Chapter 8

Optics for Engineers

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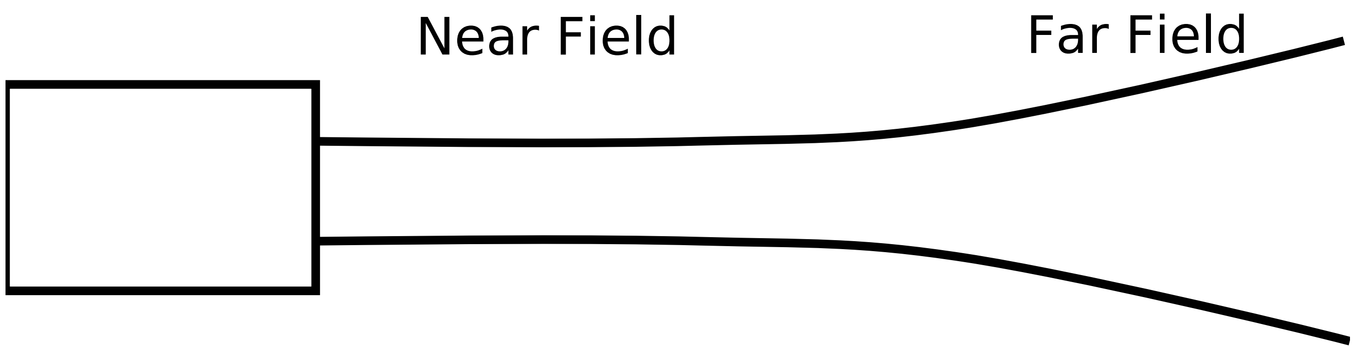
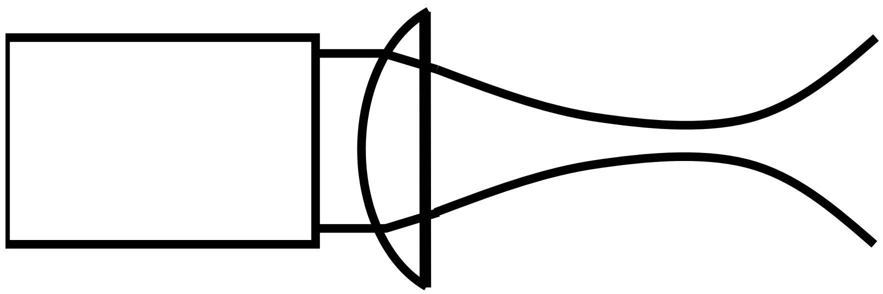
Diffraction

Collimated Beam: Divergence in Far Field

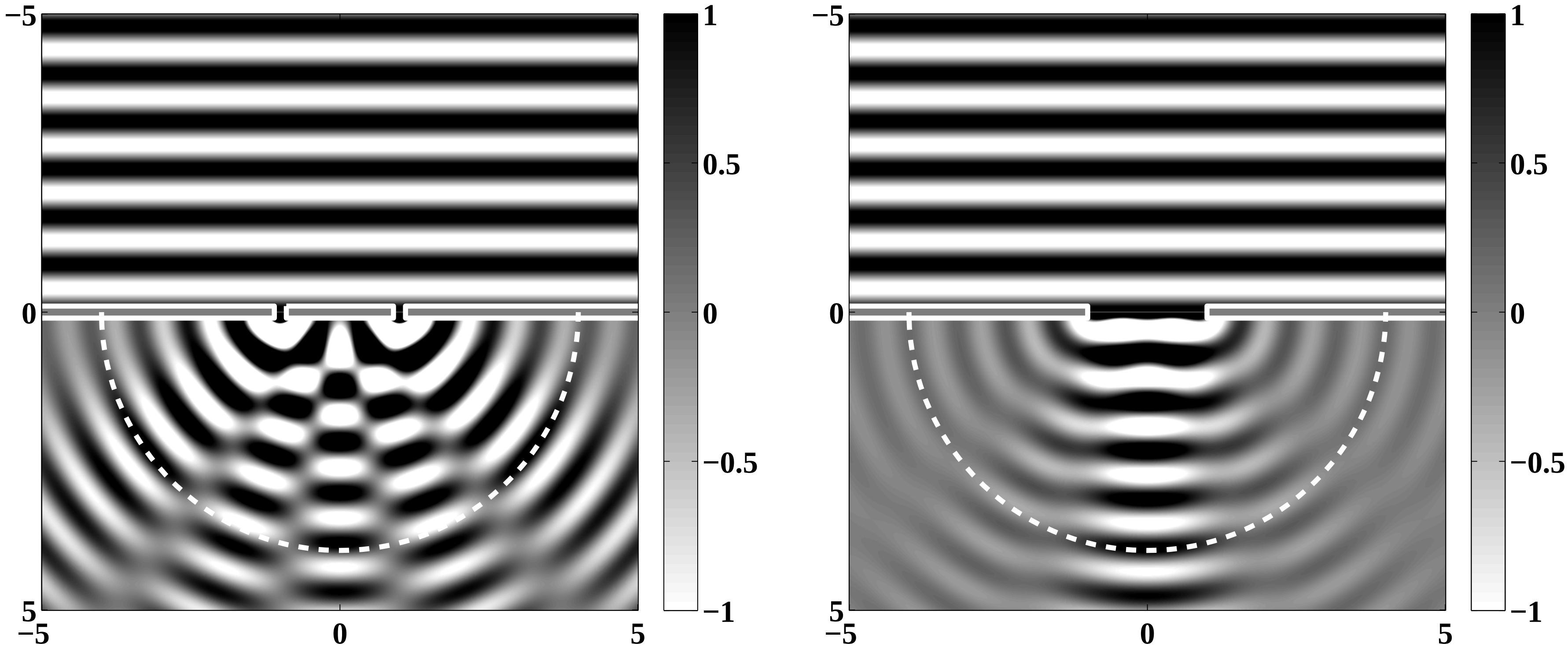


Focused Beam: Minimum Spot Size and Location



$$\alpha = C\frac{\lambda}{D}$$

Slit Experiments



A. Two slits B. Aperture

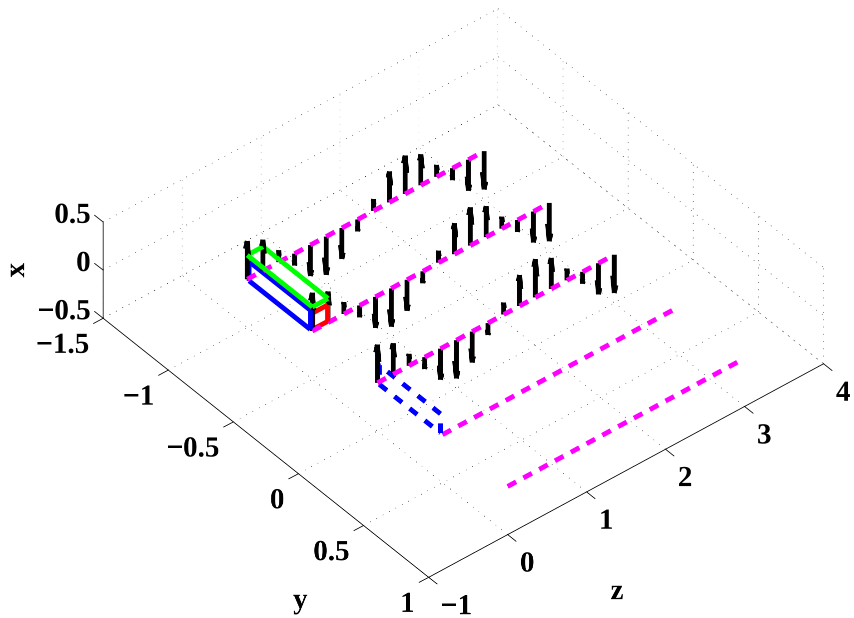
- Angular Divergence of Light Waves
- Alternating Bright and Dark Regions
- Near– and Far–Field Behavior

$\lambda = 800$nm. Axis Units are μm

Diffraction and Maxwell's Equations

- Faraday's Equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$



$$\mathbf{E} = E\hat{x}e^{jkz}$$

- Red Contour

$$\int \mathbf{E} \cdot d\mathbf{s} \neq 0 \qquad H_y \neq 0$$

  – Propagation along $\hat{z}$

- Green Contour, $E_y = E_z = 0$

$$\int \mathbf{E} \cdot d\mathbf{s} = 0 \qquad H_z = 0$$

- Blue, E_x Constant, $E_y = 0$

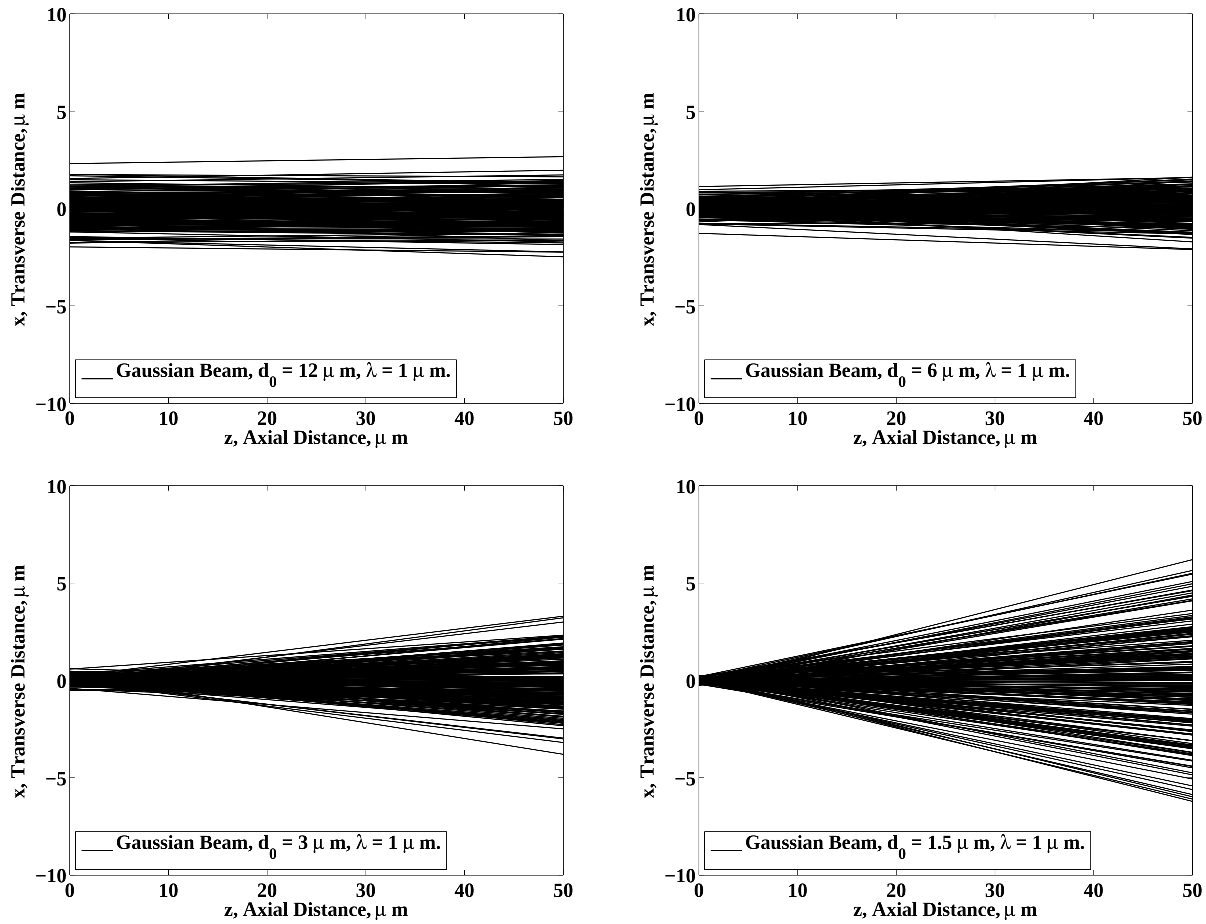
$$\int \mathbf{E} \cdot d\mathbf{s} = 0 \qquad H_z = 0$$

- Blue–Dash Contour (Edge)

$$\int \mathbf{E} \cdot d\mathbf{s} \neq 0 \qquad H_z \neq 0$$

  – Propagation Along $\hat{y}$ too

Diffraction and Quantum Mechanics



(Not to Scale in x,z)

$$\mathbf{p} = h\mathbf{k}/(2\pi)$$

$$k_x = |k| \sin\alpha$$

$$\delta x \ \delta p_x \geq h$$

$$\delta x \ \delta k_x \geq 2\pi$$

$$\delta x \ k \sin\delta\alpha \geq 2\pi$$

$$\sin\delta\alpha \geq \frac{2\pi}{\delta x \ |k|}$$

$$\sin\delta\alpha \geq \frac{\lambda}{\delta x}$$

A Peek at Gaussian Beams

- The Correct Quantum Approach
  - Position Probability Density Function, $P(x)$
  - Momentum Probability Density Function, $P(k) = FT(P(x))$
  - Uncertainty Measurement

$$(\delta x)^2 = \left\langle x^2 \right\rangle - \langle x \rangle^2 \qquad (\delta p_x)^2 = \left\langle p_x^2 \right\rangle - \langle p_x \rangle^2 ,$$

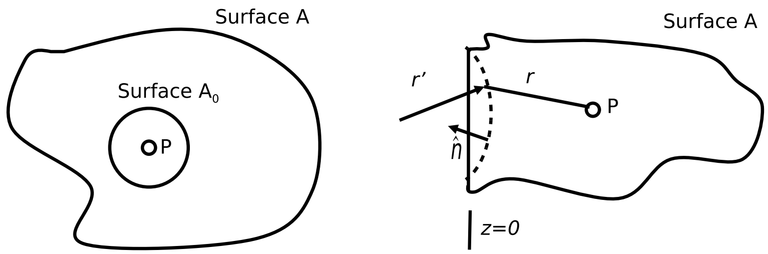
- Gaussian Beam (Minimum Uncertainty Wave)

$$\sin \delta\alpha = \frac{\lambda}{\delta x}$$

- Otherwise

$$\sin \delta\alpha > \frac{\lambda}{\delta x}$$

Fresnel–Kirchoff Integral (1)



- Auxiliary Field

$$V(x,y,z,t) = \frac{e^{j(\omega t + kr)}}{r}$$

- Electric Field

$$E(x,y,z,t) = E_s(x,y,z)\, e^{j\omega t}$$

- Green's Theorem

$$\int\int_{A+A_0} (V\nabla E - E\nabla V)\cdot \hat{n} d^2 A =$$

$$\int\int\int_{A+A_0} \nabla\cdot(V\nabla E) - \nabla\cdot(E\nabla V)\, d^3 V$$

- Expand RHS

(Cyan Terms Cancel)

$$\nabla\cdot(V\nabla E) - \nabla\cdot(E\nabla V) =$$

$$(\nabla V)(\nabla E) + V\nabla^2 E -$$

$$(\nabla E)(\nabla V) - E\nabla^2 V = 0$$

- E & V Satisfy Helmholz (Scalar Wave Eq.)

$$\nabla^2 E = \frac{\omega^2}{c^2} E$$

$$\nabla^2 V = \frac{\omega^2}{c^2} V$$

$$RHS = 0 = LHS$$

Fresnel–Kirchoff Integral (2)

- From Prev. Page

$$\int\int_{A+A_0} \left(V\nabla E - E\nabla V\right)\cdot \hat{n} d^2A = 0$$

- Split Surface

$$\int\int_{A} \left(V\nabla E - E\nabla V\right)\cdot \hat{n} d^2A +$$

$$\int\int_{A_0} \left(V\nabla E - E\nabla V\right)\cdot \hat{n} d^2A_0 = 0$$

- Radius of $A_0 \rightarrow 0$

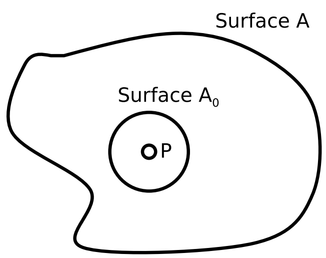
$$\int\int_{A_0} \left(V\nabla E - E\nabla V\right)\cdot \hat{n} d^2A = 4\pi E\left(x,y,z\right)$$

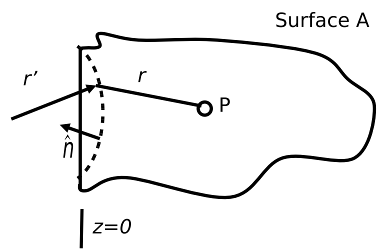
- Helmholtz–Kirchoff Integral Theorem

$$E\left(x,y,z\right) = -\frac{1}{4\pi}\int\int_A \left[E\left(\nabla\frac{e^{jkr}}{r}\right)\cdot\hat{n} - \frac{e^{jkr}}{r}\left(\nabla E\right)\cdot\hat{n}\right] d^2A$$

Fresnel–Kirchoff Integral (3)

$$E(x,y,z) = -\frac{1}{4\pi}\int\int_A \left[E\left(\nabla\frac{e^{jkr}}{r}\right)\cdot\hat{n} - \frac{e^{jkr}}{r}(\nabla E)\cdot\hat{n}\right]d^2A$$





- Irregular Surface of $A \to \infty$

Flat Surface: $E = E_0\frac{e^{-jkr'}}{r'}$

- E_0 Slowly Varying

$$(\nabla E)\cdot\hat{n} = \left(jkE - \frac{E}{r'}\right)\hat{n}\cdot\hat{r'}$$

- $r >> \lambda$, $r' >> \lambda$

$$\nabla E \to 0 \quad \frac{1}{r^2} \to 0 \quad \frac{1}{(r')^2} \to 0$$

$$E(x_1,y_1,z_1) = \frac{jk}{4\pi}\int\int_{Aperture} E(x,y,z)\frac{e^{jkr}}{r}\left(\hat{n}\cdot\hat{r} - \hat{n}\cdot\hat{r'}\right)dA$$

Obliquity Factor

- Fresnel–Kirchoff Integral Theorem (Previous Page)

$$E\left(x_1, y_1, z_1\right) = \frac{jk}{4\pi} \int\int_{Aperture} E\left(x, y, z\right) \frac{e^{jkr}}{r} \left(\hat{n} \cdot \hat{r} - \hat{n} \cdot \widehat{r'}\right) dA$$

- Numerical Calculations Possible Here, or Simpify More

- Obliquity Factor (Removes Backward Wave)

- For Small Transverse Distances . . .

$$\left(\hat{n} \cdot \hat{r} - \hat{n} \cdot \widehat{r'}\right) = 2$$

Paraxial Approximation

$$E(x_1,y_1,z_1) = \frac{jk}{2\pi}\int\int_{Aperture} E(x,y,z)\,\frac{e^{jkr}}{r}dA$$

$$r = \sqrt{(x-x_1)^2 + (y-y_1)^2 + z_1^2}$$

- Paraxial Approximation for
- r in Denominator

  $r \approx z$ Out of Integral
- r in Exponent

$$r = z_1\sqrt{\left(\frac{x-x_1}{z_1}\right)^2 + \left(\frac{y-y_1}{z_1}\right)^2 + 1}$$

$$(x-x_1) << z_1 \quad (y-y_1) << z_1$$

- Taylor's Series (First Order) (Remember Ch. 7)

$$r \approx z_1\left[1 + \frac{(x-x_1)^2}{2z_1^2} + \frac{(y-y_1)^2}{2z_1^2}\right]$$

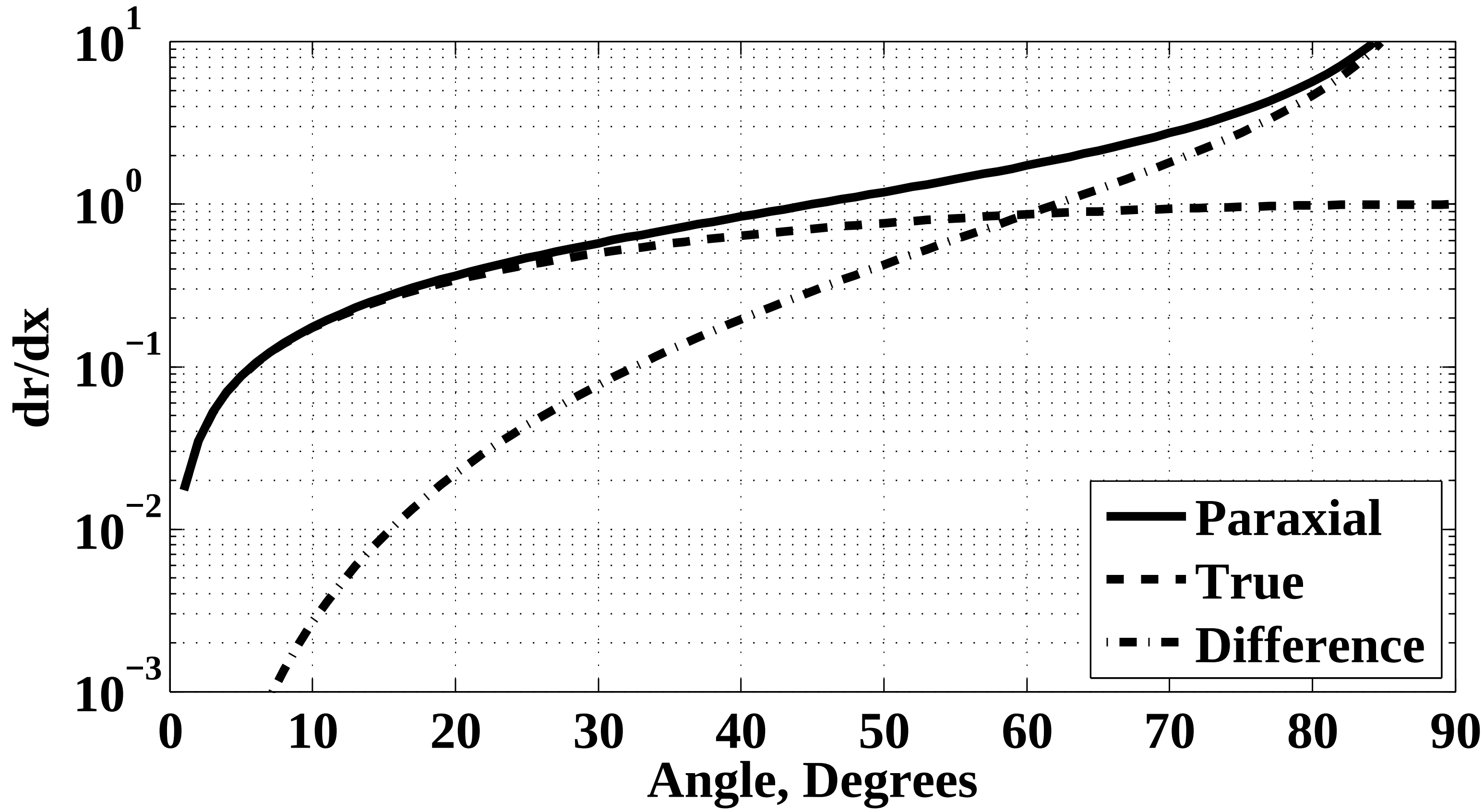
$$r \approx z_1 + \frac{(x-x_1)^2}{2z_1} + \frac{(y-y_1)^2}{2z_1}$$

- Easier Closed–Form Computation
- Amenable to Further Approximations

Error in Paraxial Approximation

$$r = \sqrt{(x-x_1)^2 + (y-y_1)^2 + z_1^2} \qquad r \approx z_1 + \frac{(x-x_1)^2}{2z_1} + \frac{(y-y_1)^2}{2z_1}$$

Very good to 30° ($NA = 0.5$) Need Vector Eq. Near 90°



Pupil Coordinates

- Pupil Position Coordinates (Length Units)

$$x_1,\ y_1$$

- Angular Coordinates (Dimensionless)

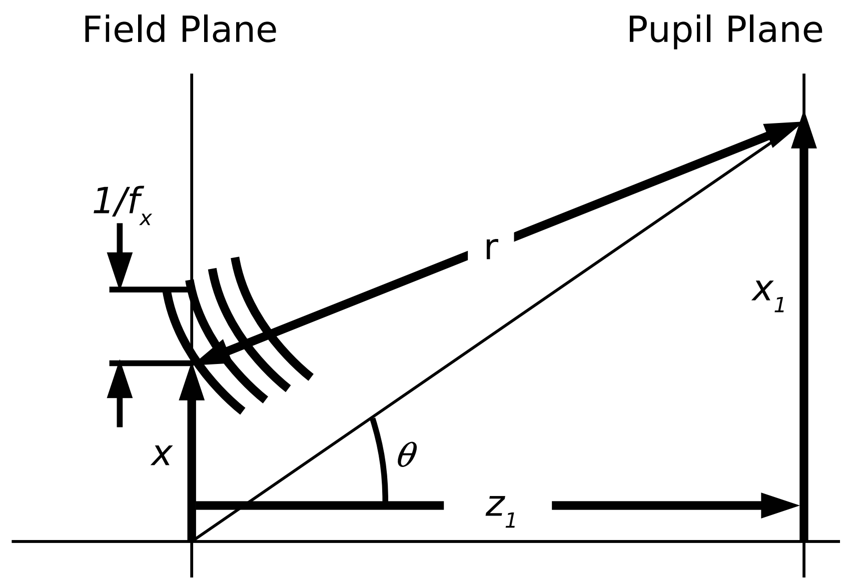
$$\arctan\frac{x_1}{z_1} = u = \sin\theta\cos\zeta$$

$$\arctan\frac{y_1}{z_1} = v = \sin\theta\sin\zeta$$

$$\max\sqrt{u^2+v^2} = NA$$

- Spatial Frequency (Inverse Length)

$$2\pi f_x = \frac{d(kr)}{dx} = \frac{2\pi}{\lambda}\frac{dr}{dx}$$



- Spatial Frequency Approximation

$$f_x \approx \frac{x_1}{r\lambda} = \frac{\sin\theta\cos\zeta}{\lambda}$$

$$f_y \approx \frac{y_1}{r\lambda} = \frac{\sin\theta\sin\zeta}{\lambda}$$

- For Fourier Optics (Ch. 11)

Coordinate Relationships

	Spatial Frequency	**Pupil Location**	**Direction Cosines**
Spatial Frequency		$x_1 = \lambda z f_x$ $y_1 = \lambda z f_y$	$u = f_x \lambda$ $v = f_y \lambda$
Pupil Location	$f_x = \frac{x_1}{\lambda z}$ $f_y = \frac{y_1}{\lambda z}$		$u = \frac{x_1}{z}$ $v = \frac{y_1}{z}$
Direction cosines	$f_x = \frac{u}{\lambda}$ $f_y = \frac{v}{\lambda}$	$x_1 = uz$ $y_1 = vz$	
Angle			$u = \sin\theta\cos\zeta$ $v = \sin\theta\sin\zeta$

Two Approaches to the Position Terms in the Exponent

- Fresnel Diffraction

$$E\left(x_1,y_1,z_1\right)=\frac{jke^{jkz_1}}{2\pi z_1}\int\int E\left(x,y,0\right)e^{jk\frac{(x-x_1)^2}{2z_1}+\frac{(y-y_1)^2}{2z_1}}dxdy$$

- Fraunhofer Diffraction (x, x_1 both small)

$$(x-x_1)^2+(y-y_1)^2=\left(x^2+y^2\right)+\left(x_1^2+y_1^2\right)-2\left(xx_1+yy_1\right)$$

$$E\left(x_1,y_1,z_1\right)=\frac{jke^{jkz_1}}{2\pi z_1}e^{jk\frac{\left(x_1^2+y_1^2\right)}{2z_1}}\int\int E\left(x,y,0\right)e^{jk\frac{\left(x^2+y^2\right)}{2z_1}}e^{-jk\frac{\left(xx_1+yy_1\right)}{z_1}}dxdy$$

Fresnel Radius and the Far Field

- Curvature Terms

$$e^{jk\frac{\left(x^2+y^2\right)}{2z_1}} \qquad e^{jk\frac{\left(x_1^2+y_1^2\right)}{2z_1}}$$

- Fresnel Radius in Integrand (See Plot on Next Page)

$$e^{jk\frac{\left(x^2+y^2\right)}{2z_1}} = e^{j2\pi\left(\frac{r}{\sqrt{2\lambda z_1}}\right)^2} = e^{j2\pi\left(\frac{r}{r_f}\right)^2} \qquad r_f = \sqrt{2\lambda z_1}$$

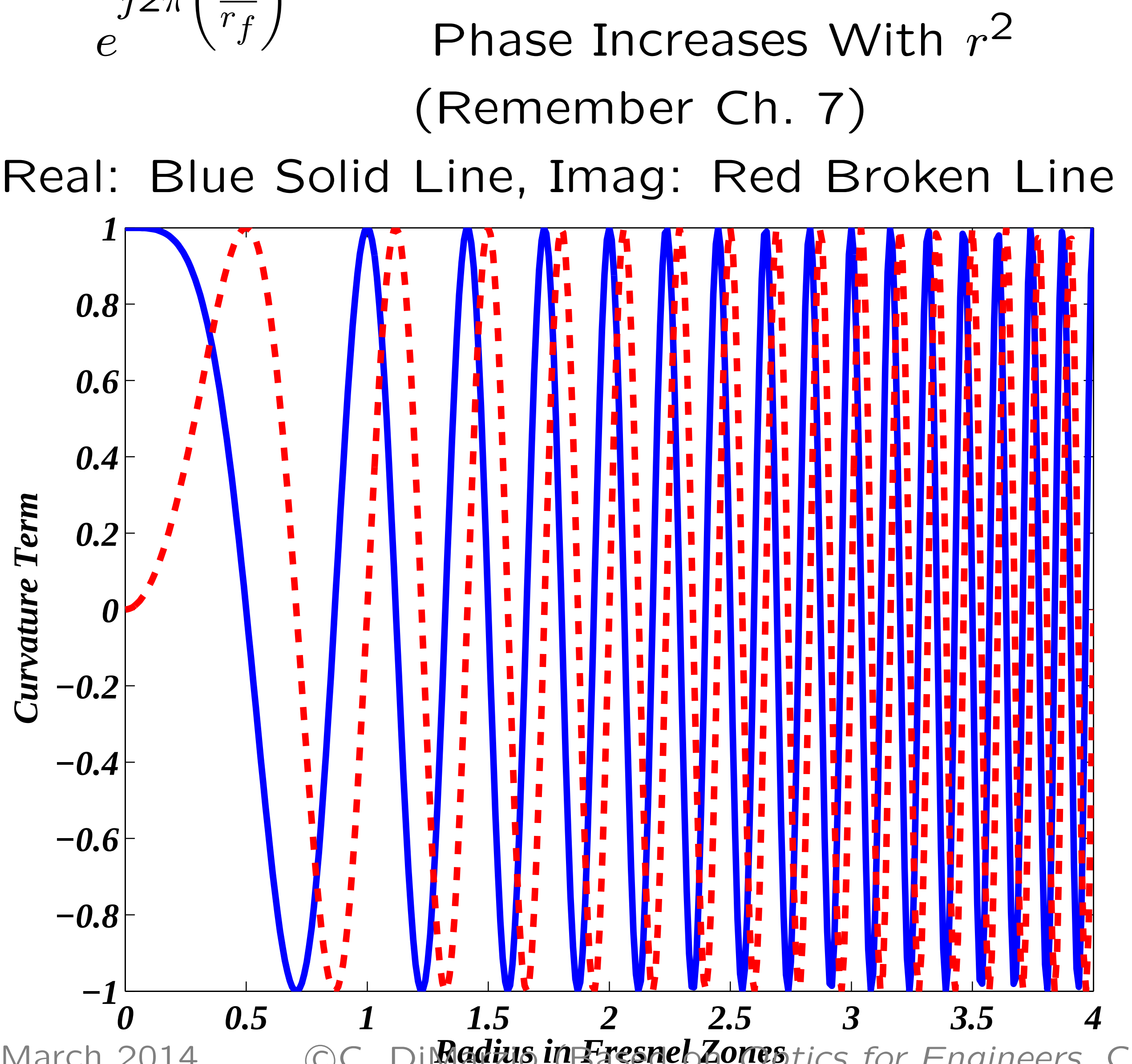
- Simplified Integral

$$E\left(x_1, y_1, z_1\right) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{\left(x_1^2+y_1^2\right)}{2z_1}} \times$$

$$\int\int E\left(x, y, 0\right) e^{j2\pi\frac{x^2+y^2}{r_f^2}} e^{-jk\frac{\left(xx_1+yy_1\right)}{z_1}} dxdy$$

Integrand in Fresnel Zones

$$e^{j2\pi\left(\frac{r}{r_f}\right)^2}$$



- Fresnel Number

$$N_f = \frac{r_{max}}{r_f}$$

  - Near Field

$$N_f \approx 0$$

  - Far Field

$$N_f \gg 0$$

  - Almost–Far Field

$$N_f \ll 0$$

- Different Approximations

Far Field: Fraunhofer Zone

$z_1 \gg D^2/\lambda$ (Far–Field Condition) $r_f = \sqrt{2z_1\lambda} \gg \sqrt{2}D$

$N_f = r/r_f \ll 1$ (Fraunhofer Zone) $e^{j2\pi\frac{(x^2+y^2)}{r_f}} \approx 1$

Fraunhofer Diffraction: Fresnel–Kirchoff
Integral is a Fourier Transform in Fraunhofer Zone

$$E(x_1, y_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{(x_1^2+y_1^2)}{2z_1}} \int\int E(x, y, 0)\, e^{-jk\frac{(xx_1+yy_1)}{z_1}} dxdy$$

Near Field: Fresnel Zone

$$z_1 \ll D^2/\lambda \qquad \text{(Near–Field Condition)}$$

$$N_f = r/r_f \gg 1$$

- Fresnel Diffraction

$$E\left(x_1, y_1, z_1\right) = \frac{jke^{jkz_1}}{2\pi z_1} \int\int E\left(x, y, 0\right) e^{jk\frac{(x-x_1)^2}{2z_1} + \frac{(y-y_1)^2}{2z_1}} dxdy$$

- Integrand Averages to Zero Except near $x_1 = x$, $y_1 = y$
  - Need Different Comp. Techniques from Fraunhofer Case (Both for Analytical and Numerical Calculations)

Almost–Far field

- Condition

N_f small but not that small

- Fraunhofer Diffraction

$$E\left(x_1,y_1,z_1\right)=\frac{jke^{jkz_1}}{2\pi z_1}e^{jk\frac{\left(x_1^2+y_1^2\right)}{2z_1}}\int\int\overset{Field}{E\left(x,y,0\right)}\ \overset{Curvature}{e^{jk\frac{\left(x^2+y^2\right)}{2z_1}}}\ \overset{Fourier}{e^{-jk\frac{\left(xx_1+yy_1\right)}{z_1}}}\,dxdy$$

- Looks Like Fourier Transform with Curvature

- Computation Feasible if N_f is Not Too Large
  (Watch for Nyquist Criterion in Sampling $E(x,y,0)$)

Fraunhofer Lens (1)

- Focusing

$$E(x,y,0) = E_0(x,y,0)\, e^{-jk\frac{\left(x^2+y^2\right)}{2f}}$$

  – Focused in Near Field of Aperture ($f << D^2/\lambda$)

$$E(x_1,y_1,z_1) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{\left(x_1^2+y_1^2\right)}{2z_1}} \int\int E(x,y,0)\, e^{jk\frac{\left(x^2+y^2\right)}{2Q}} e^{-jk\frac{\left(xx_1+yy_1\right)}{z_1}} dx_1 dy_1$$

  – Modify Fresnel Radius Equation

$r_f = \sqrt{2\lambda Q}$ $\qquad$ Q is the Defocus Parameter: Next Page

$$\frac{1}{r_f} = \frac{1}{\sqrt{2\lambda z_1}} - \frac{1}{\sqrt{2\lambda f}}$$

Fraunhofer Lens (2)

- Computational Considerations

$$E(x_1, y_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{\left(x_1^2+y_1^2\right)}{2z_1}} \int\int E(x, y, 0)\, e^{jk\frac{\left(x^2+y^2\right)}{2Q}} e^{-jk\frac{\left(xx_1+yy_1\right)}{z_1}} dx_1 dy_1$$

- Quadratic Phase Term Leads to Nyquist Issues
  - For Large Apertures
  - For Strong Defocusing
  - Practical Limit

$$Q = \frac{fz_1}{|f - z_1|} \gg \frac{D^2}{8\lambda} \qquad |f - z| \ll \frac{8z_1^2\lambda}{D^2}$$

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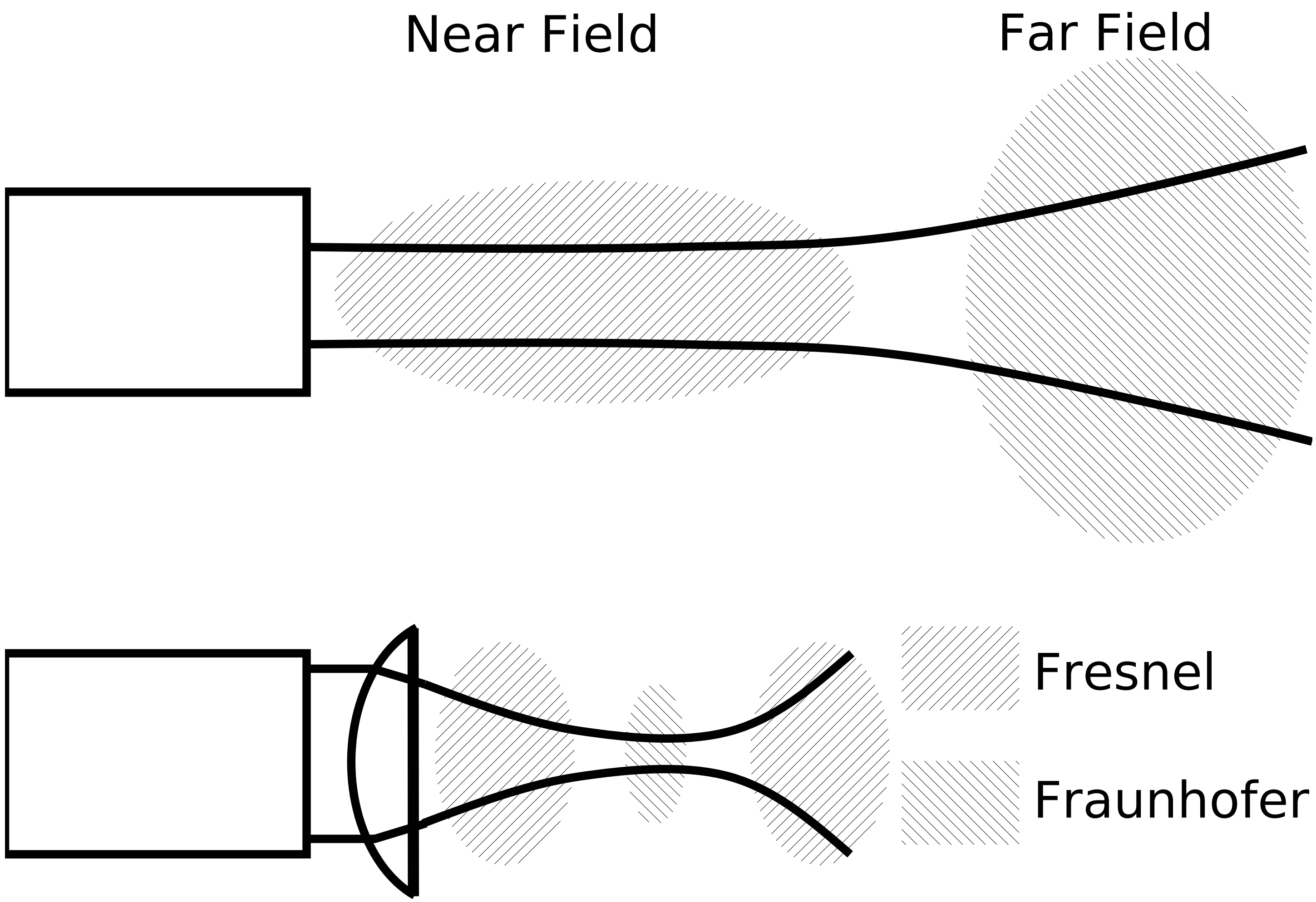
Fraunhofer and Fresnel Zones

Integrand varies slowly in Fraunhofer Zone, Fast in Fresnel



Spatial Frequency and Angle Definitions

- Spatial Frequency

$$f_x = \frac{k}{2\pi z_1} x_1 = \frac{x_1}{\lambda z_1} \qquad f_y = \frac{k}{2\pi z_1} y_1 = \frac{y_1}{\lambda z_1}$$

$$df_x = \frac{k}{2\pi}\frac{dx_1}{z_1} \qquad df_y = \frac{k}{2\pi}\frac{dy_1}{z_1}$$

- Angle or Wavefront Tilt, $E_a = E z_1 e^{-jkz_1}$,

$$E_a(u,v) = \frac{jk}{2\pi\cos\theta}\int\int E(x,y,0)\, e^{-jk\frac{xu+yv}{w}} dx dy$$

Fraunhofer Diffraction Equations

- Fraunhofer Diffraction Integral

$$E\left(x_1,y_1,z_1\right)=\frac{jke^{jkz_1}}{2\pi z_1}e^{jk\frac{\left(x_1^2+y_1^2\right)}{2z_1}}\int\int E\left(x,y,0\right)e^{-jk\frac{\left(xx_1+yy_1\right)}{z_1}}dxdy$$

- Spatial Frequency Description

$$E\left(f_x,f_y,z_1\right)= \qquad \text{(Units: V/m/(m}^{-1}\text{) = V)}$$

$$\left(\frac{k}{2\pi z_1}\right)^2\frac{jke^{jkz_1}}{2\pi z_1}e^{jk\frac{\left(x_1^2+y_1^2\right)}{2z_1}}\int\int E\left(x,y,0\right)e^{-j2\pi\left(f_xx+f_yy\right)}dxdy$$

$$E\left(f_x,f_y,z_1\right)=\frac{j2\pi z_1e^{jkz_1}}{k}e^{jk\frac{\left(x_1^2+y_1^2\right)}{z_1}}\int\int E\left(x,y,0\right)e^{-j2\pi\left(f_xx+f_yy\right)}dxdy$$

Camera Example

- Pixel Pitch: 7.4μm

$$f_{sample} = \frac{1}{7.4 \times 10^{-6}\text{m}} = 1.35 \times 10^{5}\text{m}^{-1} \qquad \text{(135 per mm.)}$$

- Nyquist Sampling: $f_{max} = f_{sample}/2$ at $u_{max} = NA$

$$NA = \frac{2 f_{sample} \lambda}{2} = f_{sample} \lambda \qquad \text{(Coherent Imaging)}$$

- Green light at 500nm,

$$NA = 0.068$$

  - Lower NA Acts as Anti–Aliasing Filter
  - Higher NA Allows Aliasing

(See Ch. 11 for Incoherent Imaging, which is More Complicated)

Some Useful Fraunhofer Patterns

- Frequently–Used Patterns
  - Uniform Illumination through Rectangular Aperture (Separable Integrals in x and y)
  - Uniform Illumination through Circular Aperture (Frequently–Encountered in Imaging)
  - Gaussian Illumination with no Aperture (Many Laser Beams, Minimum Uncertainty, Closed Form)
- Parameters of Interest
  - Some Measure of Width
  - Axial Irradiance
  - Nulls and Sidelobes if Any

Rectangular Aperture (1)

- Irradiance Pattern

$$E^2(x,y,0) = \frac{P}{D_xD_y}$$

- Fraunhofer Diffraction

$$E(x_1,y_1,z_1) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{\left(x_1^2+y_1^2\right)}{2z_1}} \int_{-D_x/2}^{D_x/2}\int_{-D_y/2}^{D_y/2} \sqrt{\frac{P}{D_xD_y}} e^{-jk\frac{\left(xx_1+yy_1\right)}{z_1}} dxdy$$

- Separable in x and y

$$E(x_1,y_1,z_1) = \sqrt{\frac{P}{D_xD_y}} \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{\left(x_1^2+y_1^2\right)}{2z_1}} \int_{-D_x/2}^{D_x/2} e^{-jk\frac{\left(xx_1\right)}{z_1}} dx \int_{-D_y/2}^{D_y/2} e^{-jk\frac{\left(yy_1\right)}{z_1}} dy$$

Rectangular Aperture (2)

- Compare Rectangular Pulse in Signal Theory

$$E\left(x_1,y_1,z_1\right)=\sqrt{P\frac{D_xD_y}{\lambda^2z_1^2}}\frac{\sin\left(kx_1\frac{D_x}{2z_1}\right)}{kx_1\frac{D_x}{2z_1}}\frac{\sin\left(ky_1\frac{D_y}{2z_1}\right)}{ky_1\frac{D_y}{2z_1}}e^{jk\frac{\left(x_1^2+y_1^2\right)}{2z_1}}$$

- Irradiance

$$I\left(x_1,y_1,z_1\right)=P\frac{D_xD_y}{\lambda^2z_1^2}\left(\frac{\sin\left(kx_1\frac{D_x}{2z_1}\right)}{kx_1\frac{D_x}{2z_1}}\frac{\sin\left(ky_1\frac{D_y}{2z_1}\right)}{ky_1\frac{D_y}{2z_1}}\right)^2$$

On Axis	First Null	First Sidelobe
$I_0=I\left(0,0,z_1\right)=P\frac{D_xD_y}{\lambda^2z_1^2}$	$\frac{d_x}{2}=\frac{\lambda}{D_x}$	$I_1=0.05I_0\ (-13dB)$

Circular Aperture (1)

- Polar Coordinates

$$E(r_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{\left(x_1^2+y_1^2\right)}{2z_1}} \int_0^{D/2} \int_0^{2\pi} E(r,0)\, e^{-j\frac{k(r\cos\zeta x + r\sin\zeta y)}{z_1}} r d\zeta dr$$

- Hankel Transform

$$E(r_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{\left(r_1^2\right)}{z_1}} \int_0^{D/2} E(r,0)\, e^{-j\frac{kr^2}{z_1}} J_0\left(\frac{krr_1}{z_1}\right) dr$$

- Fraunhofer Diffraction for Source, $E^2(r,0) = \frac{P}{\pi(D/2)^2}$

$$E(r_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{\left(r_1^2\right)}{z_1}} \int_0^{D/2} \sqrt{\frac{P}{\pi (D/2)^2}} J_0\left(\frac{krr_1}{z_1}\right) dr$$

Circular Aperture (2)

$$E\left(r_1, z_1\right) = \sqrt{P\frac{\pi D^2}{4\lambda^2 z_1^2}}\frac{J_1\left(kr_1\frac{D}{2z_1}\right)}{kr_1\frac{D}{2z_1}}e^{jk\frac{r_1^2}{2z}}$$

$$I\left(r_1, z_1\right) = P\frac{\pi D^2}{4\lambda^2 z_1^2}\left(\frac{J_1\left(kr_1\frac{D}{2z_1}\right)}{kr_1\frac{D}{2z_1}}\right)^2$$

On Axis	First Null	First Sidelobe
$I_0 = P\pi D^2/4\lambda^2 z_1^2$	$2.44 z_1 \lambda D$	$I = 0.02 I_0$ (−17dB)

Gaussian

- Plane–Wave Gaussian Source (Assume Infinite Aperture)

$$E\left(x,y,0\right)=\sqrt{\frac{2P}{\pi(w_0)^2}}e^{-r^2/w_0^2}$$

- Polar Coordinates (Arbitrary Choice)

$$E\left(r_1,z_1\right)=\frac{jke^{jkz_1}}{2\pi z_1}e^{jk\frac{\left(r_1^2\right)}{2z_1}}\int_0^\infty\int_0^{2\pi}\sqrt{\frac{2P}{\pi(w_0)^2}}e^{-r^2/w_0^2}e^{-jkr\frac{\left(r_1\cos\zeta+r_1\sin\zeta\right)}{z_1}}rd\zeta dr$$

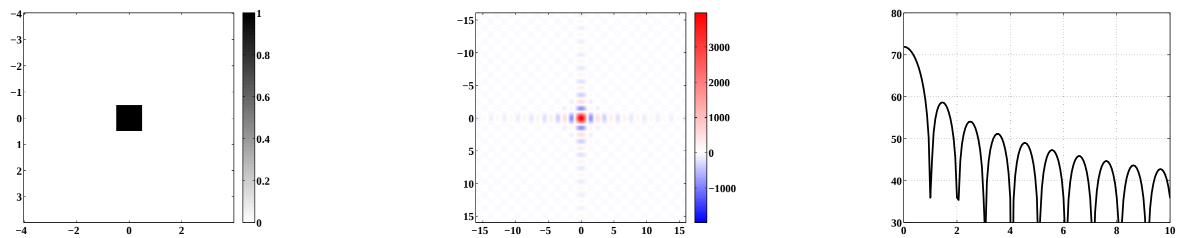
- Result: Gaussians are Forever. (More in Chapter 9)

$$E\left(x_1,y_1,z_1\right)=\sqrt{\frac{2P}{\pi(w)^2}}e^{-r_1^2/w^2}\qquad w=\frac{z_1\lambda}{\pi w_0}$$

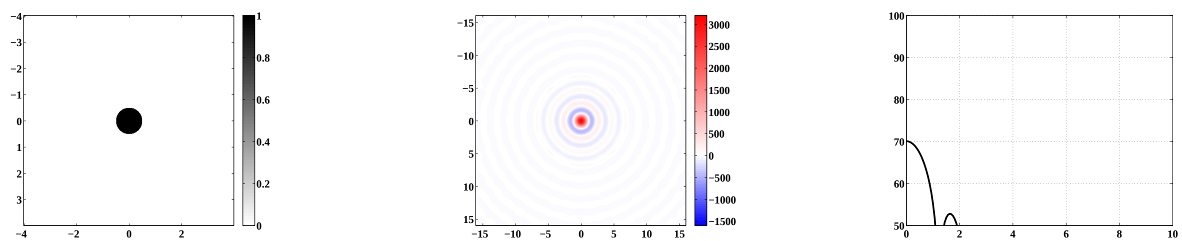
Summary of Common Fraunhofer Patterns

Pupil Pattern **Fraunhofer Pattern** **Slice**

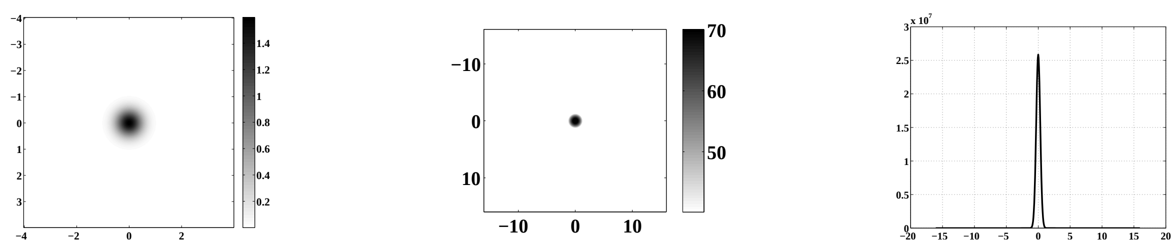
SQUARE:
$d = 2\frac{\lambda}{D}$ (1st Nulls)
$I_0 = \frac{PD^2}{\lambda^2 z_1^2}$
$I_1/I_0 = -13dB$



CIRCULAR:
$d = 2.44\frac{\lambda}{D}$ (1st Nulls)
$I_0 = \frac{\pi PD^2}{4\lambda^2 z_1^2}$
$I_1/I_0 = -17dB$



GAUSSIAN:
$d = \frac{4}{\pi}\frac{\lambda}{d_0}$ (e^{-2} Width)
$I_0 = \frac{\pi P d_0^2}{2\lambda^2 z_1^2}$
No sidelobes



Summary

Take–Away Message

- Diffraction Angle is Always Proportional to λ/D
- Proportionality Constant Depends on Beam Profile
- Proportionality Constant Depends on Definitions
  - Width of Original Field
  - Diffraction Angle
- On–Axis Irradiance Proportional to D^2/λ^2

Numerical Integration Example: Gaussian in an Aperture

- Given Aperture, D
- Choose Gaussian Size, hD
  - Fill Factor, h
  - Optimize Pattern Somehow (*e.g.* Axial Irradiance)
- Integrate in Polar Coordinates

$$E\left(x_1,y_1,z_1\right)=\frac{jke^{jkz_1}}{2\pi z_1}e^{jk\frac{\left(x_1^2+y_1^2\right)}{2z_1}}\int_0^{D/2}\int_0^{2\pi}\sqrt{\frac{2P}{\pi(hD/2)^2}}e^{-\frac{4r^2}{h^2D^2}}e^{-jkr\frac{\left(x_1\cos\zeta+y_1\sin\zeta\right)}{z_1}}rd\zeta dr$$

Truncated Gaussian Approximations

- Small Beam, $h \rightarrow 0$
  - Ignore Aperture
  - Integrate in Closed Form
  - Gaussian Pattern

$$d = \frac{4}{\pi} \frac{\lambda}{hD} z_1$$

$$I_0 = P \left(\frac{\pi h D}{\lambda z_1} \right)^2$$

  - Low Axial Irradiance

$$I_0 \propto h^2 \rightarrow 0$$

- Large Beam, $h \rightarrow \infty$
  - Near Uniform Illumination
  - Power Through Aperture

$$\frac{2P}{\pi (hD)^2/4} \pi \frac{D^2}{4} = \frac{2P}{h^2}$$

  - Use Circular Aperture

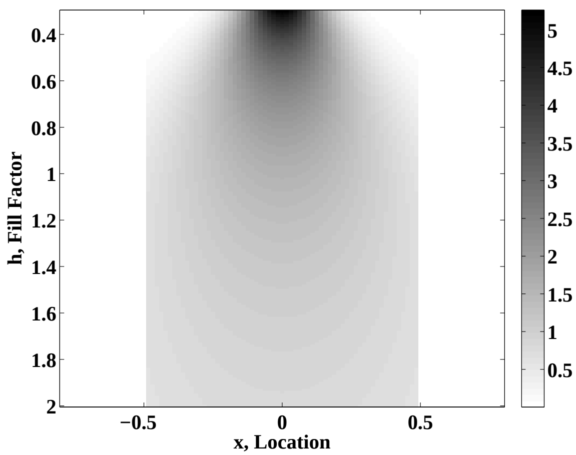
$$I_0 = \frac{P}{2\pi} \left(\frac{\pi D}{h \lambda z_1} \right)^2$$

- Low Axial Irradiance

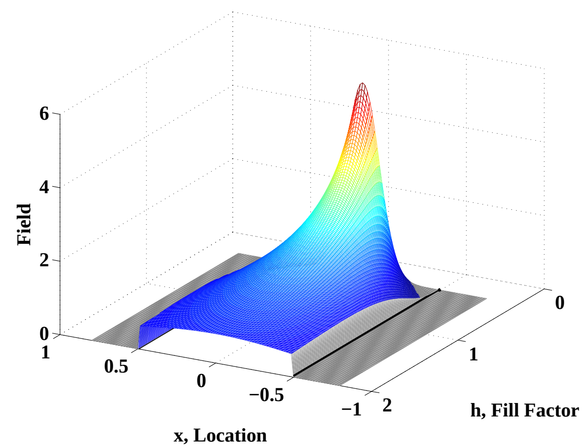
$$I_0 \propto h^{-2} \rightarrow 0$$

- Between These Limits, Use Numerical Calculation

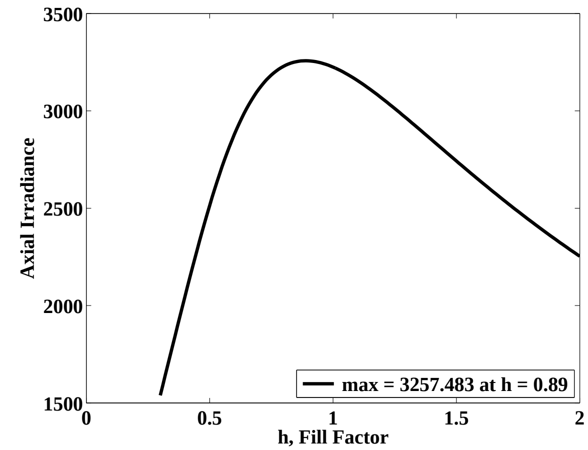
Numerical Integration Results: Gaussian in an Aperture



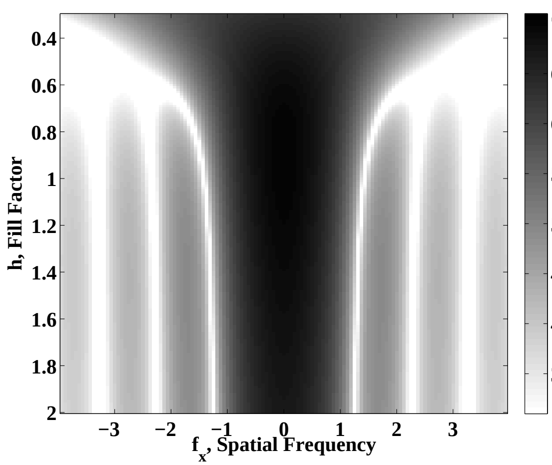
A. Source



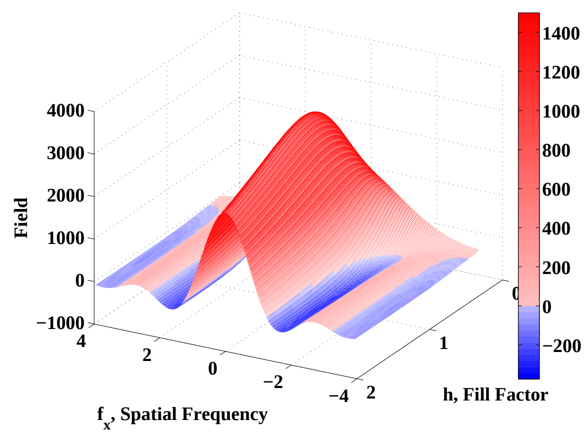
C. Source



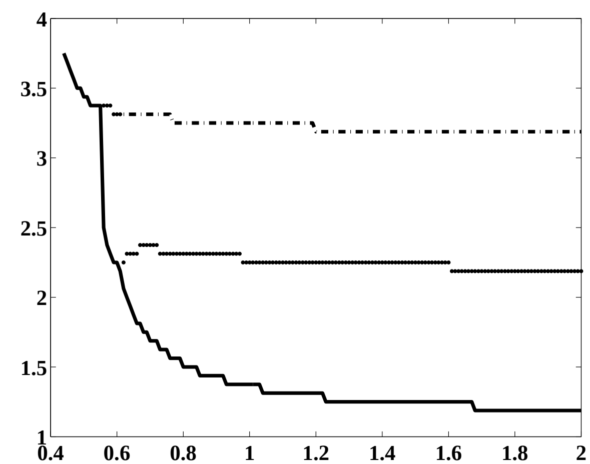
E. Axial I



B. Diffraction



D. Diffraction



F. Null Locations

 ©C. DiMarzio (Based on *Optics for Engineers*, CRC Press)

Depth of Focus

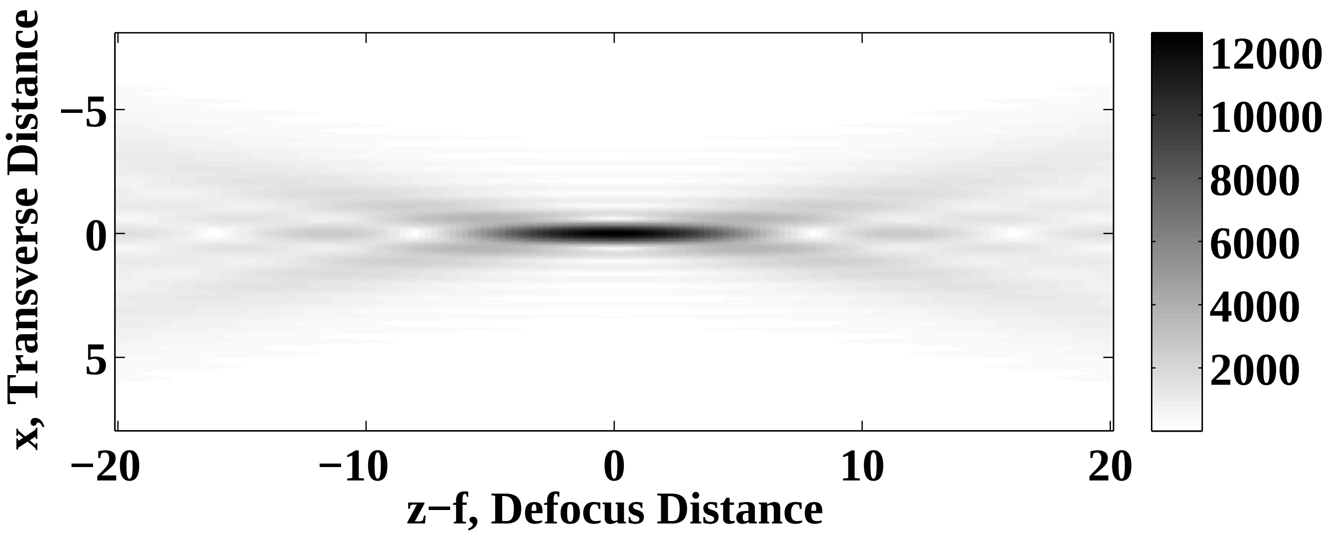
- Fraunhofer Diffraction

$$E\left(x_1, y_1, z_1\right) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{\left(x_1^2+y_1^2\right)}{2z_1}} \int\int E\left(x, y, 0\right) e^{jk\frac{\left(x^2+y^2\right)}{2Q}} e^{-jk\frac{\left(xx_1+yy_1\right)}{z_1}} dx_1 dy_1$$

- Defocus (Diopters)

$$\frac{1}{Q} = \frac{1}{z_1} - \frac{1}{f} \qquad \frac{1}{Q} \approx \frac{f - z_1}{f^2}$$

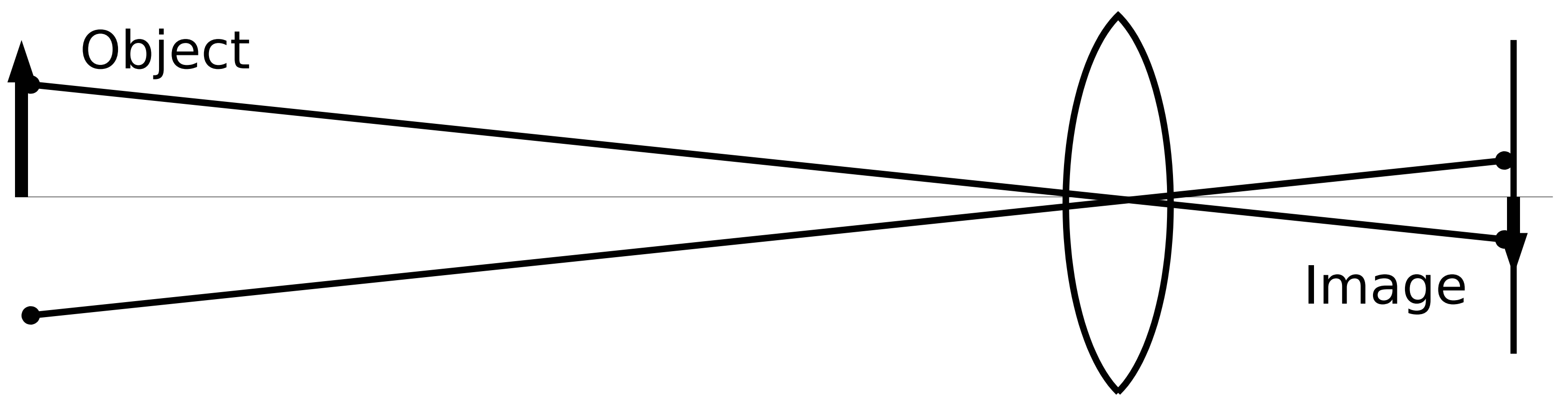
- Numerical Result



- Equation

$$DOF \approx \frac{\lambda}{NA^2}$$

Resolution of an Imaging System



- "To Resolve" or "Resolution" Defined

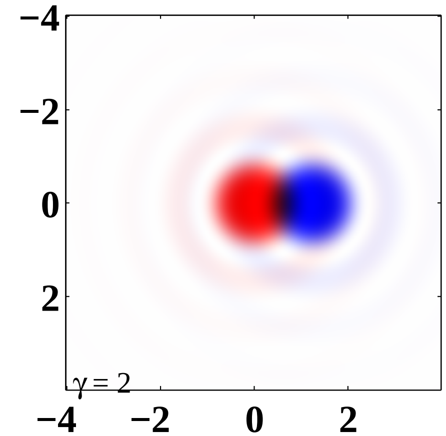
"...to distinguish parts or components of (something) that are close together in space or time; ..."

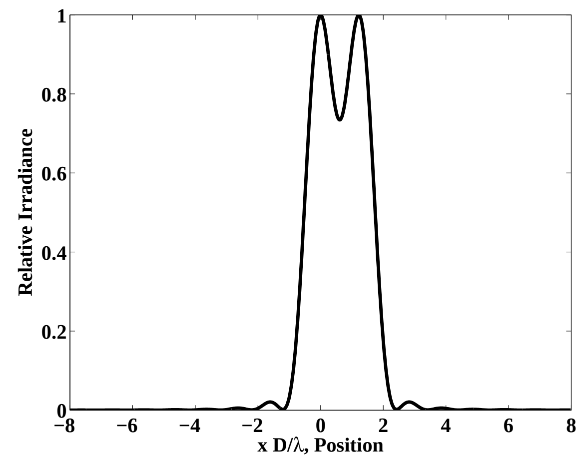
"...the process or capability of rendering distinguishable the component parts of an object or image; a measure of this, expressed as the smallest separation so distinguishable, ..."

- Resolution is Not Number of Pixels or Width of Point–Spread Function

Resolution Analysis

- Diffraction Patterns of Two "Point Objects"
  - Point–Spread Functions
  - From Fraunhofer Diffraction at Pupil
  - Fourier Transforms (See Ch. 11)
- Add
- Set Criterion for Valley
  - Noise Analysis?
  - Contrast?
  - Arbitrary Decision?





A complete and consistent definition would require knowledge we are not likely to have.

Rayleigh Criterion

- Frequently Used, but Arbitrary
- Defined by Nulls of Point–Spread Function
  - Peak of One over Valley of Other
- Produces Inconsistent Valley (Depends on Pattern)
  - Square Aperture

$$\delta = z_1 \lambda / D$$

$$\delta\theta = \lambda / D$$

  - Valley Depth

$$2\left[\frac{\sin(\pi/2)}{\pi/2}\right]^2 =$$

$$\frac{8}{\pi^2} = 0.81$$

  - Circular Aperture

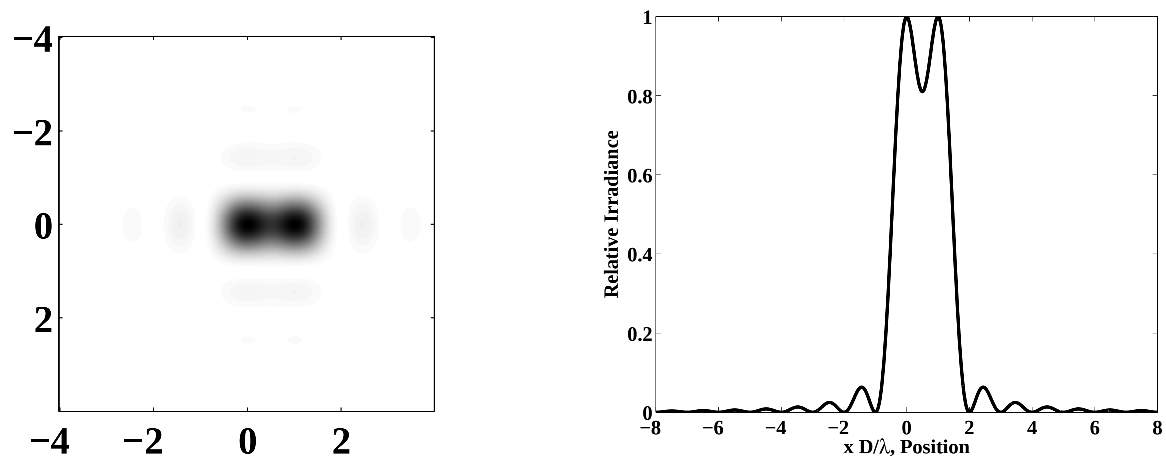
$$\delta\theta = 1.22\frac{\lambda}{D}$$

$$\delta = 0.61\frac{\lambda}{NA}$$

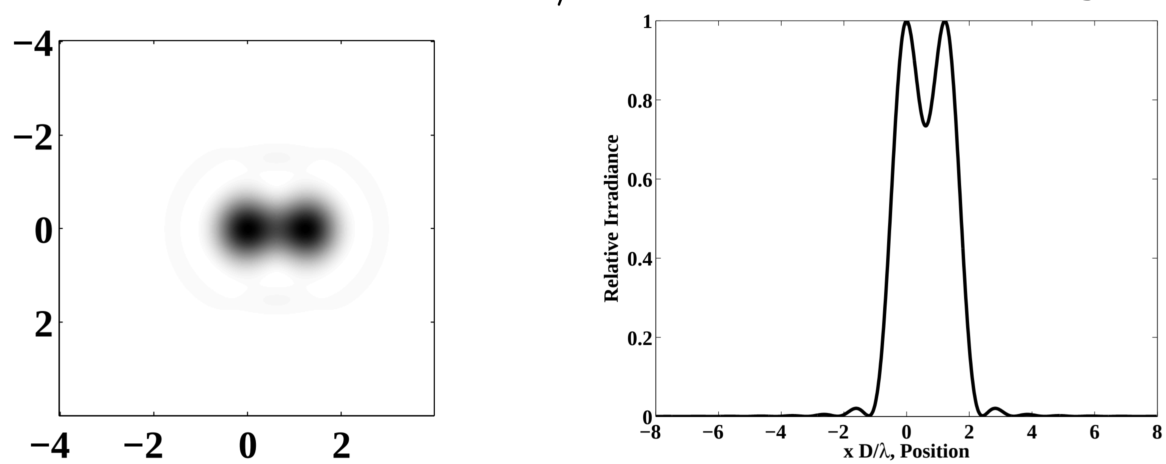
  - Valley Depth

0.73

Resolution at the Rayleigh Limit



Square Aperture, λ/D, 81% Valley

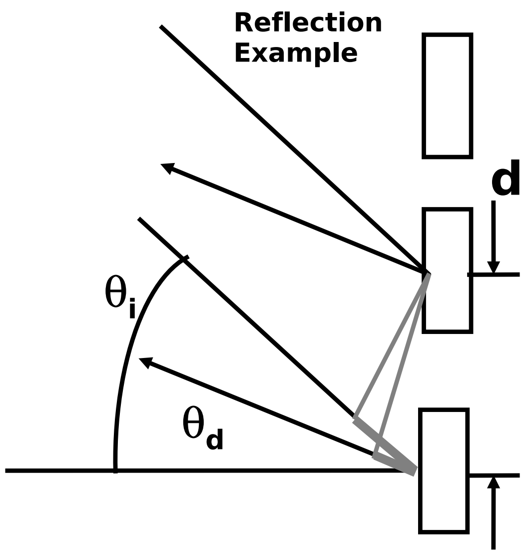


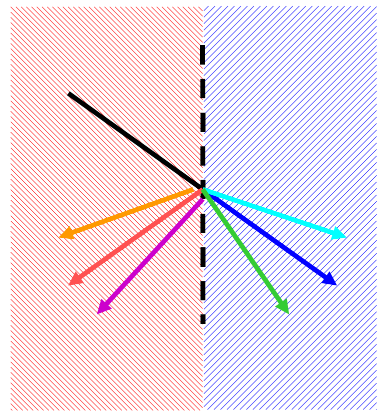
Circular Aperture, $1.22\lambda/D$, 73% Valley

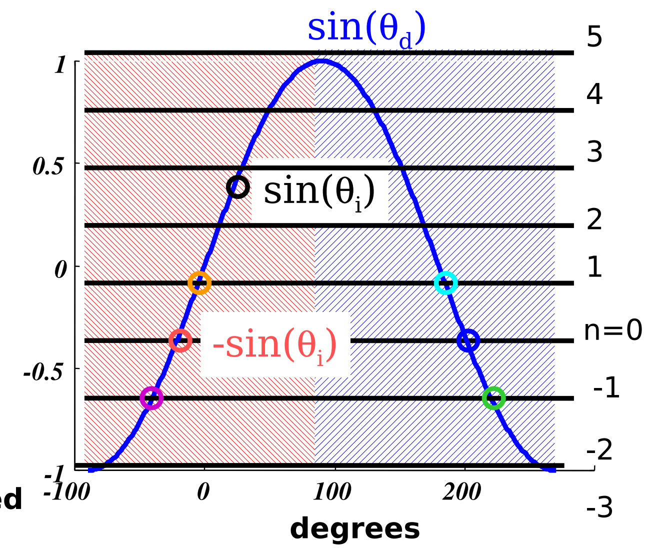
- Issues
  - Noise
  - Contrast
  - Statistics
  - Sampling*
- Other Definitions
  - MTF* (Ch. 11)
  - Any Valley (Sparrow)
  - 81% Valley (Wadsworth)
  - PSF FWHM (Houston)*

* Not really resolution

The Diffraction Grating (1)







- The Grating Equation

$$N\lambda = d\left(\sin\theta_i + \sin\theta_d\right)$$

$$\sin\theta_d = -\sin\theta_i + N\frac{\lambda}{d}$$

The Diffraction Grating (2)

- Grating Equation

$$N\lambda = d\left(\sin\theta_i + \sin\theta_d\right)$$

- Grating Dispersion

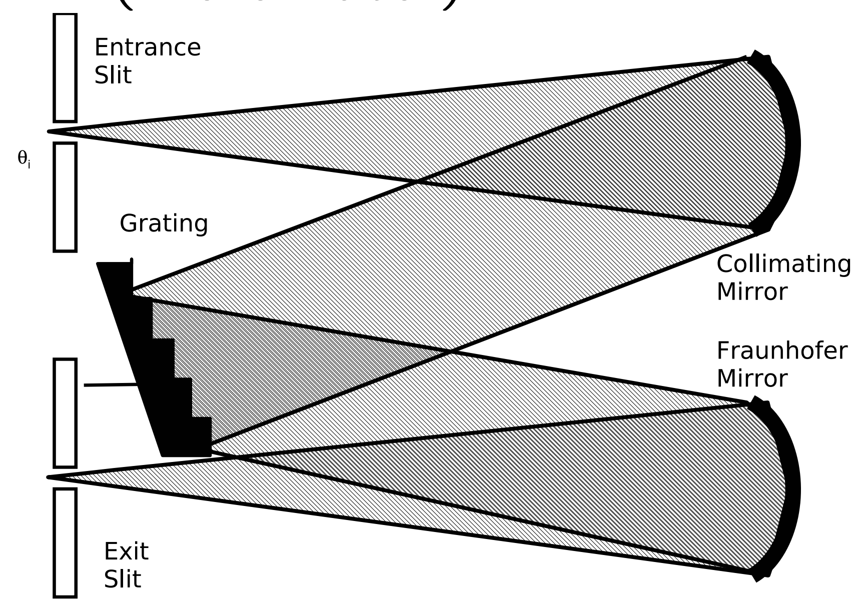
$$\delta\lambda = \frac{d}{N}\delta\left(\sin\theta_d\right)$$

- Applications
  - Monochromator
  - Spectrometer
- Aliasing Issues
  - N to $N+1$

$$\lambda_{N+1} = \lambda_N \times \frac{N}{N+1}$$

  - Maximum Width: Factor of 2

- Anti–Aliasing Filter
  - *e.g.* Colored Glass
  - *e.g.* "Filter Wheel"
- Monochromator (More Later)



Fourier Analysis of Grating

- Convolution in Grating is Multiplication in Diffraction Pattern
- Multiplication in Grating is Convolution in Diffraction Pattern

Grating	Diffraction Pattern
$g(x,0) = \int f(x-x',0)\,h(x',0)\,dx'$	$g(x_1,z_1) = f(x_1,z_1)\,h(x_1,z_1)$,
$g(x,0) = f(x,0)\,h(x,0)$	$g(x_1,z_1) = \int f\left(x_1-x_1',z_1\right)h\left(x_1',z_1\right)dx_1'$,

- Approach
  - Convolve Slit with Comb to Produce Grating
  - Multiply Grating by Illumination or Apodization Pattern

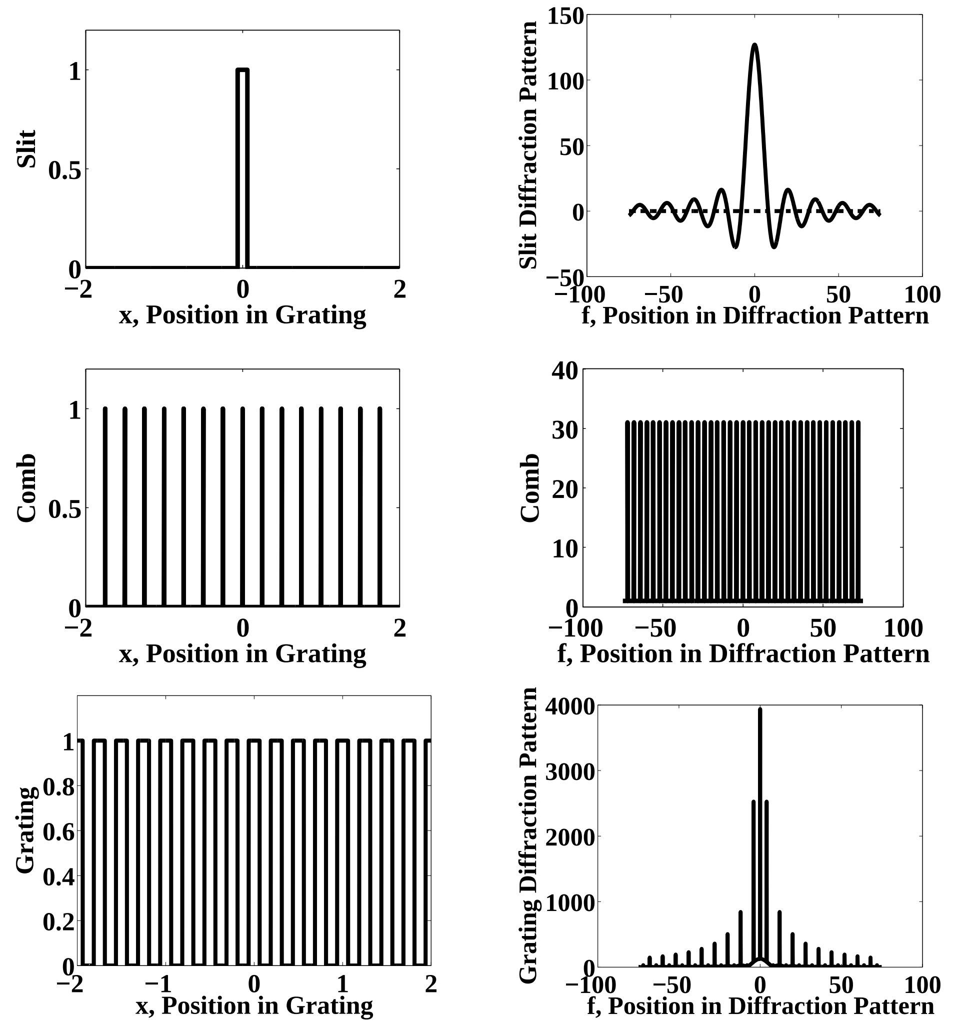
Fourier Analsysis Equations (1)

Grating Pattern **Diffraction Pattern**

Slit Pattern

Repetition Pattern

Convolve above two to obtain grating pattern.



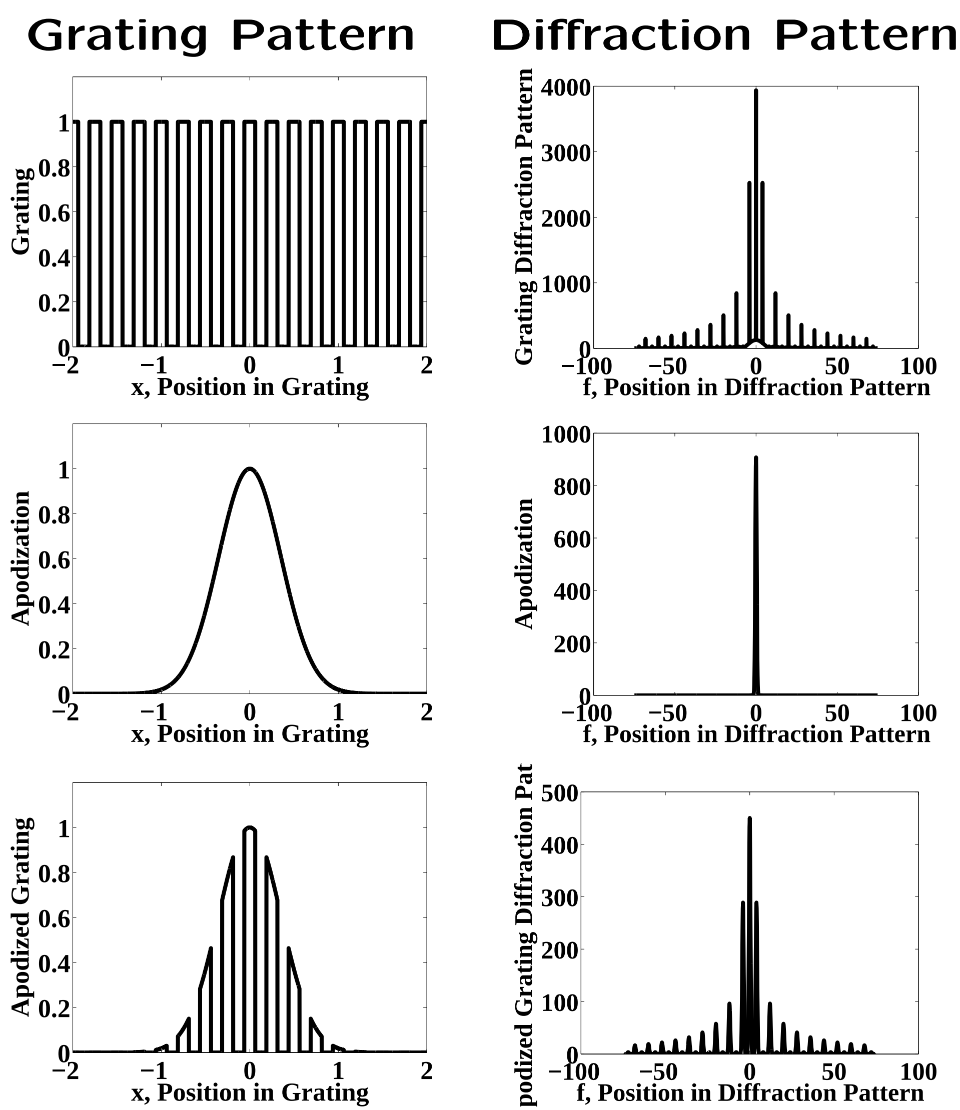
Multiply above two to obtain diffraction pattern.

Zeros Depend on Slit Width and Repetition Interval

Fourier Analsysis Equations (2)

Apodization or illumination

Multiply above two to obtain grating output.



Convolve above two to obtain diffraction pattern.

Wider Apodization Provides Better Resolution

Example of Grating Analysis Equations

Item	**Grating Pattern**	**Size**	**Diffraction Pattern**	**Angle**
Basic Shape	Uniform	D	sinc	$\lambda/D*$
Repetition	Comb	d	Comb	λ/d
Apodization	Gaussian	d_0	Gaussian	$\frac{4}{\pi}\frac{\lambda}{d_0}$

* first nulls

- Finesse: (Order Spacing)/(Width of a Single Order)

$$\frac{1}{F} = \frac{\frac{4}{\pi}\frac{\lambda}{d_0}}{\frac{\lambda}{d}} = \frac{4}{\pi}\frac{d}{d_0}$$

- Rule of Thumb
  - Roughly $F =$ Number of Grooves Illuminated

Grating Spectrometeter Example (1)

- Laser Source with Two Wavelengths

$$\lambda_1 = \lambda - \delta\lambda/2 \qquad \lambda_2 = \lambda + \delta\lambda/2$$

- Beam Diameters Equal to (Or Expanded to) d_0

- Spacing in First–Order Spectrum ($N = 1$)

$$\delta\theta = \theta_{d2} - \theta_{d1} \approx \sin\theta_{d2} - \sin\theta_{d1} = \frac{\lambda_2 - \lambda_1}{d} = \frac{\delta\lambda}{d}$$

- Width

$$\alpha = \frac{4}{\pi}\frac{\lambda}{d_0}$$

Grating Spectrometeter Example (2)

- Sum of Irradiances

$$I(\theta) = e^{-2(\theta-\theta_{d1})^2/\alpha^2} + e^{-2(\theta-\theta_{d2})^2/\alpha^2}$$

- Resolution Criterion
  - Not Rayleigh (No Nulls for Gaussian Illumination)
  - Try Sparrow: Check First Derivative

$$\frac{dI(\theta)}{d\theta} = 0 \qquad \text{1 or 3 Solutions including} \qquad \theta = \frac{\theta_{d2} + \theta_{d1}}{2}$$

  - Check Second Derivative (Positive for Valley)

$$\left.\frac{d^2 I}{d\theta^2}\right|_{\theta=(\theta_{d1}+\theta_{d2})/2} = \frac{8e^{-2\left(\frac{\delta\theta}{\alpha}\right)^2}\left[\alpha^2 - (\delta\theta)^2\right]}{\alpha^2}$$

Sparrow Criterion for Grating

- Second Derivative

$$\left.\frac{d^2I}{d\theta^2}\right|_{\theta=(\theta_{d1}+\theta_{d2})/2} = \frac{8e^{-2\left(\frac{\delta\theta}{\alpha}\right)^2}\left[\alpha^2-(\delta\theta)^2\right]}{\alpha^2}$$

Negative (Peak) for $\alpha < \delta\theta$ Positive (Valley) for $\alpha > \delta\theta$

- Boundary

$$\frac{4}{\pi}\frac{\lambda}{d_0} = \frac{\delta\lambda_0}{d}$$

Subscript on $\delta\lambda_0$ for Onset of Valley

$$\frac{\delta\lambda_0}{\lambda} = \frac{4}{\pi}\frac{d}{d_0} = \frac{1}{F} \qquad e.g. \quad \frac{4}{\pi}\frac{10^{-6}\text{m}}{10^{-3}\text{m}} \approx \frac{1}{1300}$$

For $\lambda = 830\text{nm}$, $\delta\lambda_0 = 830\text{nm}/1300 \approx 0.61\text{nm}$

Blaze Angle

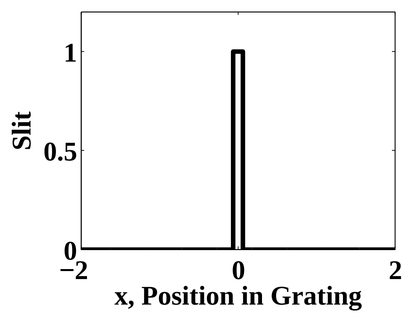
- Linear Phase Shift in Slit
  - Prism for Transmission
  - Tilt for Reflection
- Translation of Diffraction Pattern

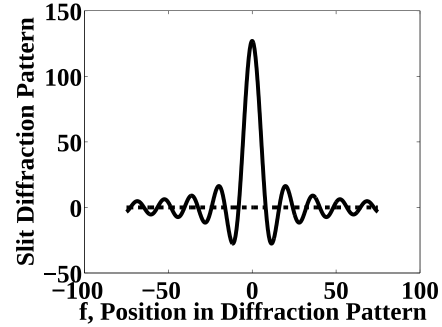
- Littrow Grating: $\theta_i = \theta_d$

$$\lambda = 2d \sin\theta_i \qquad (N = 1)$$

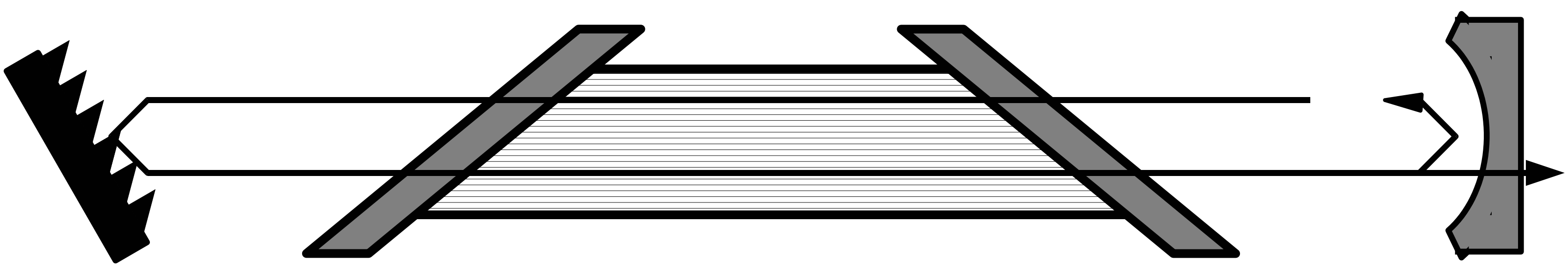
- Backward Reflection
- *e.g.* Tunable Laser

Phase Ramp Here



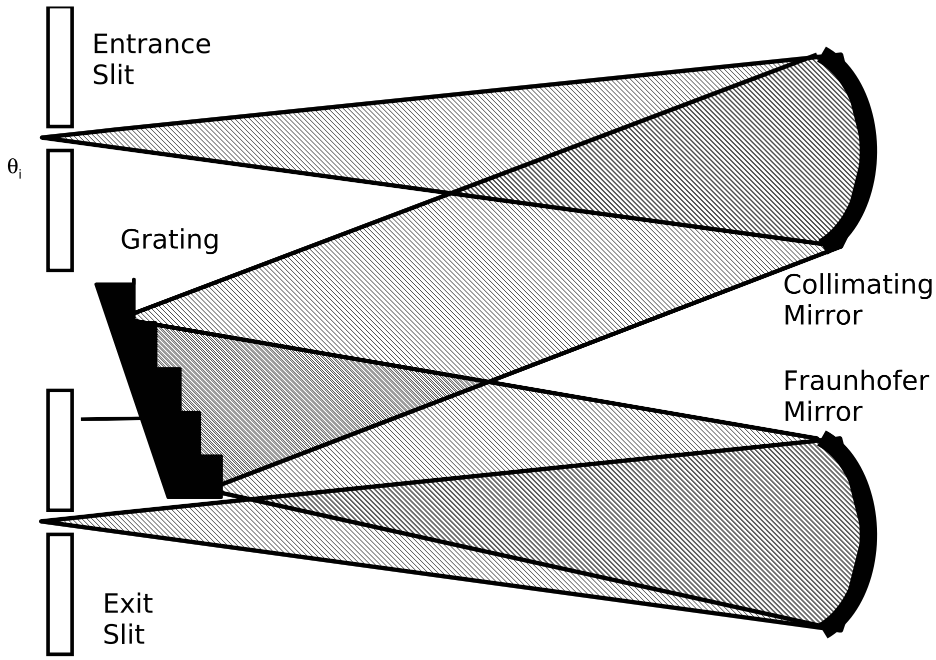


Translation Here Align Peak with First Order for Chosen λ

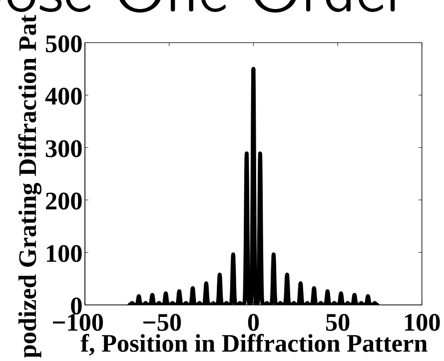


Q: Add a phase ramp to the Matlab code

Monochromator



- Entrance Slit Limits θ_i
- Exit Slit Selects Limits θ_d
- Choose One Order



- Symmetry Minimizes Aberrations
- Blazed Grating Enhances Signal
- Colored−Glass Filter Eliminates Aliasing

Monochromator Resolution

- Entrance Slit $d_i = 2w_i$
- Input Angles: $-w_i/(2f) \le \sin\theta_i \le w_i/(2f)$
- Transmission Through Entrance Slit

$$T_i(\delta\lambda) = 1 \qquad \left|\delta\lambda - \frac{d}{N}\delta\sin\theta_i\right| < w_i/2,$$

- Transmission Through Exit Slit

$$T_d(\delta\lambda) = 1 \qquad \left|\delta\lambda - \frac{d}{N}\delta\sin\theta_d\right| < w_d/2$$

- Overall Transmission Spectrum

$$T_i(\delta\lambda) \otimes I(\delta\lambda) \otimes T_d(\delta\lambda)$$

- Issues of Source Spatial Coherence (Ch. 10 and 11) in in Apodization for $I(\delta\lambda)$

Fresnel Diffraction

- Focused Light in Geometric Optics

$$\frac{|z_1 - f|}{f} D \qquad I = \frac{P}{\pi (D/2)^2} \frac{f^2}{(z_1 - f)^2}$$

- Fails at Edges of Shadow and at Focus

- Focal Region Done in Fraunhofer Diffraction

- Fresnel Diffraction Used at Edges

- Assume Integrals Separable in x and y

$$E(x_1, y_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} \int E_x(x, 0)\, e^{jk\frac{(x-x_1)^2}{2z_1}} dx \int E_y(y, 0)\, e^{jk\frac{(y-y_1)^2}{2z_1}} dy$$

Uniformly Illuminated Rectangular Aperture

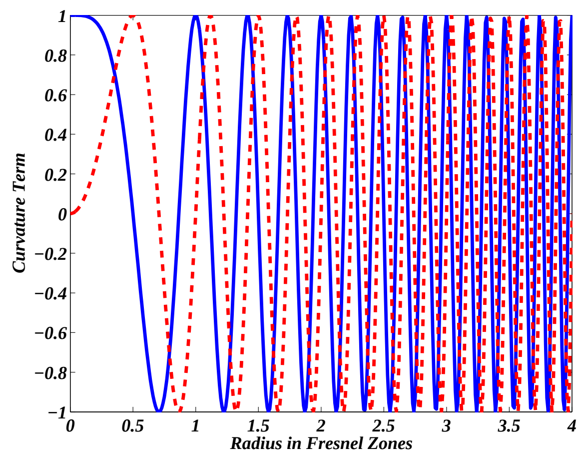
- Aperture Size $2a$ by $2b$

$$E\left(x_1,y_1,z_1\right)=\frac{jke^{jkz_1}}{2\pi z_1}\sqrt{\frac{P}{4ab}}I\left(x_1,a,z_1\right)I\left(y_1,b,z_1\right)$$

$$I\left(x_1,a,z_1\right)=\int_{-a}^{a}e^{jk\frac{(x-x_1)^2}{2z_1}}dx_1$$

$$I\left(x,a,z_1\right)=\int_{-a}^{a}\cos\left(k\frac{(x-x_1)^2}{2z_1}\right)dx+j\int_{-a}^{a}\sin\left(k\frac{(x-x_1)^2}{2z_1}\right)dx$$

- Fresnel Cosine and Sine Integral
- Cosine Interand Contributes Near $x=x_1$
- Sine is Zero at $x=x_1$
- Away from Edges, $I(x,y,z_1)=I(x,y,0)$



Fresnel Sine and Cosine Integrals (1)

- Dimensionless Coordinates

$$u = (x - x_1)\sqrt{\frac{k}{2z_1}} \qquad du = dx_1\sqrt{\frac{k}{2z_1}}$$

- Limits

$$u_1 = (x + a)\sqrt{\frac{k}{2z_1}} \qquad u_2 = (x - a)\sqrt{\frac{k}{2z_1}}$$

- Integrals

$$I\left(x_1, a, z_1\right) = \sqrt{\frac{2z_1}{k}}\int_{u_1}^{u_2} \cos^2 u \; du + j\sqrt{\frac{2z_1}{k}}\int_{u_1}^{u_2} \sin^2 u \; du$$

$$I\left(x_1, a, z_1\right) = \sqrt{\frac{2z_1}{k}}\left[C\left(u_2\right) - C\left(u_1\right) + S\left(u_2\right) - S\left(u_1\right)\right]$$

Fresnel Sine and Cosine Integrals (2)

$$I\left(x_1,a,z_1\right)=\sqrt{\frac{2z_1}{k}}\left[C\left(u_2\right)-C\left(u_1\right)+S\left(u_2\right)-S\left(u_1\right)\right]$$

$$C\left(u_1\right)=\int_0^{u_1}\cos^2 u\ du \qquad S\left(u_1\right)=\int_0^{u_1}\sin^2 u\ du$$

$$F\left(u\right)=C\left(u\right)+jS\left(u\right)$$

- Above 3 Are Odd Functions

$$C\left(u\right)=-C\left(-u\right) \qquad S\left(u\right)=-S\left(-u\right) \qquad F\left(u\right)=-F\left(-u\right)$$

- Approximations

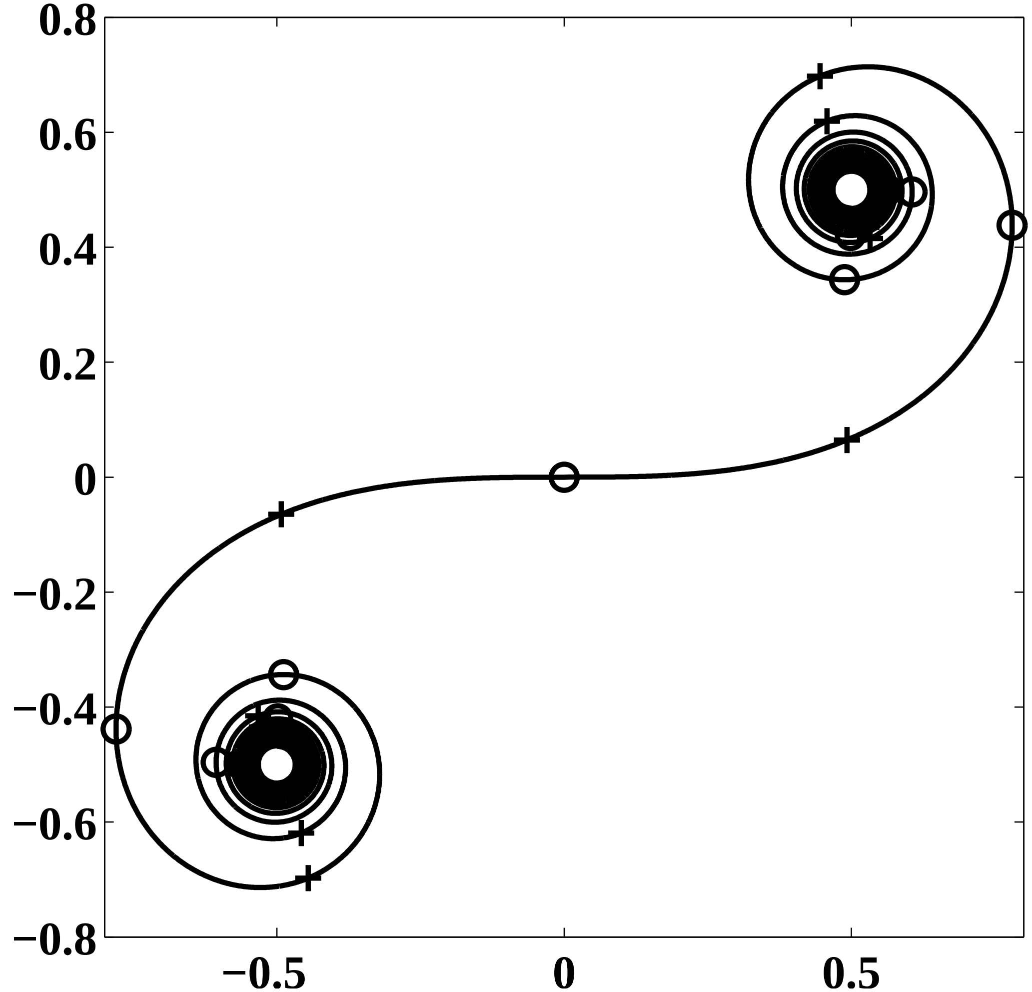
$$\begin{array}{lcl} F\left(u\right)\approx & ue^{j\pi u^2/2} & u\approx 0 \\ F\left(u\right)\approx & \frac{1+j}{2}-\frac{j}{\pi u}e^{j\pi u^2/2} & u\rightarrow\infty \end{array} \quad (1)$$

Fresnel Diffraction Results

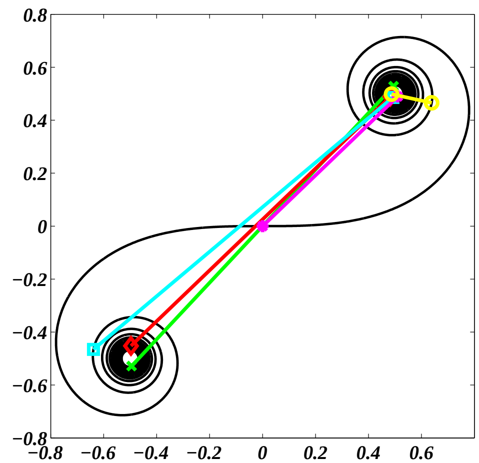
$$E(x_1,y_1,z_1) = \frac{jke^{jkz_1}}{2\pi z_1}\frac{P}{4ab}\frac{2z_1}{k}\times$$
$$\left\{F\left[(x_1-a)\sqrt{\frac{k}{2z_1}}\right]-F\left[(x_1+a)\sqrt{\frac{k}{2z_1}}\right]\right\}\times$$
$$\left\{F\left[(y_1-b)\sqrt{\frac{k}{2z_1}}\right]-F\left[(y_1+b)\sqrt{\frac{k}{2z_1}}\right]\right\}$$

- Large x_1 or y_1: $E \rightarrow 0$: Q: How Close? Use Approximations on Previous Page.
- Near the Center: $E(x_1,y_1,z_1) = E(x_1,y_1,0)$ (Using the Odd Property of F)

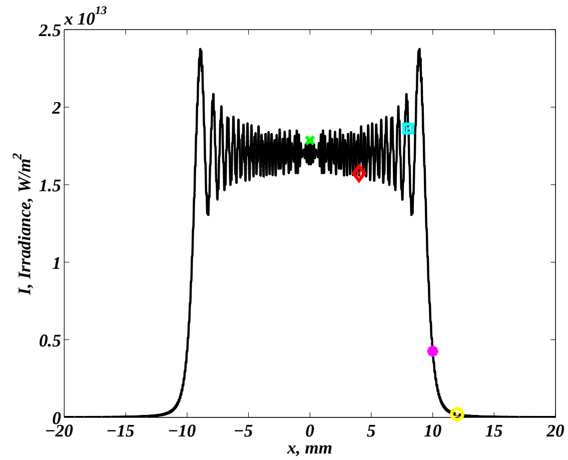
Cornu Spiral



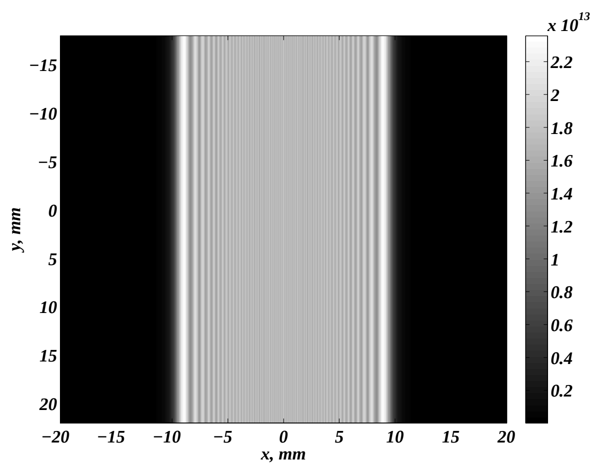
Example of Fresnel Diffraction: Near Field Diffraction



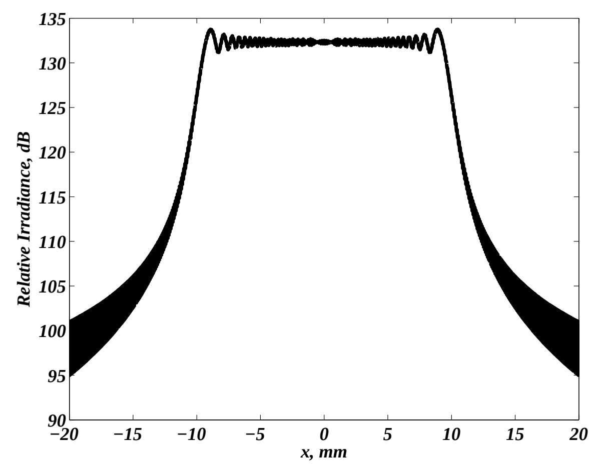
A. Cornu Spiral



B. Irradiance

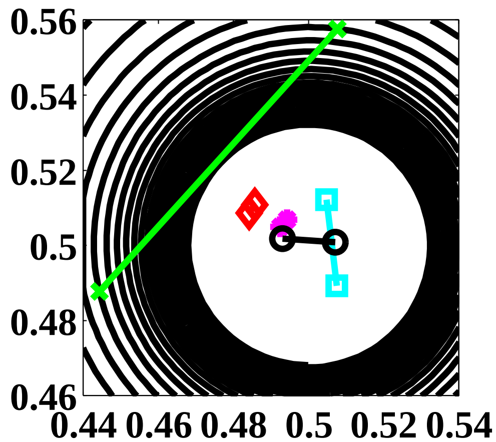


C. Image

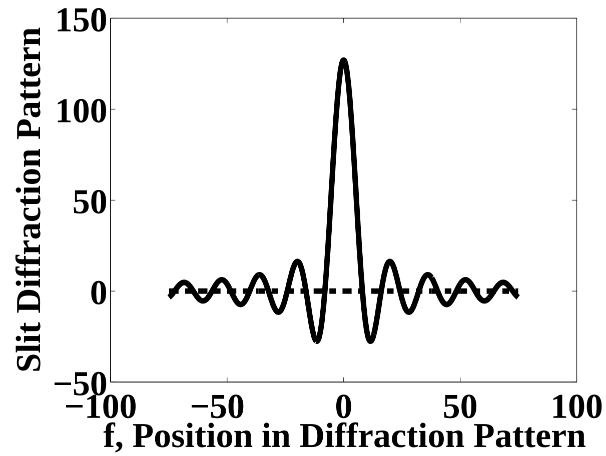


D. Irradiance, dB

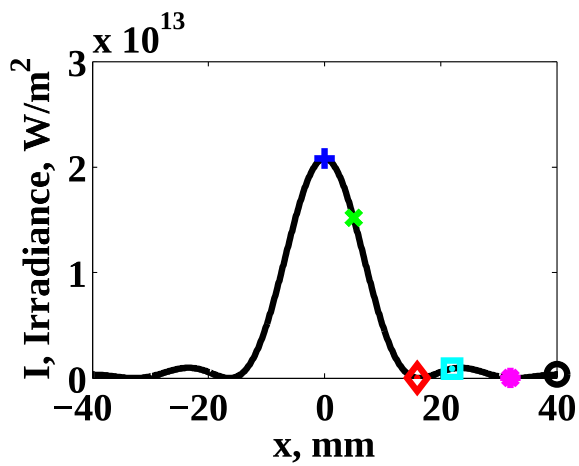
Fresnel Diffraction Includes Fraunhofer Diffraction



A. Cornu Spiral

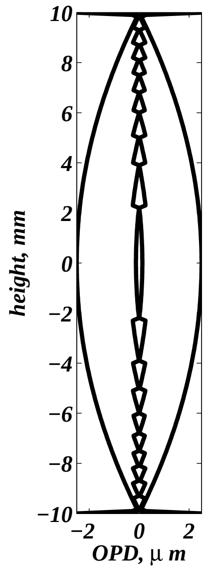


B. Field

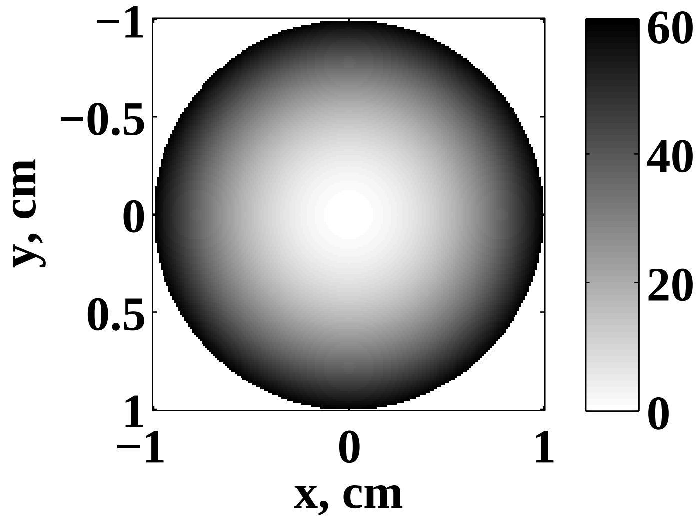


C. Irradiance

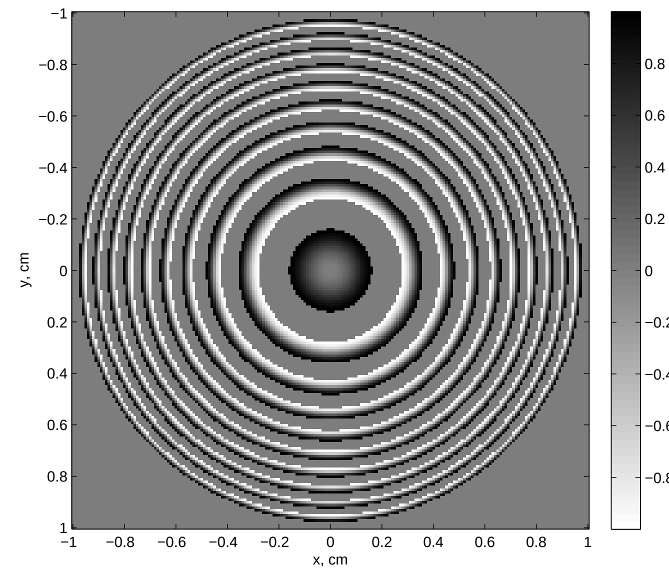
Fresnel Lens



Lens

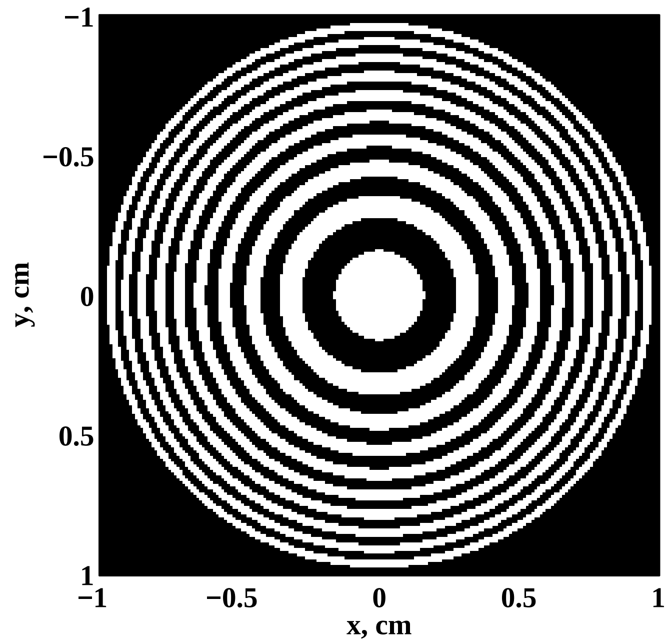


Unwrapped Phase
for Regular Lens

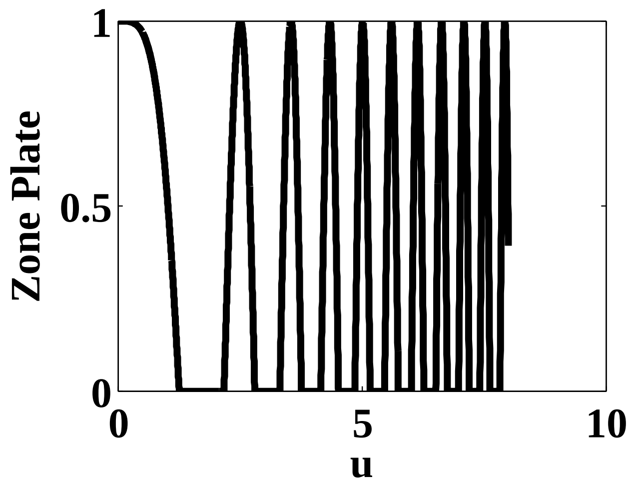


Phase for
Fresnel Lens

Fresnel Zone Plate

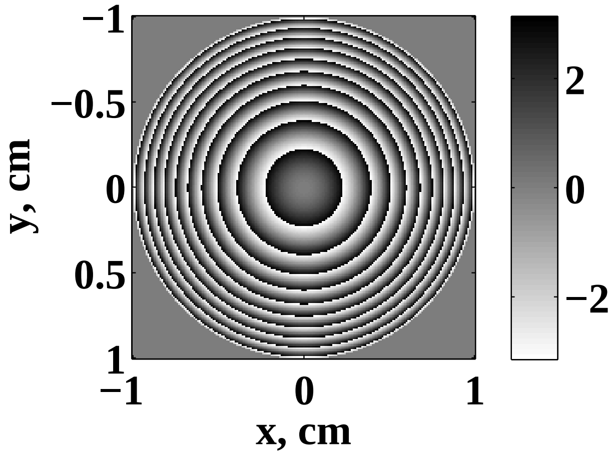


Fresnel zone plate with Diameter 2cm at distance of 10m. The zone plate acts as a lens.

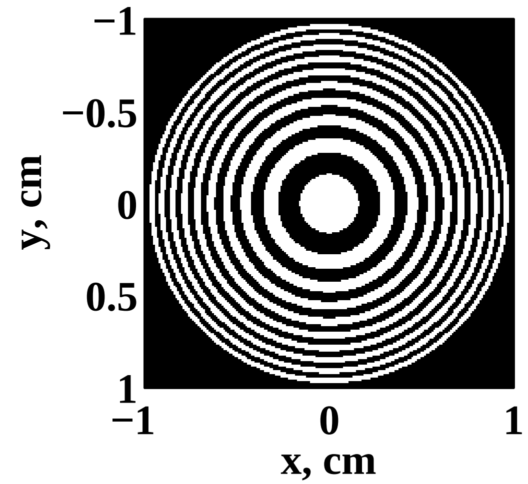


Integrand of Cosine Integral passing through zone plate. All negative contributions are blocked.

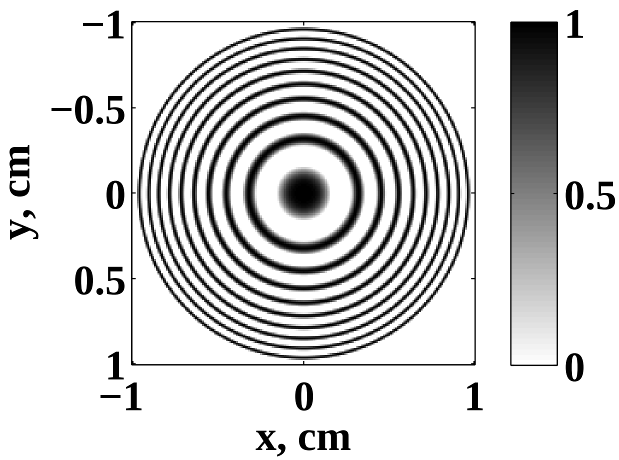
Phase in Fresnel Zone Plate



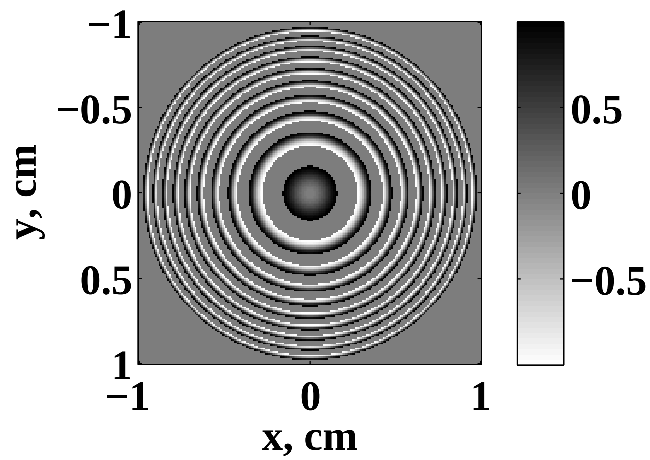
Raw Phase in Radians



Zone Plate Transmission

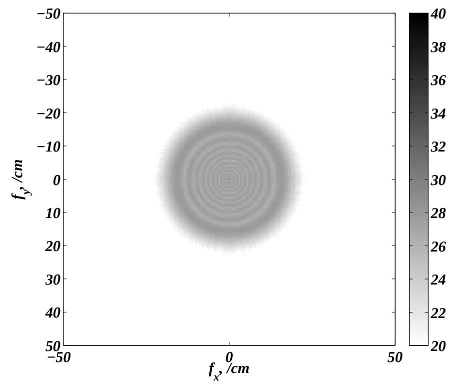


Real Transmitted Field

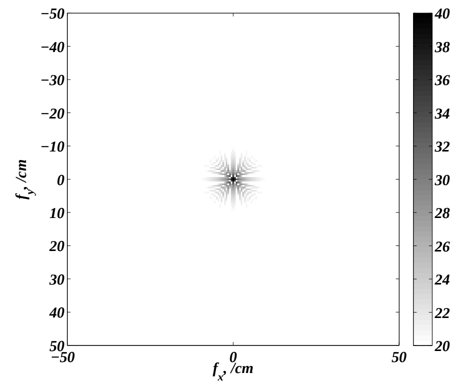


Imag. Transmitted Field

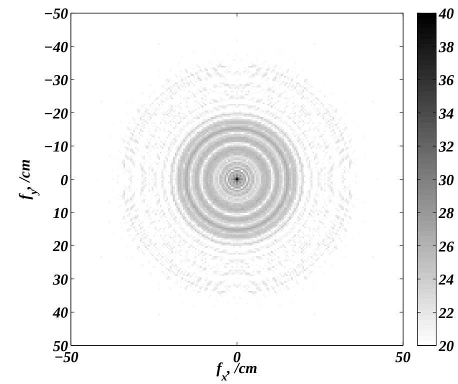
Focusing With Fresnel Zone Plate



Unfocused Pattern



Focused With Lens



Focused with Zone Plate