before.

Parallel Batteries: Superposition

$v_1 = 12\text{V},\ R_1 = 1\Omega,\ v_2 = 11.9\text{V},\ R_2 = 3\Omega$



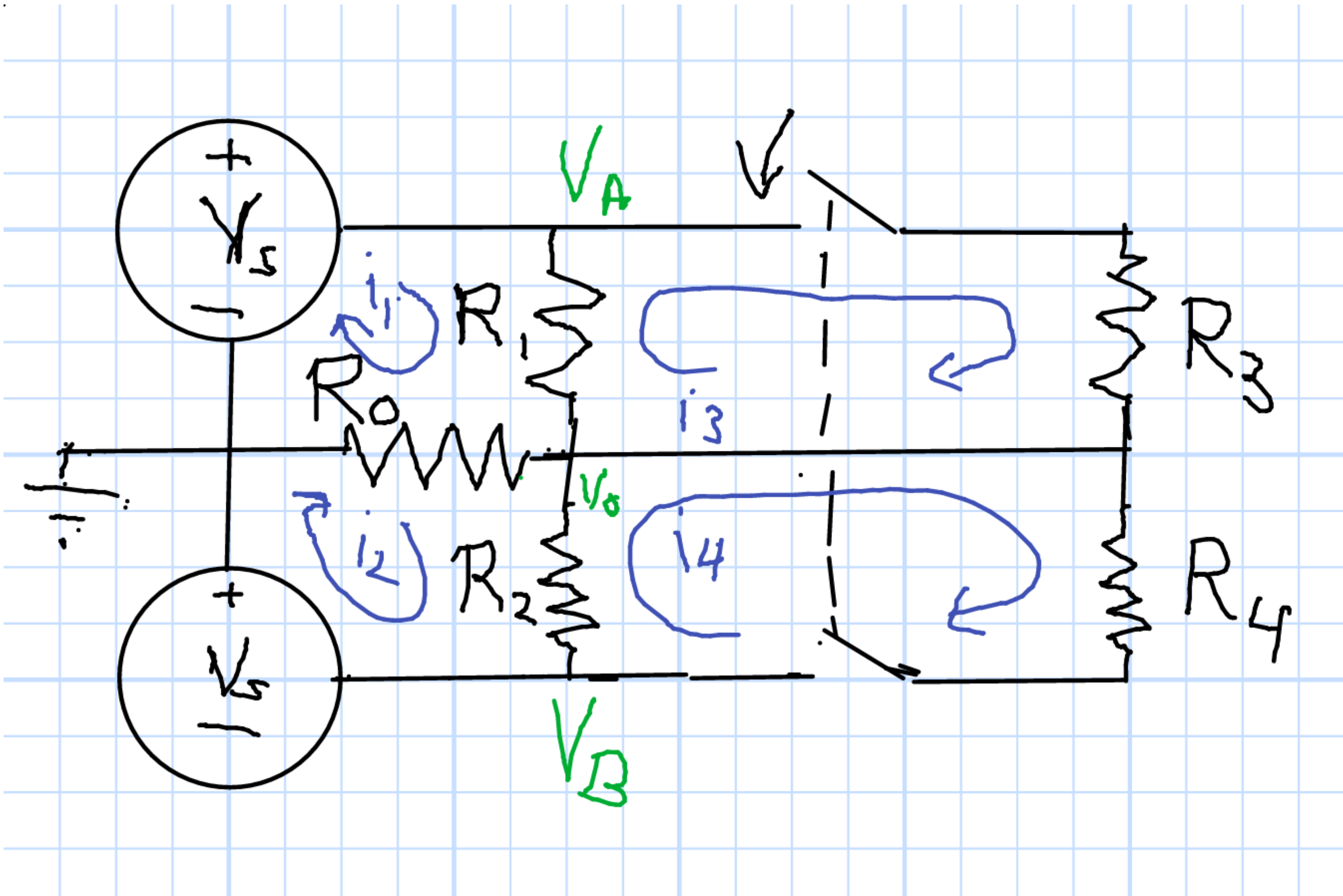


$v_L = v_1 \frac{R_L \| R_2}{R_1 + (R_L \| R_2)} = 7.83\text{V}$

$v_L = v_2 \frac{R_L \| R_1}{R_2 + (R_L \| R_1)} = 2.59\text{V}$

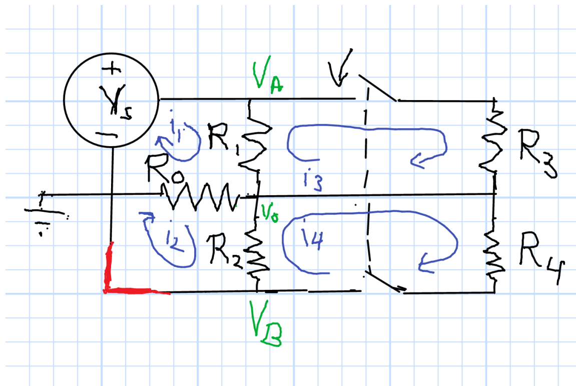
$$v_L = 7.83\text{V} + 2.59\text{V} = 10.4\text{V}$$

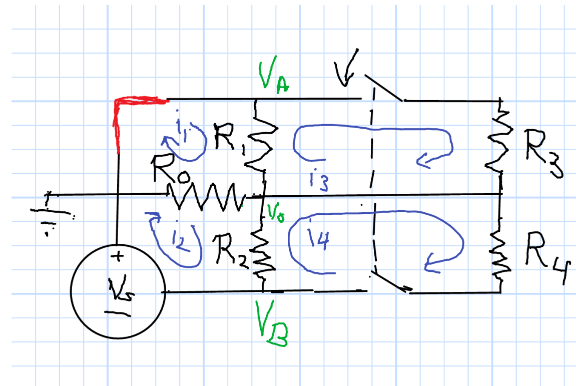
Balancing the Load



Source: $v_s = 120$V RMS. Bad Ground: $R_0 = 20\Omega$
Lights: $R_1 = 14.4\Omega$, $R_2 = 144\Omega$. Stove: $R_3 = R_4 = 7.2\Omega$.

Stove Off





Voltage Divider with Top Source: Zero Bottom so $R_2 \parallel R_0$

$$v_1 = v_s \frac{R_1}{R_1 + (R_0 \parallel R_2)} \qquad v_2 = v_s \frac{R_0 \parallel R_2}{R_1 + (R_0 \parallel R_2)}$$

Bottom Source: Zero Top so $R_1 \parallel R_0$

$$v_1 = -v_s \frac{R_1 \parallel R_0}{R_2 + (R_0 \parallel R_1)} \qquad v_2 = -v_s \frac{R_2}{R_2 + (R_0 \parallel R_1)}$$

Total

$$v_1 = 60.7\text{V} \qquad v_2 = 179.3\text{V} \qquad \text{Check } v_1 + v_2 = 240\text{V}$$

$$v_A = 120\text{V} \qquad v_B = -120\text{V} \qquad v_0 = v_A - v_1 = 59.3\text{V}$$

Stove On: Balance Load

Same as Previous Page, but with $R_1 \parallel R_3$ and $R_2 \parallel R_4$

Top Source

$$v_1 = v_s \frac{R_1 \| R_3}{(R_1 \| R_3) + (R_0 \| R_2 \| R_4)}$$

$$v_2 = v_s \frac{R_0 \| R_2 \| R_4}{(R_1 \| R_3) + (R_0 \| R_2 \| R_4)}$$

Bottom Source

$$v_1 = -v_s \frac{R_1 \| R_3 \| R_0}{(R_2 \| R_4) + (R_0 \| R_1 \| R_3)}$$

$$v_2 = -v_s \frac{R_2 \| R_4}{(R_2 \| R_4) + (R_0 \| R_1 \| R_3)}$$

Total

$v_1 = 101.4\text{V}$ $\quad v_2 = 138.6\text{V}$ $\quad$ Check $v_1 + v_2 = 240\text{V}$

$v_A = 120\text{V}$ $\quad v_B = -120\text{V}$ $\quad v_0 = v_A - v_1 = 18.6\text{V}$

Comparison

Stove Off

$$v_1 = 60.7\text{V} \qquad v_2 = 179.3\text{V}$$

$$v_A = 120\text{V} \qquad v_B = -120\text{V} \qquad v_0 = v_A - v_1 = 59.3\text{V}$$

Lights Represented by R_1 Are Dim
Lights Represented by R_2 Are Too Bright

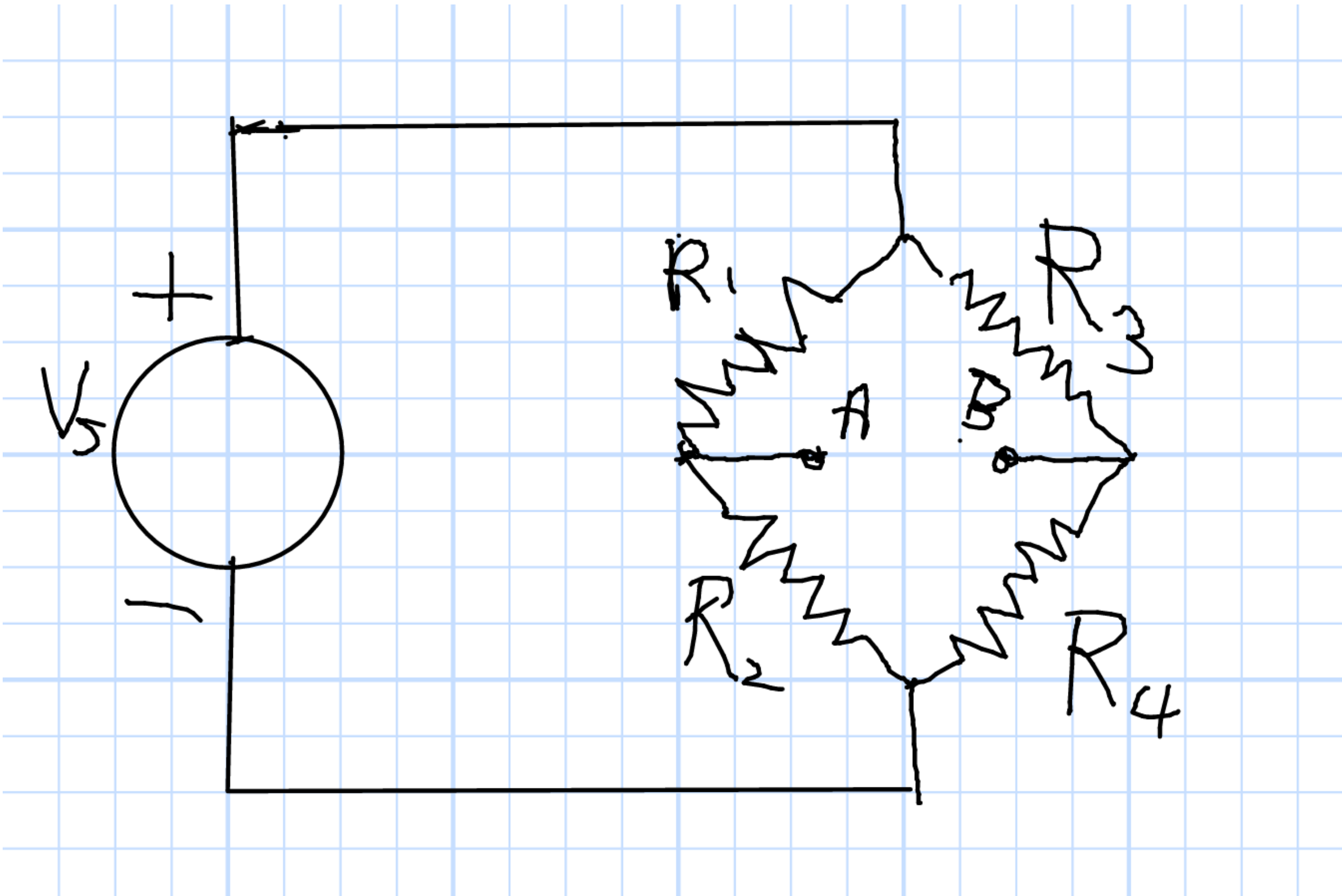
Stove On

$$v_1 = 101.4\text{V} \qquad v_2 = 138.6\text{V}$$

$$v_A = 120\text{V} \qquad v_B = -120\text{V} \qquad v_0 = v_A - v_1 = 18.6\text{V}$$

Lights Represented by R_1 Become Brighter
Lights Represented by R_2 Become Dimmer
Better Balance
"Ground" Voltage, v_0, Closer to Zero

Wheatstone Bridge

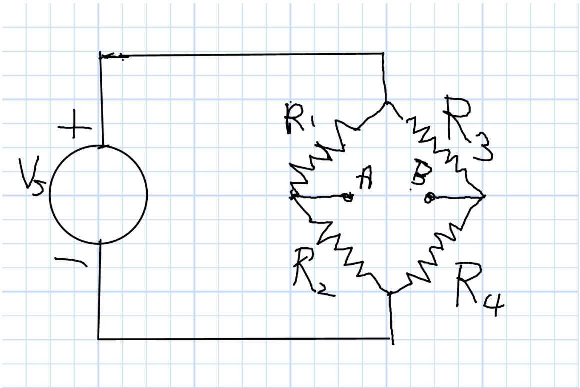


Balancing the Bridge

$$v_A = v_s \frac{R_2}{R_1 + R_2}$$

$$v_B = v_s \frac{R_3}{R_3 + R_4}$$

$$v_{BA} = v_s \left(\frac{R_2}{R_1 + R_2} - \frac{R_3}{R_3 + R_4} \right)$$



Application:

- R_4 is Unknown
- R_2 is Variable and Calibrated
- R_1 and R_3 are Precision Resistors
- Adjust for $v_{BA} = 0$
- Solve for R_4

Strain Gauge

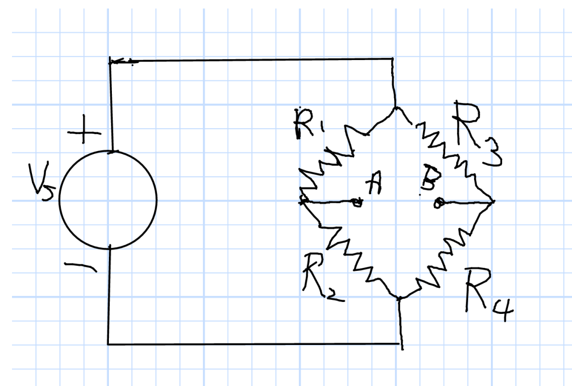
- R_4 is a Strain Guage
- $R_4 = R_0 + \Delta R$
- $R_{1:3} = R_0$ Precision Resistors
- Calculate v_{AB}

$$v_{BA} = v_s \left(\frac{R_0}{2R_0} - \frac{R_0 + \Delta R}{2R_0 + \Delta R} \right)$$

$$v_{BA} = v_s \left(\frac{1}{2} - \frac{R_0 + \Delta R}{2R_0 + \Delta R} \right)$$

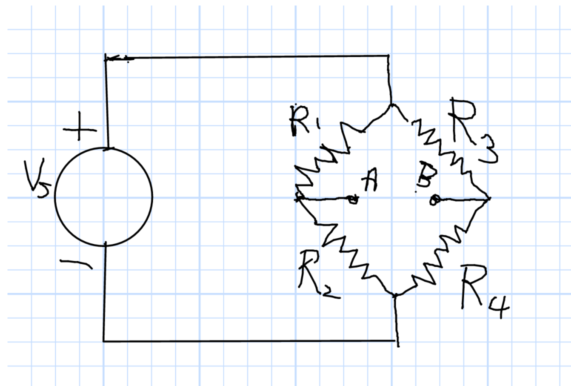
$$v_{BA} \approx v_s \left(\frac{1}{2} - \frac{R_0 + \Delta R}{2R_0} \right)$$

$$v_{BA} \approx -v_s \frac{\Delta R}{2R_0}$$



4 Balanced Strain Gauges

- $R_1 = R_4 = R_0 + \Delta R$ Are Strain Guage
- $R_2 = R_3 = R_0 + \Delta R$ Measure Opposite Strain
- Calculate v_{AB}



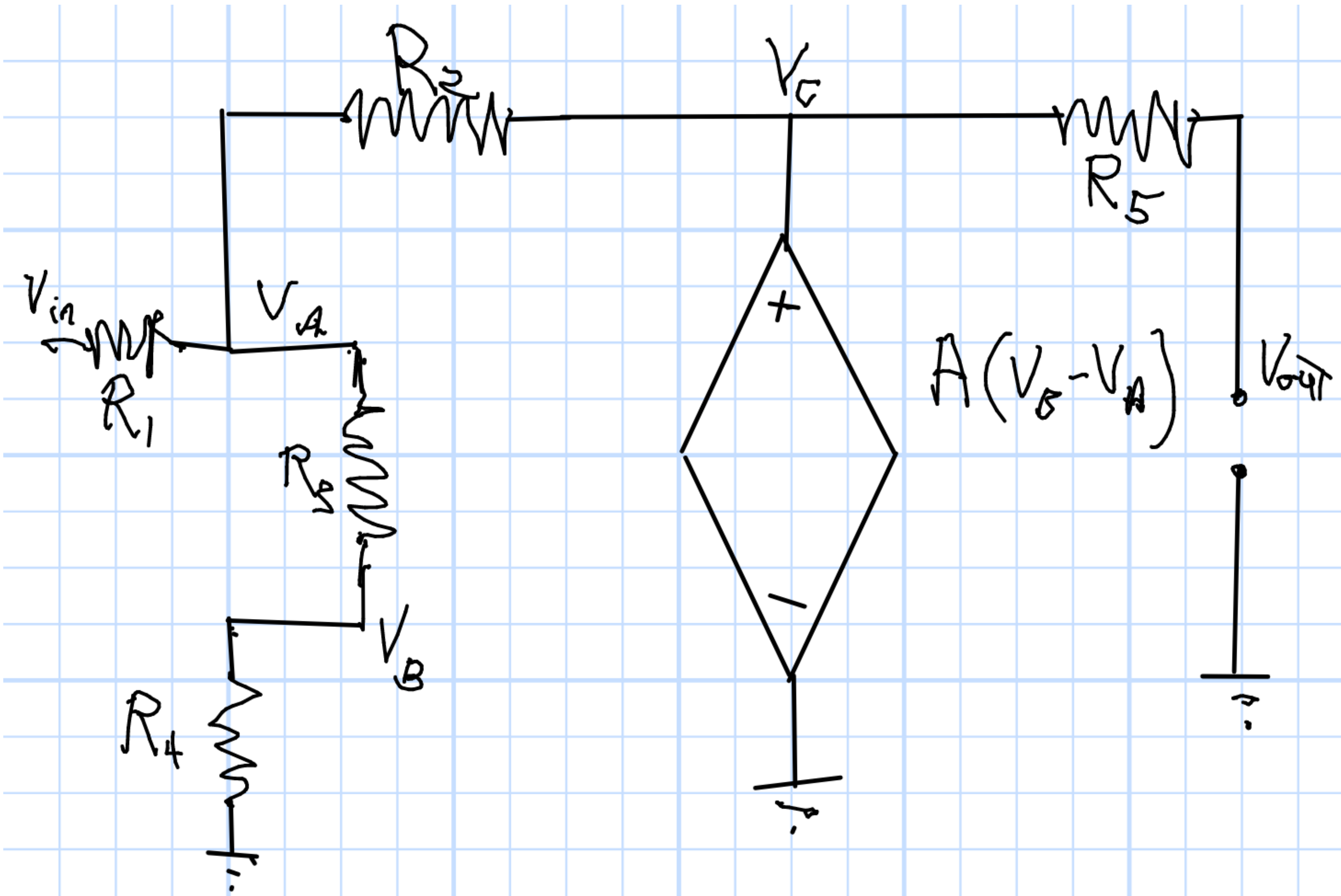
$$v_{BA} = v_s \left(\frac{R_0 + \Delta R}{2R_0} - \frac{R_0 - \Delta R}{2R_0} \right)$$

$$v_{BA} = v_s \frac{2\Delta R}{2R_0}$$

$$v_{BA} = v_s \frac{\Delta R}{R_0}$$

May Need a Variable to Get a Good Null

In–Class Exercise (1)



In–Class Exercise (2)

$$R_1 = 1\text{k}\Omega,\ R_2 = 10\text{k}\Omega,\ R_3 = 10\text{M}\Omega,\ R_4 = 10\Omega,\ R_5 = 50\Omega,$$
$$A = 10^5.$$

- Let $v_{in} = 1$
- Solve Symbolically for v_A, v_B, v_C with output open and shorted.
- Draw and label the Thévenin equivalent circuit for the output.
- What is the input resistance?
- Simplify the symbolic solution for the case where $R_3 \rightarrow \infty$ and $A \rightarrow \infty$.
- Next week we will see why this circuit is important.

Symbolic Solution

$$\frac{v_{in}-v_A}{R_1}+\frac{v_C-v_A}{R_2}+\frac{v_B-v_A}{R_3}=0$$

$$v_A-v_B=v_A\frac{R_3}{R_3+R_4} \qquad v_C=A\left(v_B-v_A\right)=-v_AA\frac{R_3}{R_3+R_4}$$

$$\frac{v_{in}-v_A}{R_1}+\frac{-v_AA\frac{R_3}{R_3+R_4}-v_A}{R_2}+\frac{-v_A}{R_3+R_4}=0$$

$$\left[\frac{-1}{R_1}+\frac{-A\frac{R_3}{R_3+R_4}-1}{R_2}+\frac{-1}{R_3+R_4}\right]v_A=\frac{-v_{in}}{R_1}$$

$$v_A=v_{in}\times\frac{1}{1+R_1\frac{A\frac{R_3}{R_3+R_4}+1}{R_2}+R_1\frac{1}{R_3+R_4}}$$

Numerical Solution

$$v_A = 9.9989 \times 10^{-5} \approx 0$$

$$v_B = v_A \frac{R_3}{R_3 + R_4} = 9.9989 \times 10^{-11} \approx 0$$

$$v_C = -v_A A \frac{R_3}{R_3 + R_4} = -9.9989$$

$$v_C \approx -\frac{R_2}{R_1} v_{in} \qquad \text{(Interesting?)}$$

$$R_{out} = R_5 = 50\Omega \qquad \text{because} \qquad i_{sc} = v_{oc}/R_5$$

$$R_{in} = R_1 = 1\text{k}\Omega \qquad \text{because} \qquad v_A \approx 0$$

Approximate Solution

- Let

$$R_3 \to \infty \qquad A \to \infty$$

- Exact Equation

$$v_C = -A\frac{R_3}{R_3+R_4}v_{in} \times \frac{1}{1+R_1\frac{A\frac{R_3}{R_3+R_4}+1}{R_2}+\frac{R_1}{R_3+R_4}}$$

- Approximation

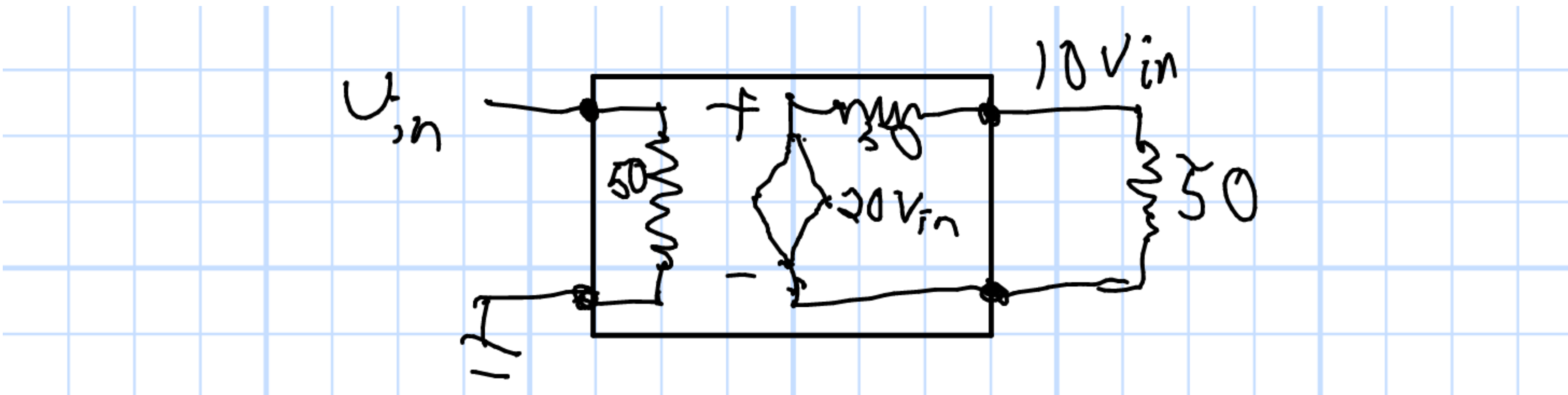
$$v_C \approx -A\frac{R_3}{R_3+R_4}v_{in} \times \frac{1}{0+R_1\frac{A\frac{R_3}{R_3+R_4}+0}{R_2}+0} = v_{in}\frac{-R_1}{R_2}$$

Using Thévenin Circuits

- Source: Thévenin Source, $v_s = 1\text{mV} \cos \omega t$, with $R_T = 150\Omega$
- First Amplifier: Voltage Gain of 10, 50Ω in and out, designed for 50Ω circuits.
- Second Amplifier: Voltage Gain of 20, 50Ω in and out, designed for 50Ω circuits.
- Measurement device: Your oscilloscope.
- What is the measured voltage?

Solution: First Amplifier

Gain of 10 with 50Ω resistances.



Solution

- First amplifier mismatch going in; $\frac{50}{200} = \frac{1}{4}$

- Amplifiers $10 * 20 = 200$

- But v_T of final amplifier is $40\times$ input to produce gain of 20 into 50Ω but scope is $1\text{M}\Omega$. Actual gain is 40 instead of 20.

- Total gain $\frac{1}{4} \times 10 \times 40 = 100$

- Output $0.1\text{V} \cos \omega t$

- $0.2\text{V}_{\text{p}-\text{p}}$

Solution (2)