

Chapter 8

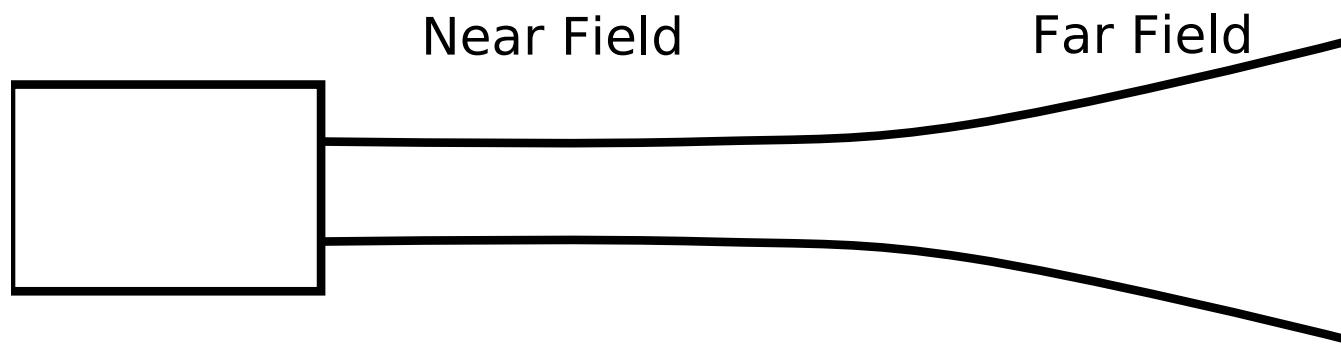
Optics for Engineers

Charles A. DiMarzio
Northeastern University

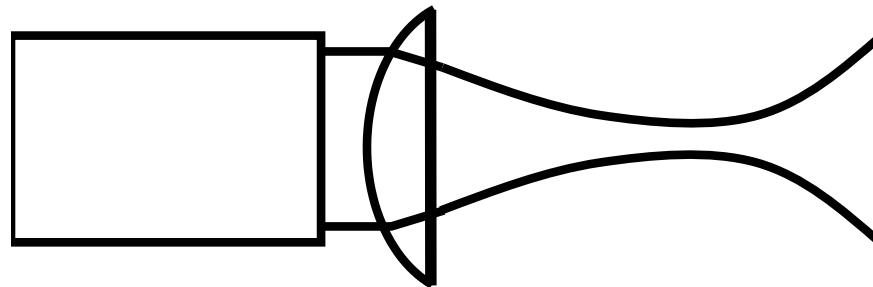
Sept. 2012

Diffraction

Collimated Beam: Divergence in Far Field

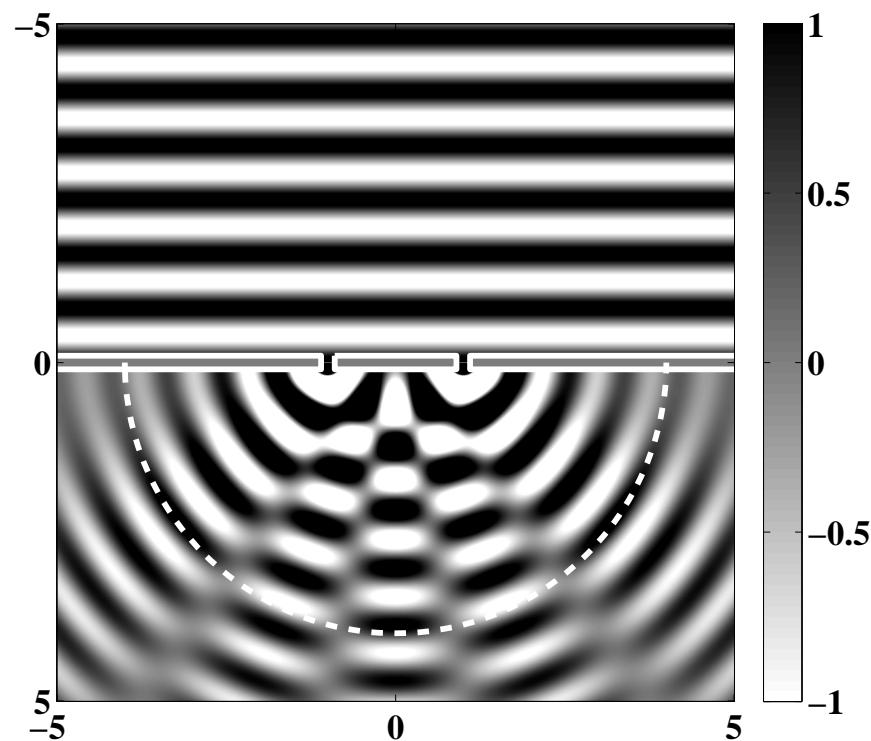


Focused Beam: Minimum Spot Size and Location

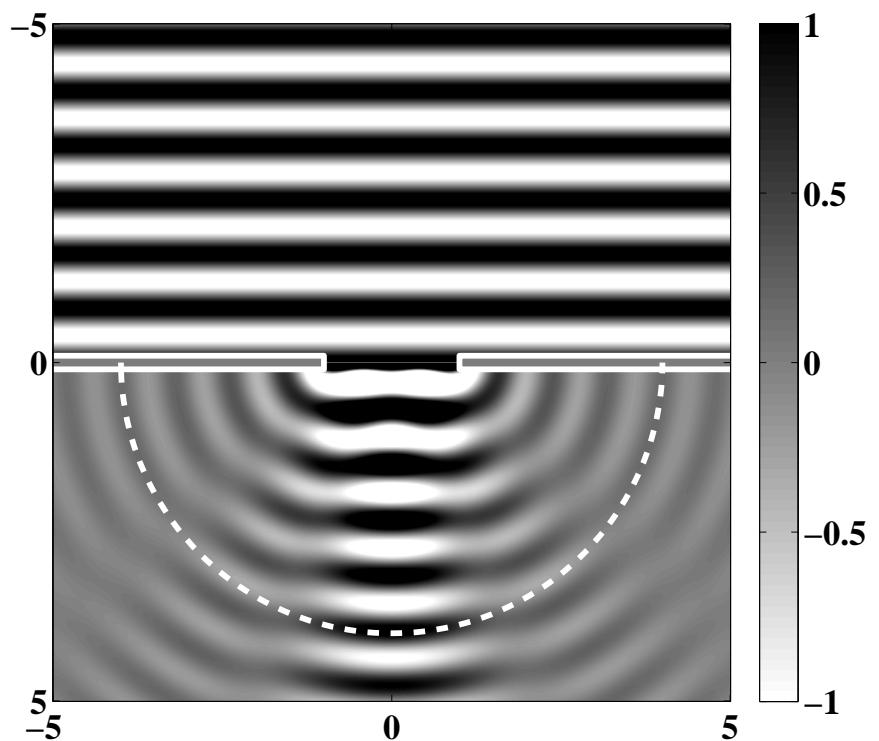


$$\alpha = C \frac{\lambda}{D}$$

Slit Experiments



A. Two slits



B. Aperture

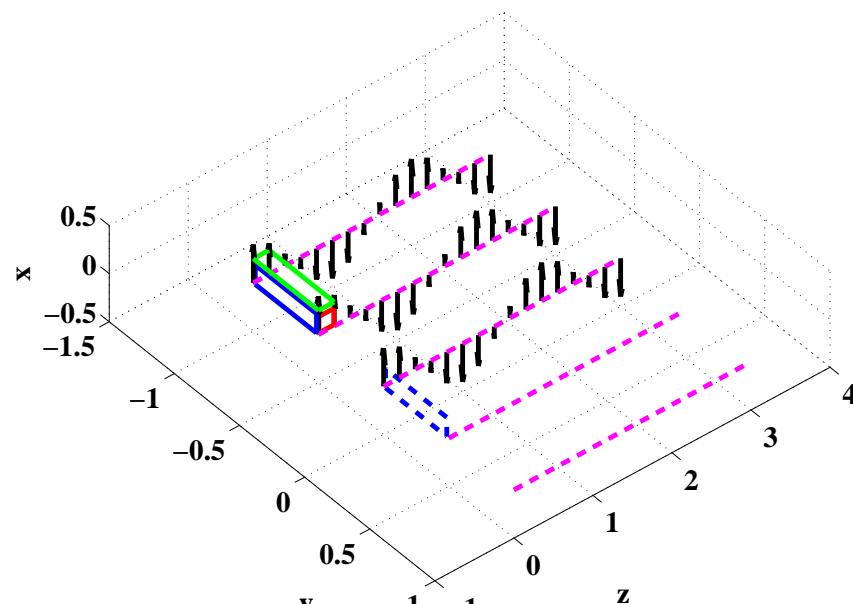
- Angular Divergence of Light Waves
- Alternating Bright and Dark Regions
- Near– and Far–Field Behavior

$\lambda = 800\text{nm}$. Axis Units are μm

Diffraction and Maxwell's Equations

- Faraday's Equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$



$$\mathbf{E} = E \hat{x} e^{jkz}$$

- Red Contour

$$\int \mathbf{E} \cdot d\mathbf{s} \neq 0 \quad H_y \neq 0$$

– Propagation along \hat{z}

- Green Contour, $E_y = E_z = 0$

$$\int \mathbf{E} \cdot d\mathbf{s} = 0 \quad H_z = 0$$

- Blue, E_x Constant, $E_y = 0$

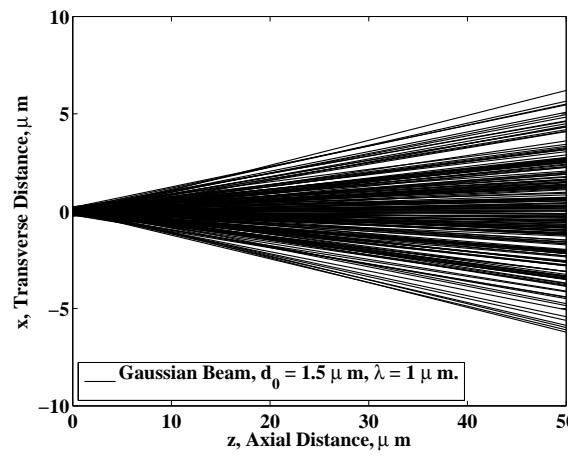
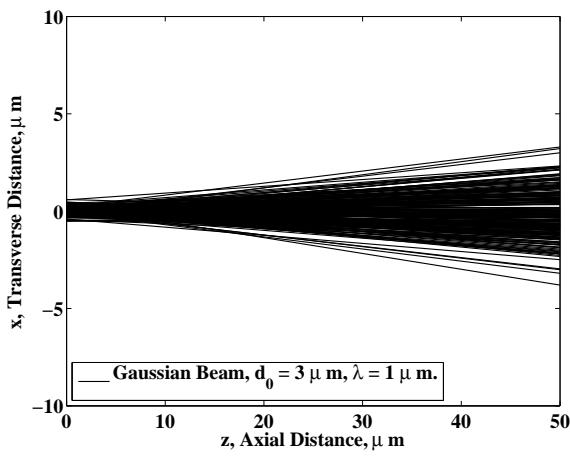
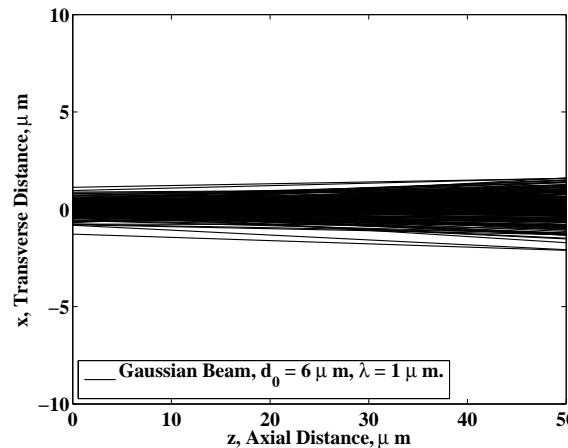
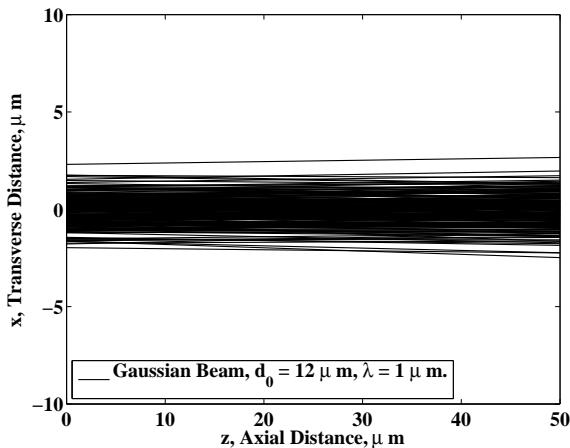
$$\int \mathbf{E} \cdot d\mathbf{s} = 0 \quad H_z = 0$$

- Blue-Dash Contour (Edge)

$$\int \mathbf{E} \cdot d\mathbf{s} \neq 0 \quad H_z \neq 0$$

– Propagation Along \hat{y} too

Diffraction and Quantum Mechanics



$$p = h\mathbf{k}/(2\pi)$$

$$k_x = |k| \sin \alpha$$

$$\delta x \ \delta p_x \geq h$$

$$\delta x \ \delta k_x \geq 2\pi$$

$$\delta x \ k \sin \delta \alpha \geq 2\pi$$

$$\sin \delta \alpha \geq \frac{2\pi}{\delta x \ |k|}$$

$$\sin \delta \alpha \geq \frac{\lambda}{\delta x}$$

A Peek at Gaussian Beams

- The Correct Quantum Approach

- Position Probability Density Function, $P(x)$
 - Momentum Probability Density Function, $P(k) = FT(P(x))$
 - Uncertainty Measurement

$$(\delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 \quad (\delta p_x)^2 = \langle p_x^2 \rangle - \langle p_x \rangle^2,$$

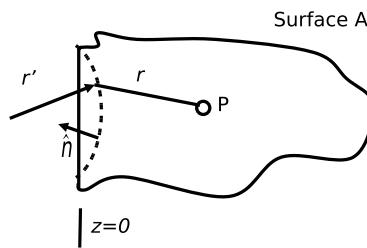
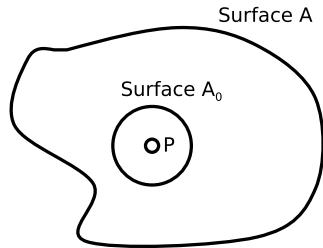
- Gaussian Beam (Minimum Uncertainty Wave)

$$\sin \delta\alpha = \frac{\lambda}{\delta x}$$

- Otherwise

$$\sin \delta\alpha > \frac{\lambda}{\delta x}$$

Fresnel–Kirchoff Integral (1)



- Auxiliary Field

$$V(x, y, z, t) = \frac{e^{j(\omega t + kr)}}{r}$$

- Electric Field

$$E(x, y, z, t) = E_s(x, y, z) e^{j\omega t}$$

- Green's Theorem

$$\iint_{A+A_0} (V\nabla E - E\nabla V) \cdot \hat{n} d^2A =$$

$$\iint_{A+A_0} \nabla \cdot (V\nabla E) - \nabla \cdot (E\nabla V) d^3V \quad RHS = 0 = LHS$$

- Expand RHS

$$\nabla \cdot (V\nabla E) - \nabla \cdot (E\nabla V) =$$

$$(\nabla V)(\nabla E) + V\nabla^2 E -$$

$$(\nabla E)(\nabla V) - E\nabla^2 V = 0$$

- E & V Satisfy Helmholtz

$$\nabla^2 E = \frac{\omega^2}{c^2} E$$

$$\nabla^2 V = \frac{\omega^2}{c^2} V$$

Fresnel–Kirchoff Integral (2)

- From Prev. Page

$$\int \int_{A+A_0} (V \nabla E - E \nabla V) \cdot \hat{n} d^2 A = 0$$

- Split Surface

$$\int \int_A (V \nabla E - E \nabla V) \cdot \hat{n} d^2 A +$$

$$\int \int_{A_0} (V \nabla E - E \nabla V) \cdot \hat{n} d^2 A_0 = 0$$

- Radius of $A_0 \rightarrow 0$

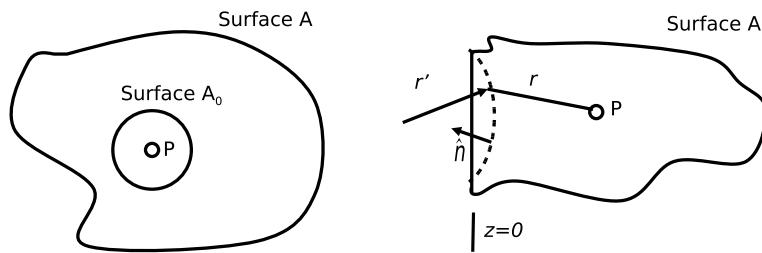
$$\int \int_{A_0} (V \nabla E - E \nabla V) \cdot \hat{n} d^2 A = 4\pi E(x, y, z)$$

- Helmholtz–Kirchoff Integral Theorem

$$E(x, y, z) = -\frac{1}{4\pi} \int \int_A \left[E \left(\nabla \frac{e^{jkr}}{r} \right) \cdot \hat{n} - \frac{e^{jkr}}{r} (\nabla E) \cdot \hat{n} \right] d^2 A$$

Fresnel–Kirchoff Integral (3)

$$E(x, y, z) = -\frac{1}{4\pi} \int \int_A \left[E \left(\nabla \frac{e^{jkr}}{r} \right) \cdot \hat{n} - \frac{e^{jkr}}{r} (\nabla E) \cdot \hat{n} \right] d^2 A$$



- Irregular Surface of $A \rightarrow \infty$

$$E = E_0 \frac{e^{-jkr'}}{r'}$$

- E_0 Slowly Varying

$$(\nabla E) \cdot \hat{n} = \left(jkE - \frac{E}{r'} \right) \hat{n} \cdot \hat{r}'$$

- $r \gg \lambda, r' \gg \lambda$

$$\nabla E \rightarrow 0 \quad 1/r^2 \rightarrow 0 \quad 1/(r')^2 \rightarrow 0$$

$$E(x_1, y_1, z_1) = \frac{jk}{4\pi} \int \int_{Aperture} E(x, y, z) \frac{e^{jkr}}{r} (\hat{n} \cdot \hat{r} - \hat{n} \cdot \hat{r}') dA$$

Obliquity Factor

- Fresnel–Kirchoff Integral Theorem (Previous Page)

$$E(x_1, y_1, z_1) = \frac{jk}{4\pi} \int \int_{Aperture} E(x, y, z) \frac{e^{jkr}}{r} (\hat{n} \cdot \hat{r} - \hat{n} \cdot \hat{r}') dA$$

- Numerical Calculations Possible Here, or Simplify More
- Obliquity Factor (Removes Backward Wave)
- For Small Transverse Distances . . .

$$(\hat{n} \cdot \hat{r} - \hat{n} \cdot \hat{r}') = 2$$

Paraxial Approximation

$$E(x_1, y_1, z_1) = \frac{jk}{2\pi} \int \int_{Aperture} E(x, y, z) \frac{e^{jkr}}{r} dA$$

$$r = \sqrt{(x - x_1)^2 + (y - y_1)^2 + z_1^2}$$

- Paraxial Approximation for
- Taylor's Series (First Order)

- r in Denominator

$$r \approx z_1 \left[1 + \frac{(x - x_1)^2}{2z_1^2} + \frac{(y - y_1)^2}{2z_1^2} \right]$$

- r in Exponent

$$r = z_1 \sqrt{\left(\frac{x - x_1}{z_1}\right)^2 + \left(\frac{y - y_1}{z_1}\right)^2 + 1}$$

$$(x - x_1) \ll z_1 \quad (y - y_1) \ll z_1$$

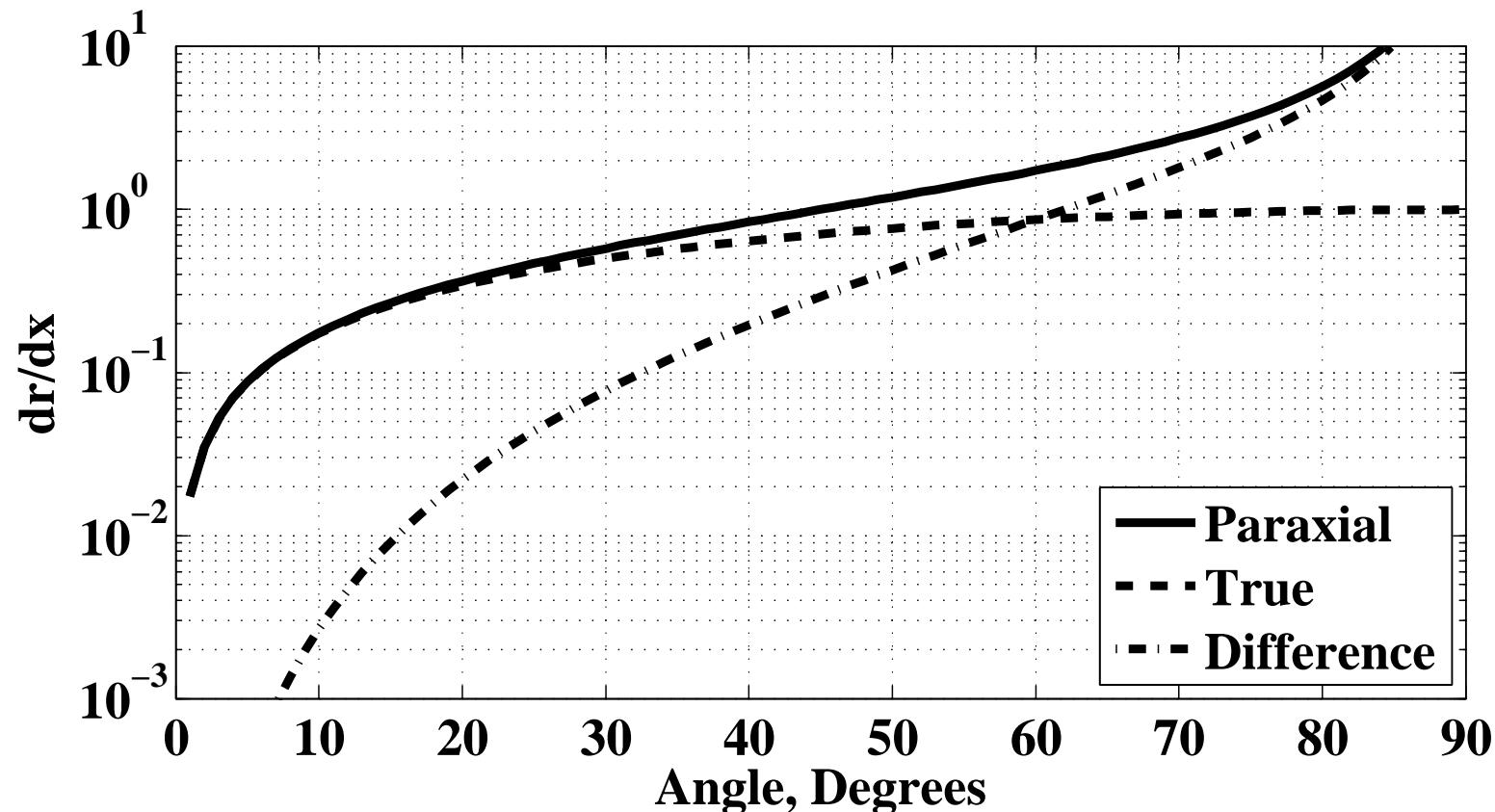
$$r \approx z_1 + \frac{(x - x_1)^2}{2z_1} + \frac{(y - y_1)^2}{2z_1}$$

- Easier Closed-Form Computation
- Amenable to Further Approximations

Error in Paraxial Approximation

$$r = \sqrt{(x - x_1)^2 + (y - y_1)^2 + z_1^2}$$

$$r \approx z_1 + \frac{(x - x_1)^2}{2z_1} + \frac{(y - y_1)^2}{2z_1}$$



Pupil Coordinates

- Pupil Position Coordinates (Length Units)

$$x_1, y_1$$

- Angular Coordinates (Dimensionless)

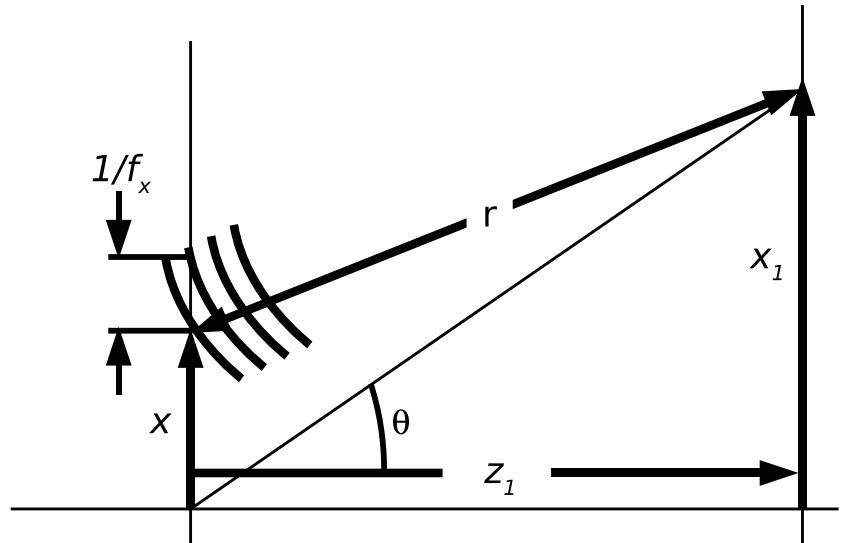
$$u = \sin \theta \cos \zeta$$

$$v = \sin \theta \sin \zeta$$

$$\max \sqrt{u^2 + v^2} = NA$$

- Spatial Frequency (Inverse Length)

$$2\pi f_x = \frac{d(kr)}{dx} = \frac{2\pi}{\lambda} \frac{dr}{dx}$$



- Spatial Frequency Approximation

$$f_x \approx \frac{x_1}{r\lambda} = \frac{\sin \theta \cos \zeta}{\lambda}$$

$$f_y \approx \frac{y_1}{r\lambda} = \frac{\sin \theta \sin \zeta}{\lambda}$$

- For Fourier Optics (Ch. 11)

Coordinate Relationships

	Spatial Frequency	Pupil Location	Direction Cosines
Spatial Frequency		$x_1 = \lambda z f_x$ $y_1 = \lambda z f_y$	$u = f_x \lambda$ $v = f_y \lambda$
Pupil Location	$f_x = \frac{x_1}{\lambda z}$ $f_y = \frac{y_1}{\lambda z}$		$u = \frac{x_1}{z}$ $v = \frac{y_1}{z}$
Direction cosines	$f_x = \frac{u}{\lambda}$ $f_y = \frac{v}{\lambda}$	$x_1 = u z$ $y_1 = v z$	
Angle			$u = \sin \theta \cos \zeta$ $v = \sin \theta \sin \zeta$

Two Approaches

- Fresnel Diffraction

$$E(x_1, y_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} \int \int E(x, y, 0) e^{jk\frac{(x-x_1)^2}{2z_1} + \frac{(y-y_1)^2}{2z_1}} dx dy$$

- Fraunhofer Diffraction (x, x_1 both small)

$$(x - x_1)^2 + (y - y_1)^2 = (x^2 + y^2) + (x_1^2 + y_1^2) - 2(xx_1 + yy_1)$$

$$E(x_1, y_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{(x_1^2 + y_1^2)}{2z_1}} \int \int E(x, y, 0) e^{jk\frac{(x^2 + y^2)}{2z_1}} e^{-jk\frac{(xx_1 + yy_1)}{z_1}} dx dy$$

Fresnel Radius and the Far Field

- Curvature Terms

$$e^{jk\frac{(x^2+y^2)}{2z_1}} \quad e^{jk\frac{(x_1^2+y_1^2)}{2z_1}}$$

- Fresnel Radius in Integrand (See Plot on Next Page)

$$e^{jk\frac{(x^2+y^2)}{2z_1}} = e^{j2\pi\left(\frac{r}{\sqrt{2\lambda z_1}}\right)^2} = e^{j2\pi\left(\frac{r}{r_f}\right)^2} \quad r_f = \sqrt{2\lambda z_1}$$

- Simplified Integral

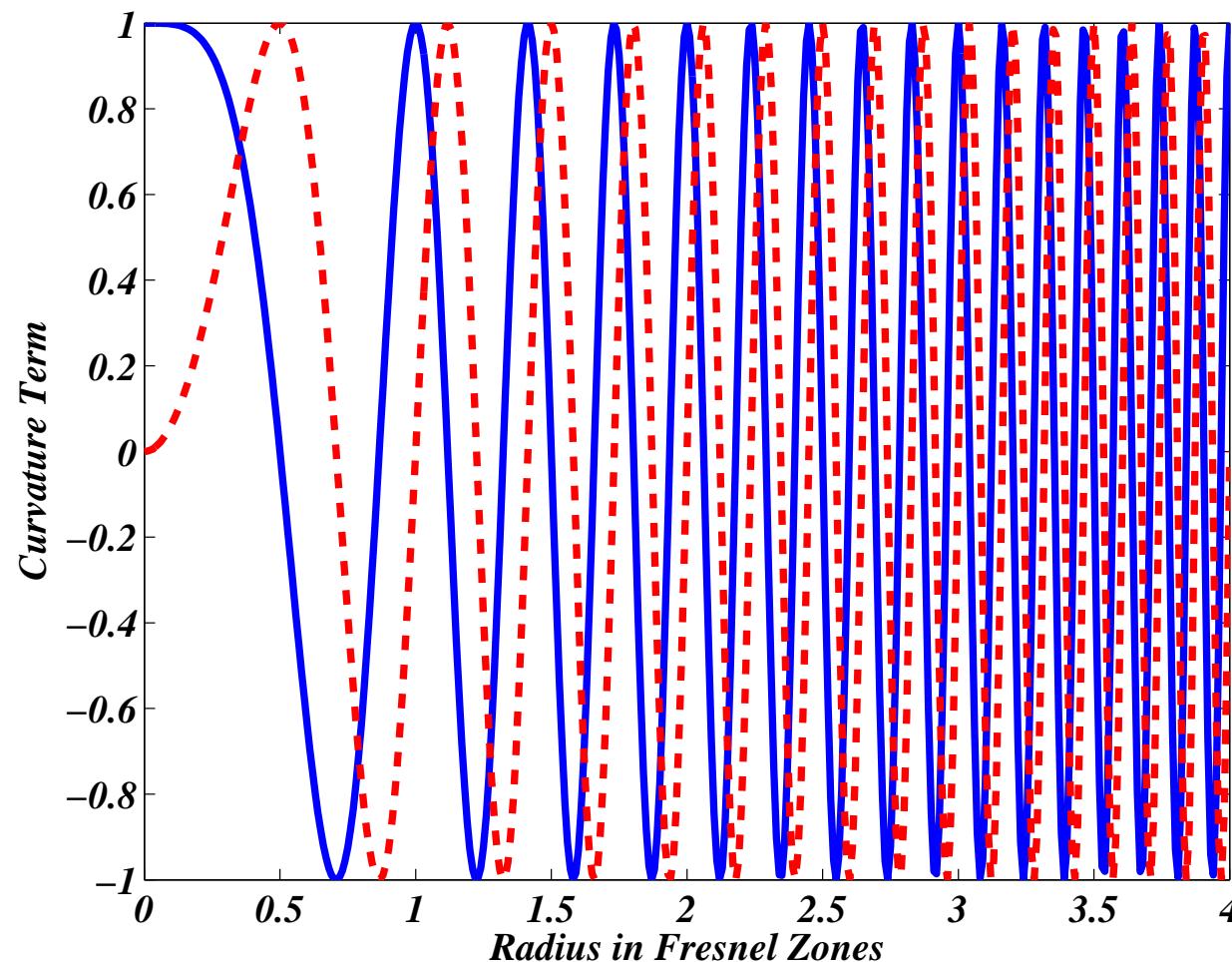
$$E(x_1, y_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{(x_1^2+y_1^2)}{2z_1}} \times \\ \int \int E(x, y, 0) e^{j2\pi\frac{x^2+y^2}{r_f^2}} e^{-jk\frac{(xx_1+yy_1)}{z_1}} dx dy$$

Integrand in Fresnel Zones

$$e^{j2\pi \left(\frac{r}{r_f}\right)^2}$$

Phase Increases With r^2

Real: Blue Solid Line, Imag: Red Broken Line



- Fresnel Number

$$N_f = \frac{r_{max}}{r_f}$$

– Near Field

$$N_f \approx 0$$

– Far Field

$$N_f \gg 0$$

– Almost–Far Field

$$N_f \ll 0$$

- Different Approximations

Far Field: Fraunhofer Zone

$$z_1 \gg D^2/\lambda \quad (\text{Far-Field Condition}) \quad r_f = \sqrt{2z_1\lambda} \gg \sqrt{2}D$$

$$N_f = r/r_f \ll 1 \quad (\text{Fraunhofer Zone}) \quad e^{j2\pi \frac{(x^2+y^2)}{r_f}} \approx 1$$

Fraunhofer Diffraction

Fresnel–Kirchoff Integral is a Fourier Transform

$$E(x_1, y_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{(x_1^2+y_1^2)}{2z_1}} \int \int E(x, y, 0) e^{-jk\frac{(xx_1+yy_1)}{z_1}} dx dy$$

Near Field: Fresnel Zone

$$z_1 \ll D^2/\lambda \quad (\text{Near-Field Condition})$$

$$N_f = r/r_f \gg 1$$

- Fresnel Diffraction

$$E(x_1, y_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} \int \int E(x, y, 0) e^{jk\frac{(x-x_1)^2}{2z_1} + \frac{(y-y_1)^2}{2z_1}} dx dy$$

- Integrand Averages to Zero Except near $x_1 = x, y_1 = y$

Almost–Far field

- Condition

N_f small but not that small

- Fraunhofer Diffraction

$$E(x_1, y_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{(x_1^2+y_1^2)}{2z_1}} \int \int E(x, y, 0) e^{jk\frac{(x^2+y^2)}{2z_1}} e^{-jk\frac{(xx_1+yy_1)}{z_1}} dx dy$$

- Looks Like Fourier Transform with Curvature
- Computation Feasible if N_f is Not Too Large
(Watch for Nyquist Criterion in Sampling $E(x, y, 0)$)

Fraunhofer Lens (1)

- Focusing

$$E(x, y, 0) = E_0(x, y, 0) e^{-jk\frac{(x^2+y^2)}{2f}}$$

- Focused in Near Field of Aperture ($f \ll D^2/\lambda$)

$$E(x_1, y_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{(x_1^2+y_1^2)}{2z_1}} \int \int E(x, y, 0) e^{jk\frac{(x^2+y^2)}{2Q}} e^{-jk\frac{(xx_1+yy_1)}{z_1}} dx_1 dy_1$$

- Modify Fresnel Radius Equation

$$r_f = \sqrt{2\lambda Q} \quad Q \text{ is the Defocus Parameter}$$

$$\frac{1}{r_f} = \frac{1}{\sqrt{2\lambda z_1}} - \frac{1}{\sqrt{2\lambda f}}$$

Fraunhofer Lens (2)

- Computational Considerations

$$E(x_1, y_1, z_1) = \frac{jke^{j kz_1}}{2\pi z_1} e^{jk \frac{(x_1^2 + y_1^2)}{2z_1}} \int \int E(x, y, 0) e^{jk \frac{(x^2 + y^2)}{2Q}} e^{-jk \frac{(xx_1 + yy_1)}{z_1}} dx_1 dy_1$$

- Quadratic Phase Term Leads to Nyquist Issues
 - For Large Apertures
 - For Strong Defocusing
 - Practical Limit

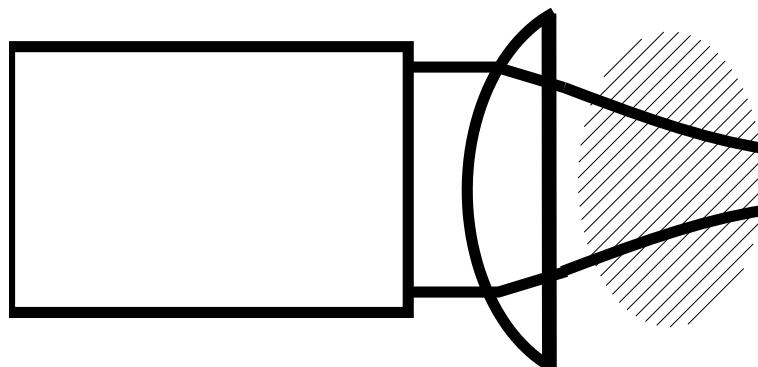
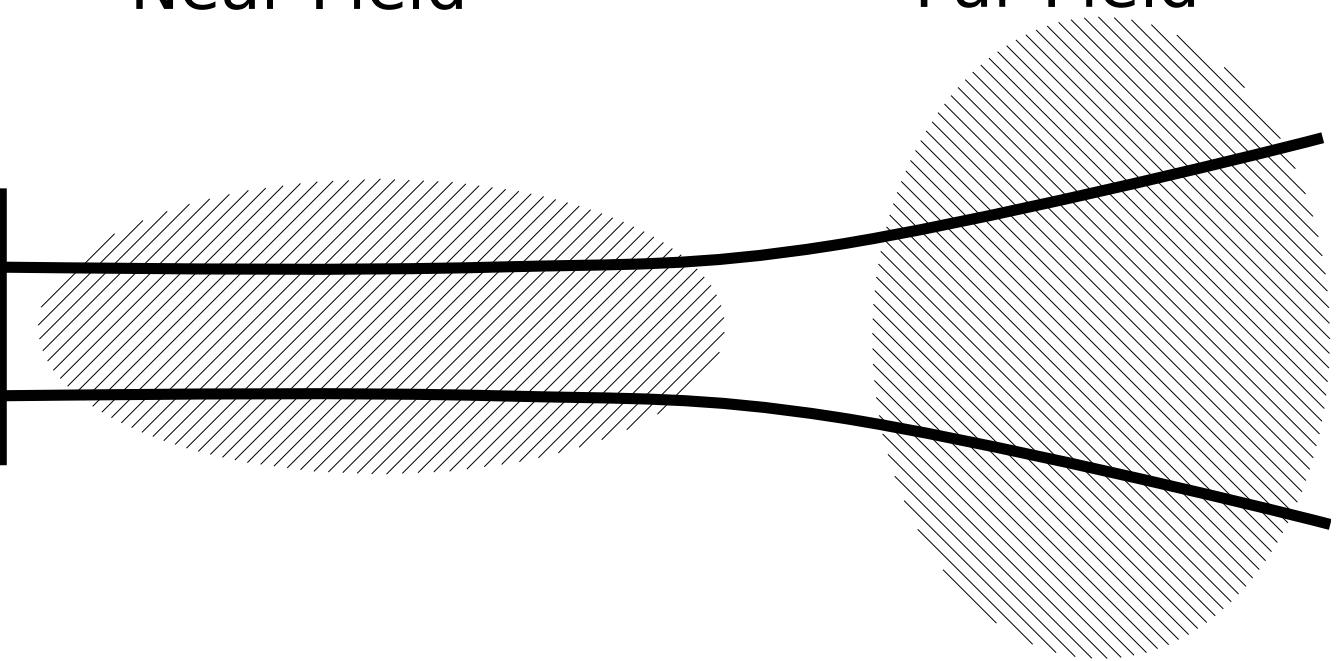
$$Q = \frac{fz_1}{|f - z_1|} \gg \frac{D^2}{8\lambda} \quad |f - z| \ll \frac{8z_1^2\lambda}{D^2}$$

Fraunhofer and Fresnel Zones

Near Field



Far Field



Fresnel

Fraunhofer

Spatial Frequency and Angle Definitions

- Spatial Frequency

$$f_x = \frac{k}{2\pi z_1} x_1 = \frac{x_1}{\lambda z_1} \quad f_y = \frac{k}{2\pi z_1} y_1 = \frac{y_1}{\lambda z_1}$$

$$df_x = \frac{k}{2\pi} \frac{dx_1}{z_1} \quad df_y = \frac{k}{2\pi} \frac{dy_1}{z_1}$$

- Angle or Wavefront Tilt, $E_a = Ez_1 e^{-jkz_1}$,

$$E_a(u, v) = \frac{jk}{2\pi \cos \theta} \int \int E(x, y, 0) e^{-jk \frac{xu+yv}{w}} dx dy$$

Fraunhofer Diffraction Equations

- Fraunhofer Diffraction Integral

$$E(x_1, y_1, z_1) = \frac{jke^{j kz_1}}{2\pi z_1} e^{jk \frac{(x_1^2 + y_1^2)}{2z_1}} \int \int E(x, y, 0) e^{-jk \frac{(xx_1 + yy_1)}{z_1}} dx dy$$

- Spatial Frequency Description

$$E(f_x, f_y, z_1) = \left(\frac{k}{2\pi z_1} \right)^2 \frac{jke^{j kz_1}}{2\pi z_1} e^{jk \frac{(x_1^2 + y_1^2)}{2z_1}} \int \int E(x, y, 0) e^{-j2\pi(f_x x + f_y y)} dx dy$$
$$E(f_x, f_y, z_1) = \frac{j2\pi z_1 e^{j kz_1}}{k} e^{jk \frac{(x_1^2 + y_1^2)}{z_1}} \int \int E(x, y, 0) e^{-j2\pi(f_x x + f_y y)} dx dy$$

Camera Example

- Pixel Pitch: $7.4\mu\text{m}$

$$f_{sample} = \frac{1}{7.4 \times 10^{-6}\text{m}} = 1.35 \times 10^5 \text{m}^{-1} \quad (\text{135 per mm.})$$

- Nyquist Sampling: $f_{max} = f_{sample}/2$ at $u_{max} = NA$

$$NA = \frac{2f_{sample}\lambda}{2} = f_{sample}\lambda \quad (\text{Coherent Imaging})$$

- Green light at 500nm,

$$NA = 0.068$$

(See Ch. 11 for Incoherent Imaging, which is More Complicated)

Some Useful Fraunhofer Patterns

- Frequently–Used Patterns
 - Uniform Illumination through Rectangular Aperture
(Separable Integrals in x and y)
 - Uniform Illumination through Circular Aperture
(Frequently–Encountered in Imaging)
 - Gaussian Illumination with no Aperture
(Many Laser Beams, Minimum Uncertainty, Closed Form)
- Parameters of Interest
 - Some Measure of Width
 - Axial Irradiance
 - Nulls and Sidelobes if Any

Rectangular Aperture (1)

- Irradiance Pattern

$$E^2(x, y, 0) = \frac{P}{D_x D_y}$$

- Fraunhofer Diffraction

$$E(x_1, y_1, z_1) = \frac{jke^{j kz_1}}{2\pi z_1} e^{jk \frac{(x_1^2 + y_1^2)}{2z_1}} \int_{-D_x/2}^{D_x/2} \int_{-D_y/2}^{D_y/2} \sqrt{\frac{P}{D_x D_y}} e^{-jk \frac{(xx_1 + yy_1)}{z_1}} dx dy$$

- Separable in x and y

$$E(x_1, y_1, z_1) = \sqrt{\frac{P}{D_x D_y}} \frac{jke^{j kz_1}}{2\pi z_1} e^{jk \frac{(x_1^2 + y_1^2)}{2z_1}} \int_{-D_x/2}^{D_x/2} e^{-jk \frac{(xx_1)}{z_1}} dx \int_{-D_y/2}^{D_y/2} e^{-jk \frac{(yy_1)}{z_1}} dy$$

Rectangular Aperture (2)

- Compare Rectangular Pulse in Signal Theory

$$E(x_1, y_1, z_1) = \sqrt{P \frac{D_x D_y}{\lambda^2 z_1^2} \frac{\sin(kx_1 \frac{D_x}{2z_1}) \sin(ky_1 \frac{D_y}{2z_1})}{kx_1 \frac{D_x}{2z_1} ky_1 \frac{D_y}{2z_1}}} e^{jk \frac{(x_1^2 + y_1^2)}{2z_1}}$$

- Irradiance

$$I(x_1, y_1, z_1) = P \frac{D_x D_y}{\lambda^2 z_1^2} \left(\frac{\sin(kx_1 \frac{D_x}{2z_1}) \sin(ky_1 \frac{D_y}{2z_1})}{kx_1 \frac{D_x}{2z_1} ky_1 \frac{D_y}{2z_1}} \right)^2$$

On Axis	First Null	First Sidelobe
$I_0 = I(0, 0, z_1) = P \frac{D_x D_y}{\lambda^2 z_1^2}$	$\frac{d_x}{2} = \frac{\lambda}{D_x}$	$I_1 = 0.0174 I_0$ (-13dB) Check Math

Circular Aperture (1)

- Polar Coordinates

$$E(r_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{(x_1^2+y_1^2)}{z_1}} \int_0^{D/2} \int_0^{2\pi} E(r, 0) e^{-j\frac{k(r \cos \zeta x + r \sin \zeta y)}{z_1}} r d\zeta dr$$

- Hankel Transform

$$E(r_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{(r_1^2)}{z_1}} \int_0^{D/2} E(r, 0) e^{-j\frac{kr^2}{z_1}} J_0\left(\frac{krr_1}{z_1}\right) dr$$

- Fraunhofer Diffraction for Source, $E^2(r, 0) = \frac{P}{\pi(D/2)^2}$

$$E(r_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{(r_1^2)}{z_1}} \int_0^{D/2} \sqrt{\frac{P}{\pi(D/2)^2}} J_0\left(\frac{krr_1}{z_1}\right) dr$$

Circular Aperture (2)

$$E(r_1, z_1) = \sqrt{P \frac{\pi D^2}{4\lambda^2 z_1^2} \frac{J_1\left(kr_1 \frac{D}{2z_1}\right)}{kr_1 \frac{D}{2z_1}}} e^{jk \frac{r_1^2}{2z}}$$

$$I(r_1, z_1) = P \frac{\pi D^2}{4\lambda^2 z_1^2} \left(\frac{J_1\left(kr_1 \frac{D}{2z_1}\right)}{kr_1 \frac{D}{2z_1}} \right)^2$$

On Axis	First Null	First Sidelobe
$I_0 = P\pi D^2 / 4\lambda^2 z_1^2$	$2.44z_1\lambda D$	$I = 0.02I_0$ (-17dB)

Gaussian

- Plane-Wave Gaussian Source (Assume Infinite Aperture)

$$E(x, y, 0) = \sqrt{\frac{2P}{\pi(w_0)^2}} e^{-r^2/w_0^2}$$

- Polar Coordinates (Arbitrary Choice)

$$E(r_1, z_1) = \frac{jke^{jkr_1}}{2\pi z_1} e^{jk\frac{(r_1^2)}{2z_1}} \int_0^\infty \int_0^{2\pi} \sqrt{\frac{2P}{\pi(w_0)^2}} e^{-r^2/w_0^2} e^{-jkr\frac{(r_1 \cos \zeta + r_1 \sin \zeta)}{z_1}} rd\zeta dr$$

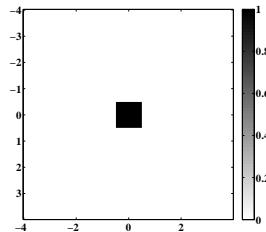
- Result: Gaussians are Forever. (More in Chapter 9)

$$E(x_1, y_1, z_1) = \sqrt{\frac{2P}{\pi(w)^2}} e^{-r_1^2/w^2} \quad w = \frac{z_1 \lambda}{\pi w_0}$$

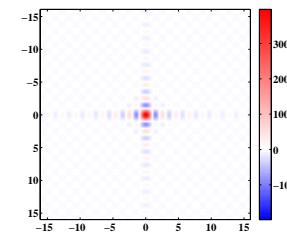
Summary of Common Fraunhofer Patterns

SQUARE:
 $d = 2\frac{\lambda}{D}$ (1st Nulls)
 $I_0 = \frac{PD^2}{\lambda^2 z_1^2}$
 $I_1/I_0 = -13dB$

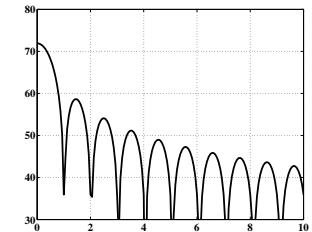
Pupil Pattern



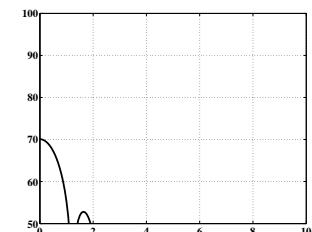
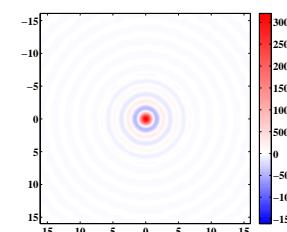
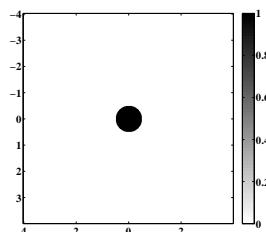
Fraunhofer Pattern



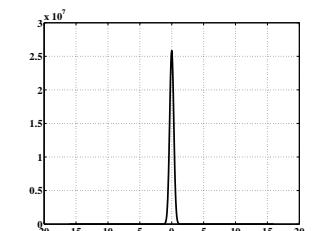
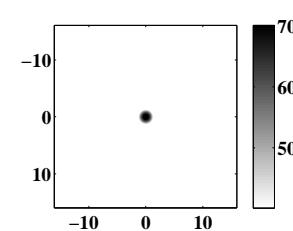
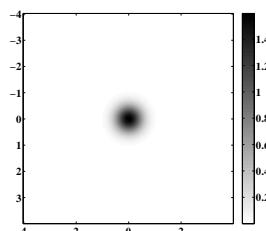
Slice



CIRCULAR:
 $d = 2.44\frac{\lambda}{D}$ (1st Nulls)
 $I_0 = \frac{\pi PD^2}{4\lambda^2 z_1^2}$
 $I_1/I_0 = -17dB$



GAUSSIAN:
 $d = \frac{4\lambda}{\pi d_0}$ (e^{-2} Width)
 $I_0 = \frac{\pi P d_0^2}{2\lambda^2 z_1^2}$
 No sidelobes



Numerical Integration Example: Gaussian in an Aperture

- Given Aperture, D
- Choose Gaussian Size, hD
 - Fill Factor, h
 - Optimize Pattern Somehow (e.g. Axial Irradiance)
- Integrate in Polar Coordinates

$$E(x_1, y_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{(x_1^2+y_1^2)}{2z_1}} \int_0^{D/2} \int_0^{2\pi} \sqrt{\frac{2P}{\pi(hD/2)^2}} e^{-\frac{4r^2}{h^2D^2}} e^{-jkr\frac{(x_1 \cos \zeta + y_1 \sin \zeta)}{z_1}} rd\zeta dr$$

Truncated Gaussian Approximations

- Small Beam, $h \rightarrow 0$

- Ignore Aperture
 - Integrate in Closed Form
 - Gaussian Pattern

$$d = \frac{4}{\pi} \frac{\lambda}{hD} z_1$$

- Low Axial Irradiance

$$I_0 \propto h^2 \rightarrow 0$$

- Large Beam, $h \rightarrow \infty$

- Near Uniform Illumination
 - Power Through Aperture

$$\frac{2P}{\pi(hD)^2/4} \pi \frac{D^2}{4} = \frac{2P}{h^2}$$

- Use Circular Aperture

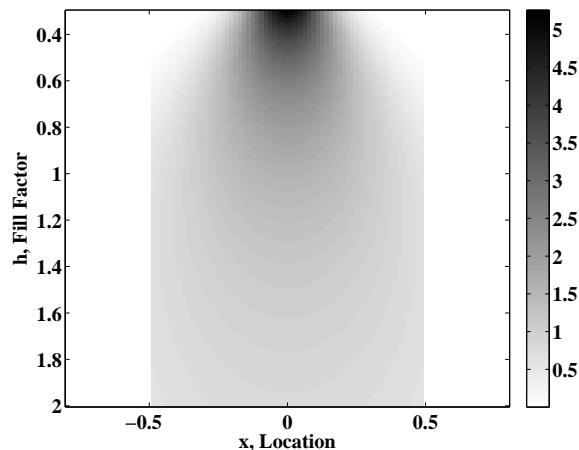
$$I_0 = \frac{P}{2\pi} \left(\frac{\pi D}{h\lambda z_1} \right)^2$$

- Low Axial Irradiance

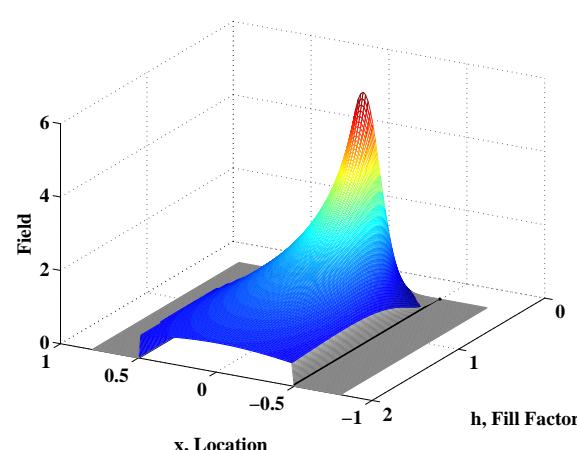
$$I_0 \propto h^{-2} \rightarrow 0$$

- Between These Limits, Use Numerical Calculation

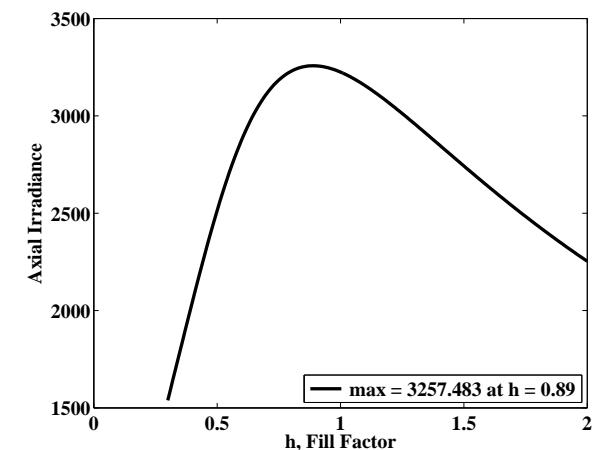
Numerical Integration Results: Gaussian in an Aperture



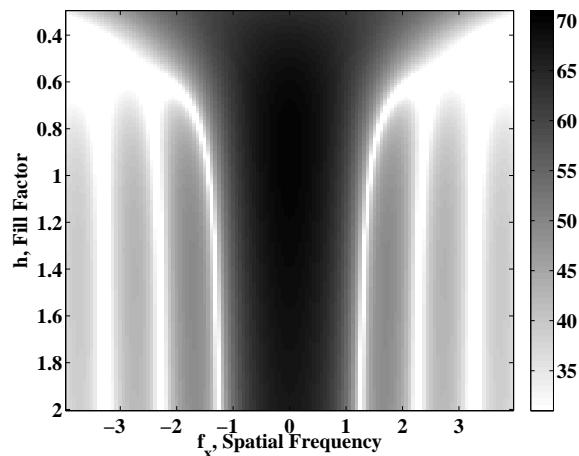
A. Source



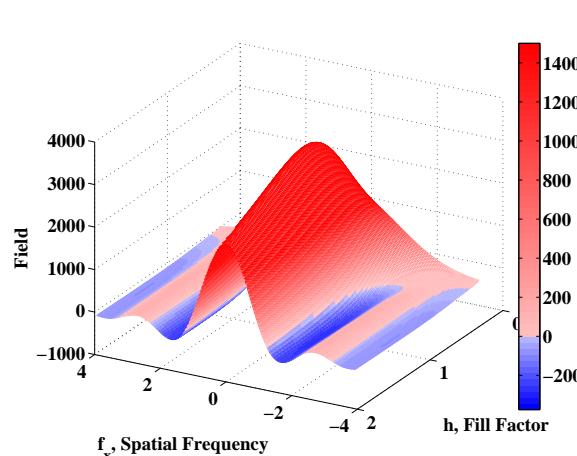
C. Source



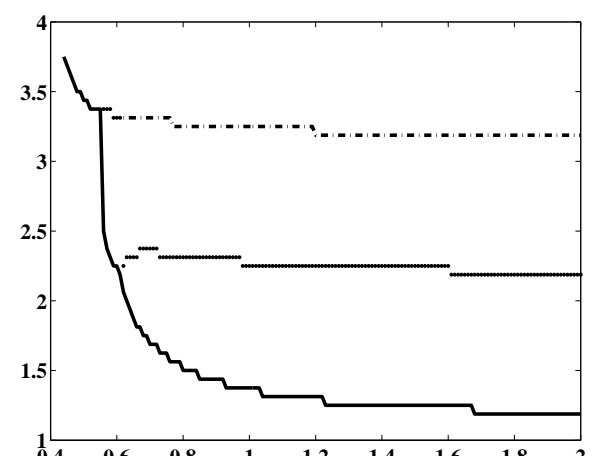
E. Axial I



B. Diffraction



D. Diffraction



F. Null Locations

Depth of Focus

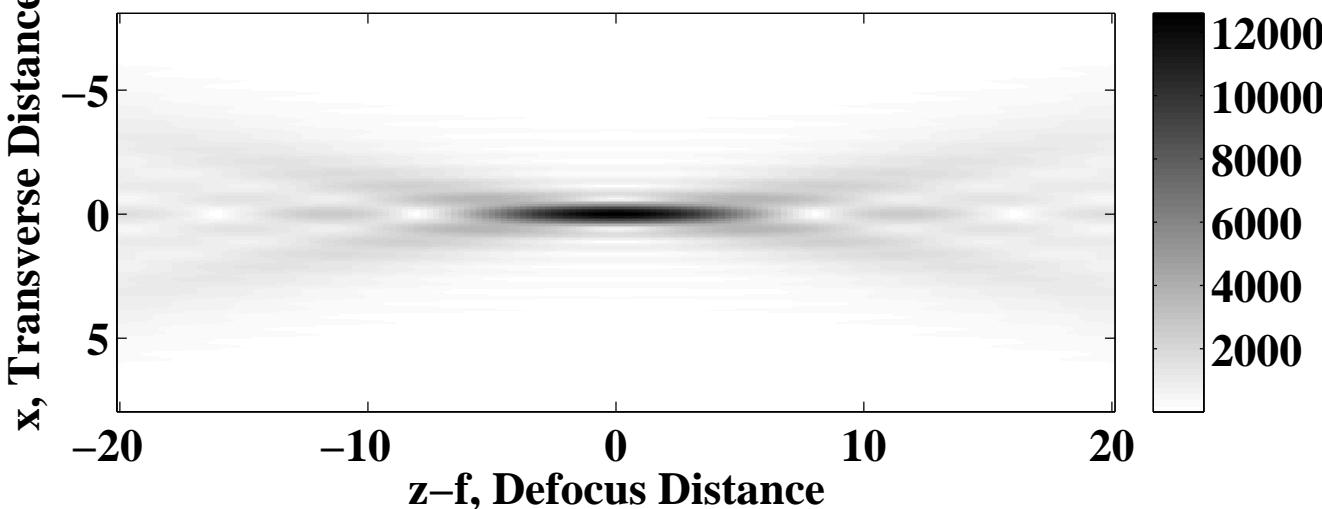
- Fraunhofer Diffraction

$$E(x_1, y_1, z_1) = \frac{jke^{j kz_1}}{2\pi z_1} e^{jk \frac{(x_1^2 + y_1^2)}{2z_1}} \int \int E(x, y, 0) e^{jk \frac{(x^2 + y^2)}{2Q}} e^{-jk \frac{(xx_1 + yy_1)}{z_1}} dx_1 dy_1$$

- Defocus (Diopters)

$$\frac{1}{Q} = \frac{1}{z_1} - \frac{1}{f} \quad \frac{1}{Q} \approx \frac{f - z_1}{f^2}$$

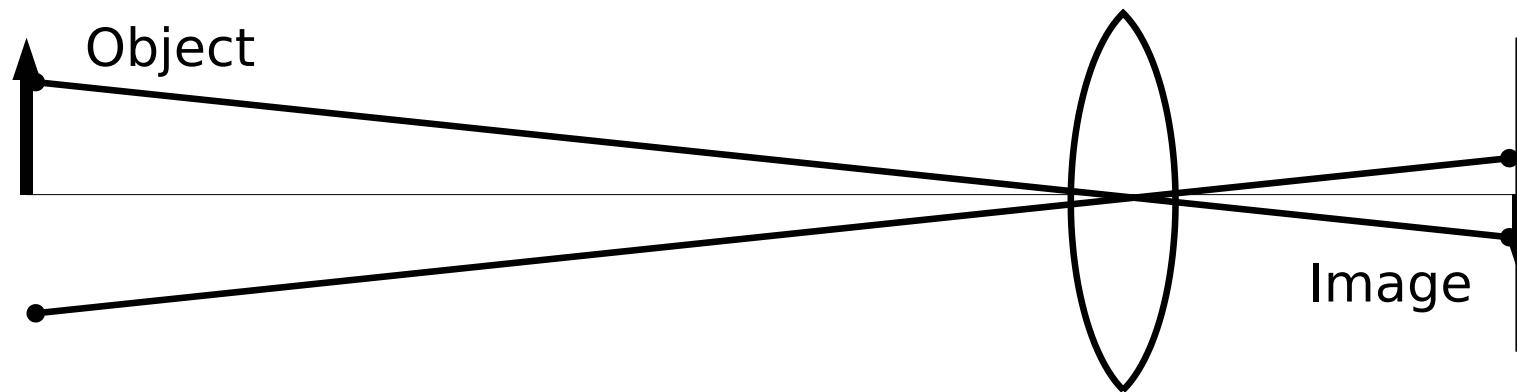
- Numerical Result



- Equation

$$DOF \approx \frac{\lambda}{NA^2}$$

Resolution of an Imaging System



- “To Resolve” or “Resolution” Defined

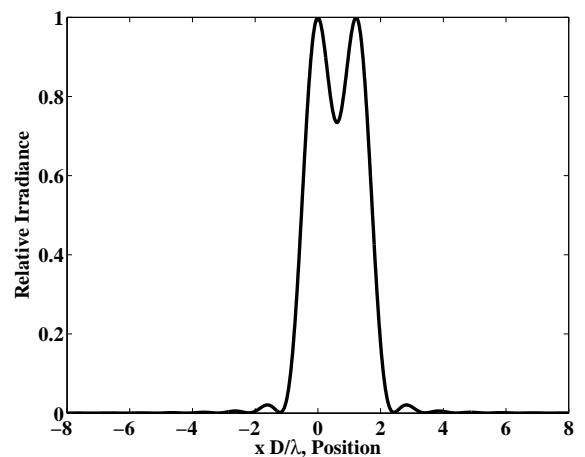
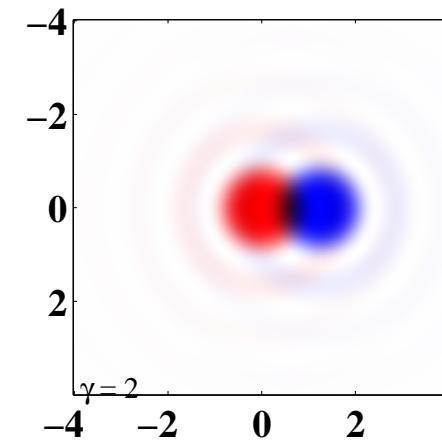
“...to distinguish parts or components of (something) that are close together in space or time; ...”

“...the process or capability of rendering distinguishable the component parts of an object or image; a measure of this, expressed as the smallest separation so distinguishable, ...”

- Resolution is Not Number of Pixels or Width of Point-Spread Function

Resolution Analysis

- Diffraction Patterns of Two “Point Objects”
 - Point-Spread Functions
 - From Fraunhofer Diffraction at Pupil
 - Fourier Transforms (See Ch. 11)
- Add
- Set Criterion for Valley
 - Noise Analysis?
 - Contrast?
 - Arbitrary Decision?



A complete and consistent definition would require knowledge we are not likely to have.

Rayleigh Criterion

- Frequently Used, but Arbitrary
- Defined by Nulls of Point-Spread Function
 - Peak of One over Valley of Other
- Produces Inconsistent Valley
 - Square Aperture
 - Circular Aperture

$$\delta = z_1 \lambda / D$$

$$\delta\theta = \lambda / D$$

– Valley Depth

$$2 \left[\frac{\sin(\pi/2)}{\pi/2} \right]^2 =$$

$$\frac{8}{\pi^2} = 0.81$$

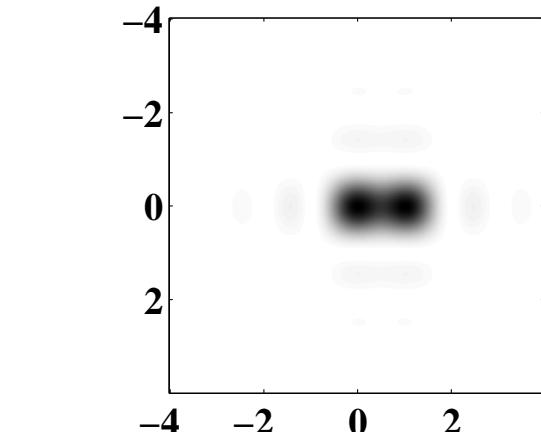
$$\delta\theta = 1.22 \frac{\lambda}{D}$$

$$\delta = 0.61 \frac{\lambda}{NA}$$

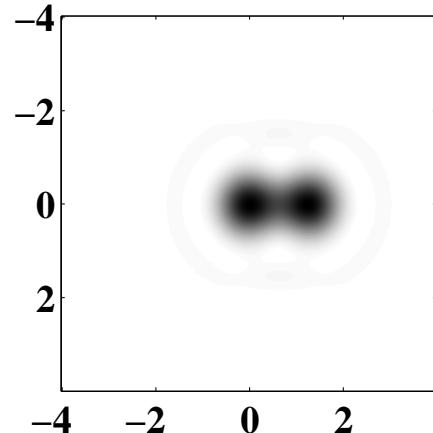
– Valley Depth

0.73

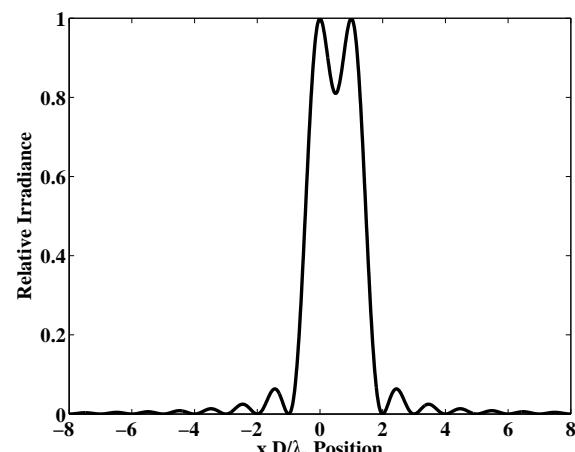
Resolution at the Rayleigh Limit



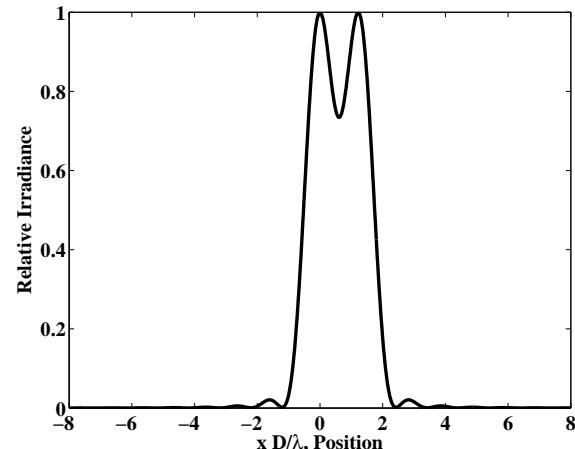
Square Aperture,



Circular Aperture,



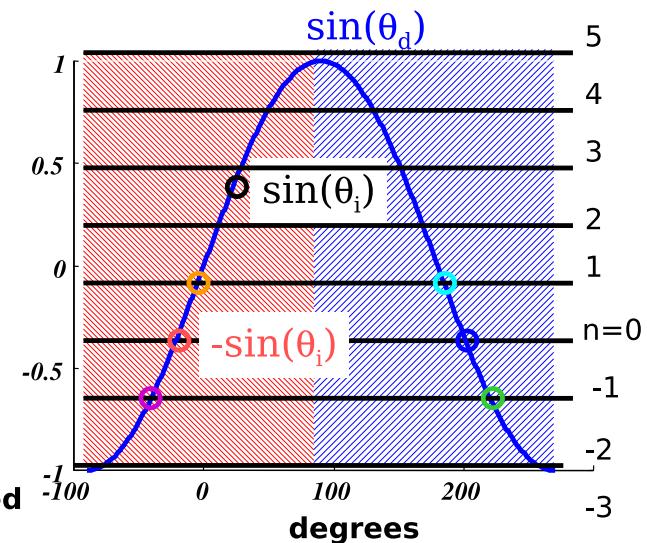
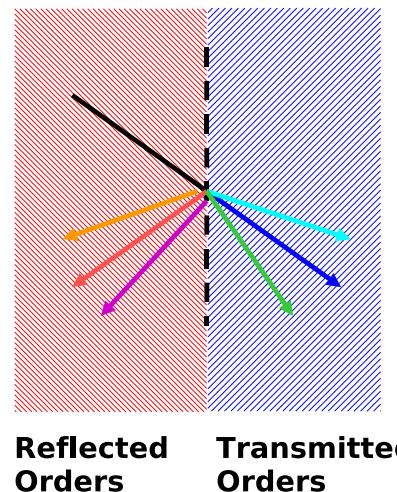
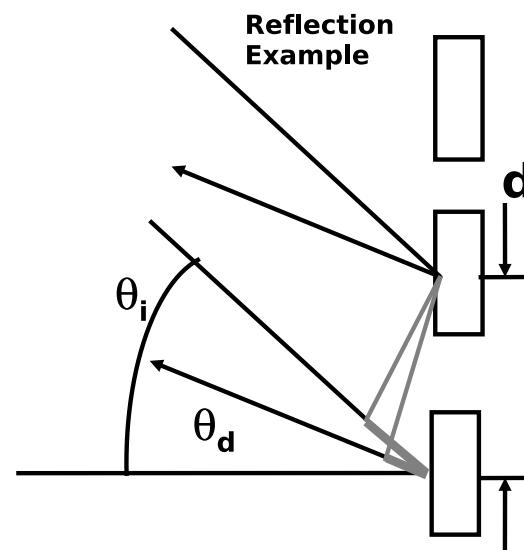
λ/D , 81% Valley



$1.22\lambda/D$, 73% Valley

- Issues
 - Noise
 - Contrast
 - Statistics
 - Sampling*
 - Other Definitions
 - MTF (Ch. 11)
 - Any Valley (Sparrow)
 - 81% Valley (Wadsworth)
 - PSF FWHM (Houston)*
- * Not really resolution

The Diffraction Grating (1)



- The Grating Equation

$$N\lambda = d(\sin \theta_i + \sin \theta_d)$$

$$\sin \theta_d = -\sin \theta_i + N \frac{\lambda}{d}$$

The Diffraction Grating (2)

- Grating Equation

$$N\lambda = d(\sin \theta_i + \sin \theta_d)$$

- Grating Dispersion

$$\delta\lambda = \frac{d}{N}\delta(\sin \theta_d)$$

- Applications

- Monochromator
 - Spectrometer

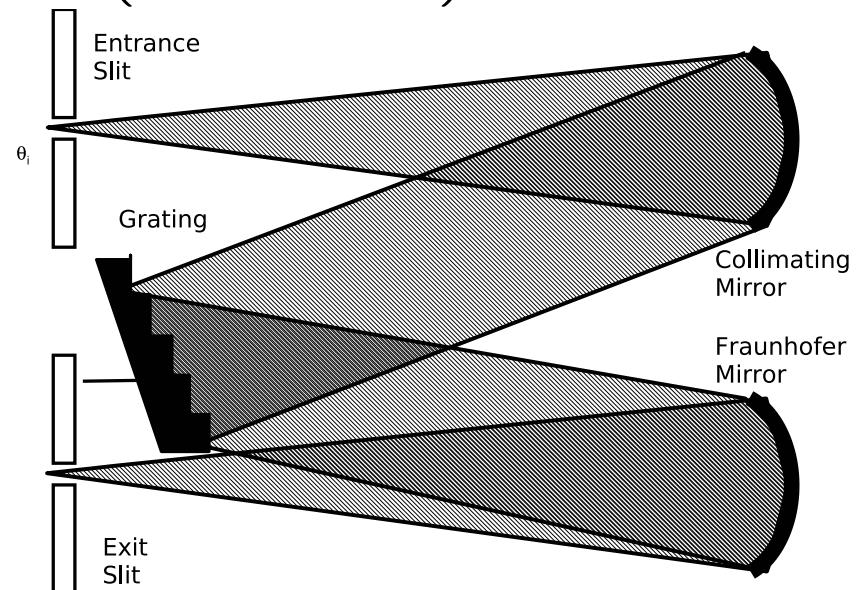
- Aliasing Issues

- N to $N + 1$

$$\lambda_{N+1} = \lambda_N \times \frac{N}{N + 1}$$

- Maximum Width:
Factor of 2

- Anti-Aliasing Filter
 - e.g. Colored Glass
 - e.g. “Filter Wheel”
- Monochromator
(More Later)



Fourier Analysis of Grating

- Convolution in Grating is Multiplication in Diffraction Pattern
- Multiplication in Grating is Convolution in Diffraction Pattern

Grating

$$g(x, 0) = \int f(x - x', 0) h(x', 0) dx'$$

$$g(x, 0) = f(x, 0) h(x, 0)$$

Diffraction Pattern

$$g(x_1, z_1) = f(x_1, z_1) h(x_1, z_1),$$

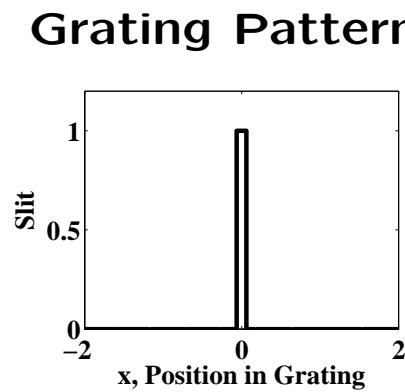
$$g(x_1, z_1) = \int f(x_1 - x'_1, z_1) h(x'_1, z_1) dx'_1,$$

- Approach

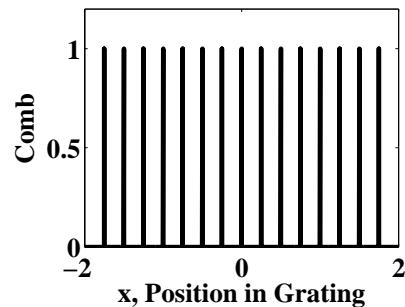
- Convolve Slit with Comb to Produce Grating
 - Multiply Grating by Illumination or Apodization Pattern

Fourier Analysis Equations (1)

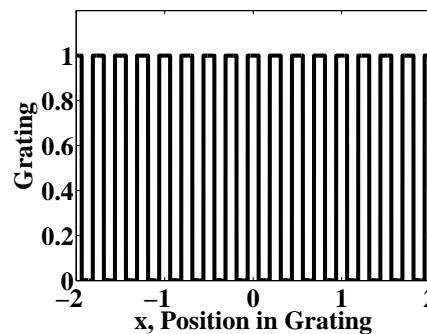
Slit Pattern



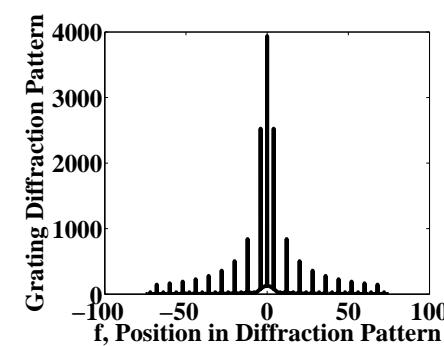
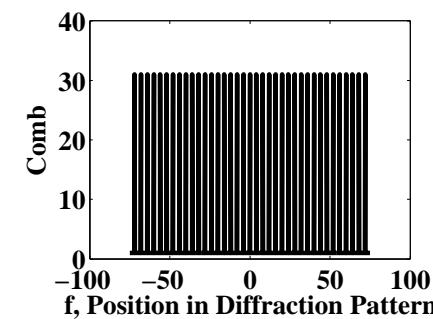
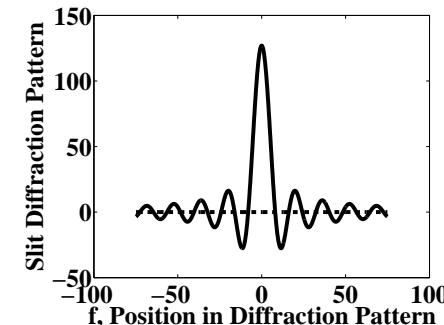
Repetition Pattern



Convolve above two to obtain grating pattern.



Diffraction Pattern

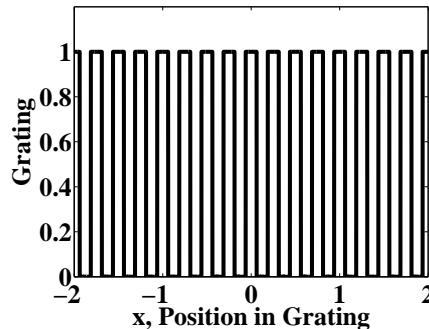


Multiply above two to obtain diffraction pattern.

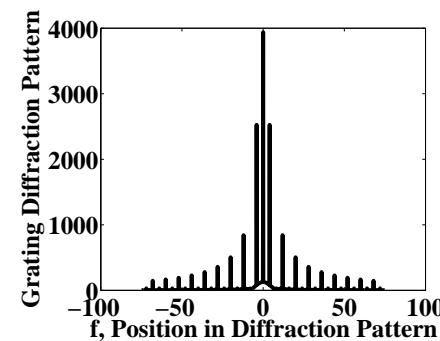
Zeros Depend on Slit Width and Repetition Interval

Fourier Analysis Equations (2)

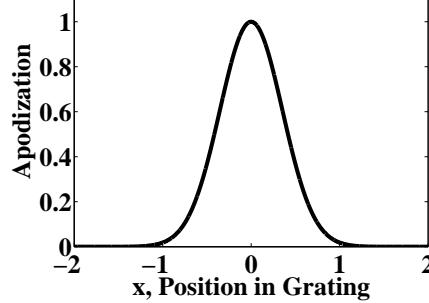
Grating Pattern



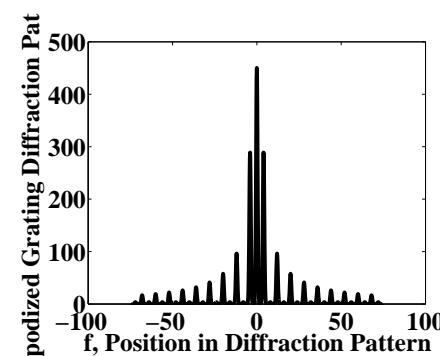
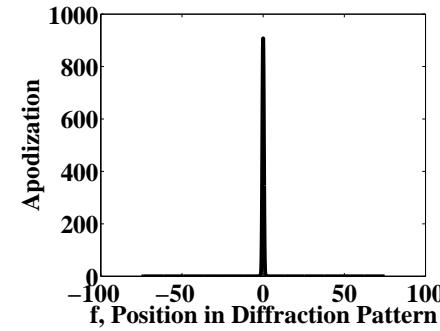
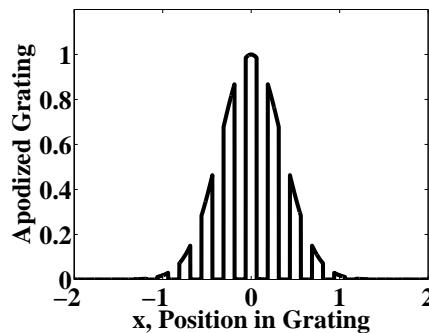
Diffraction Pattern



Apodization or
illumination



Multiply above
two to obtain
grating output.



Convolve above
two to obtain
**diffraction
pattern.**

Wider Apodization Provides Better Resolution

Example of Grating Analysis Equations

Item	Grating Pattern	Size	Diffraction Pattern	Angle
Basic Shape	Uniform	D	sinc	λ/D^*
Repetition	Comb	d	Comb	λ/d
Apodization	Gaussian	d_0	Gaussian	$\frac{4}{\pi} \frac{\lambda}{d_0}$

* first nulls

- Finesse: (Order Spacing)/(Width of a Single Order)

$$\frac{1}{F} = \frac{\frac{4}{\pi} \frac{\lambda}{d_0}}{\frac{\lambda}{d}} = \frac{4}{\pi} \frac{d}{d_0}$$

- Rule of Thumb
 - Roughly $F = \text{Number of Grooves Illuminated}$

Grating Spectrometer Example (1)

- Laser Source with Two Wavelengths

$$\lambda_1 = \lambda - \delta\lambda/2 \quad \lambda_2 = \lambda + \delta\lambda/2$$

- Beam Diameters Equal to (Or Expanded to) d_0
- Spacing in First-Order Spectrum ($N = 1$)

$$\delta\theta = \theta_{d2} - \theta_{d1} \approx \sin \theta_{d2} - \sin \theta_{d1} = \frac{\lambda_2 - \lambda_1}{d} = \frac{\delta\lambda}{d}$$

- Width

$$\alpha = \frac{4}{\pi} \frac{\lambda}{d_0}$$

Grating Spectrometer Example (2)

- Sum of Irradiances

$$I(\theta) = e^{-2(\theta-\theta_{d1})^2/\alpha^2} + e^{-2(\theta-\theta_{d2})^2/\alpha^2}$$

- Resolution Criterion

- Not Rayleigh (No Nulls)
 - Try Sparrow: Check First Derivative

$$\frac{dI(\theta)}{d\theta} = 0 \quad \text{1 or 3 Solutions including } \theta = \frac{\theta_{d2} + \theta_{d1}}{2}$$

- Check Second Derivative (Positive for Valley)

$$\left. \frac{d^2 I}{d\theta^2} \right|_{\theta=(\theta_{d1}+\theta_{d2})/2} = \frac{8e^{-2\left(\frac{\delta\theta}{\alpha}\right)^2} [\alpha^2 - (\delta\theta)^2]}{\alpha^2}$$

Sparrow Criterion for Grating

- Second Derivative

$$\left. \frac{d^2 I}{d\theta^2} \right|_{\theta=(\theta_{d1}+\theta_{d2})/2} = \frac{8e^{-2\left(\frac{\delta\theta}{\alpha}\right)^2} [\alpha^2 - (\delta\theta)^2]}{\alpha^2}$$

Negative (Peak) for $\alpha < \delta\theta$ Positive (Valley) for $\alpha > \delta\theta$

- Boundary

$$\frac{4 \lambda}{\pi d_0} = \frac{\delta\lambda_0}{d} \quad \text{Subscript on } \delta\lambda_0 \text{ for Onset of Valley}$$

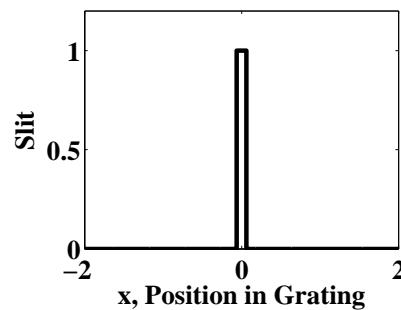
$$\frac{\delta\lambda_0}{\lambda} = \frac{4 d}{\pi d_0} = \frac{1}{F} \quad \text{e.g. } \frac{4 \cdot 10^{-6} \text{m}}{\pi \cdot 10^{-3} \text{m}} \approx \frac{1}{1300}$$

For $\lambda = 830\text{nm}$, $\delta\lambda_0 = 830\text{nm}/1300 \approx 0.61\text{nm}$

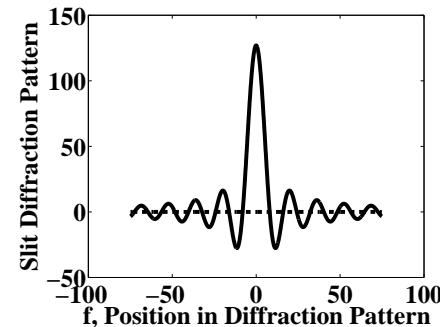
Blaze Angle

- Linear Phase Shift in Slit
 - Prism for Transmission
 - Tilt for Reflection
- Translation of Diffraction Pattern

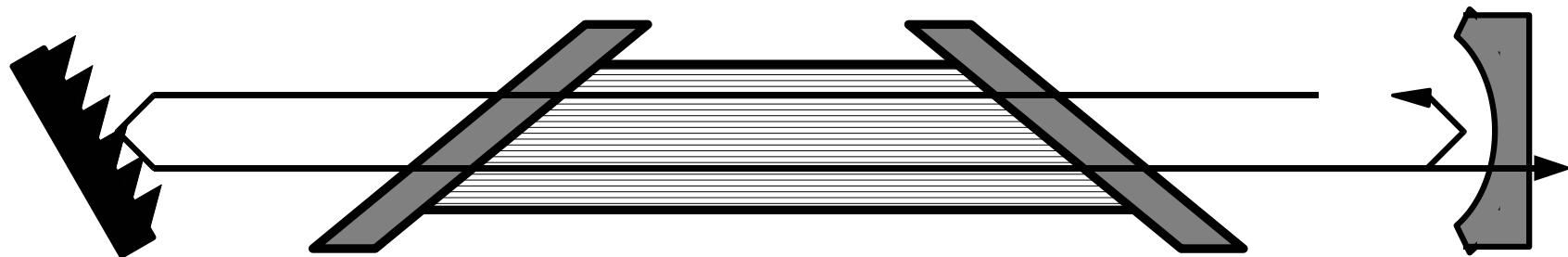
Phase Ramp Here



- Littrow Grating: $\theta_i = \theta_d$
- $$\lambda = 2d \sin \theta_i \quad (N = 1)$$
- Backward Reflection
- e.g. Tunable Laser

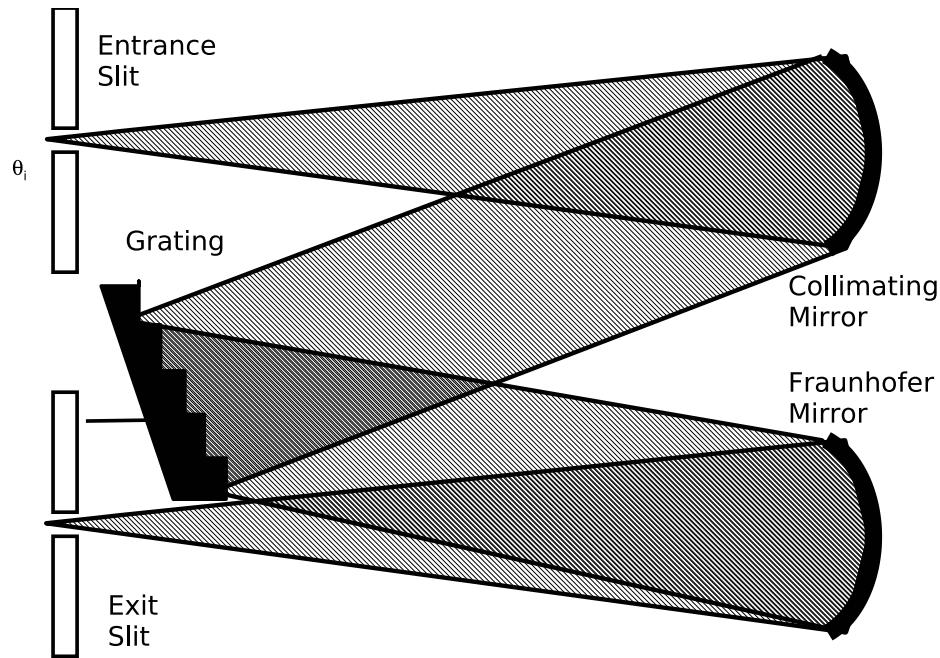


Translation Here
Align Peak with
First Order for
Chosen λ

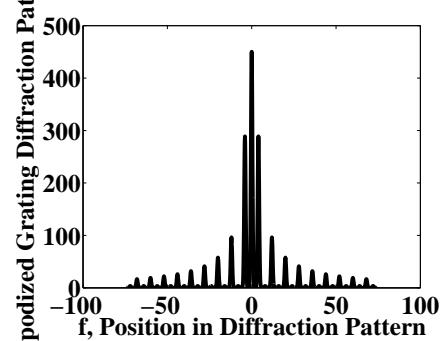


Q: Add a phase ramp to the Matlab code

Monochromator



- Entrance Slit Limits θ_i
- Exit Slit Selects Limits θ_d
- Choose One Order



- Symmetry Minimizes Aberrations
- Blazed Grating Enhances Signal
- Colored-Glass Filter Eliminates Aliasing

Monochromator Resolution

- Entrance Slit $d + i = 2w_i$
- Input Angles: $-w_i/(2f) \leq \sin \theta_i \leq w_i/(2f)$
- Transmission Through Entrance Slit

$$T_i(\delta\lambda) = 1 \quad \left| \delta\lambda - \frac{d}{N} \delta \sin \theta_i \right| < w_i/2,$$

- Transmission Through Exit Slit

$$T_d(\delta\lambda) = 1 \quad \left| \delta\lambda - \frac{d}{N} \delta \sin \theta_d \right| < w_d/2$$

- Overall Transmission Spectrum

$$T_i(\delta\lambda) \otimes I(\delta\lambda) \otimes T_d(\delta\lambda)$$

- Issues of Source Spatial Coherence (Ch. 10 and 11) in
Apodization for $I(\delta\lambda)$

Fresnel Diffraction

- Focused Light in Geometric Optics

$$\frac{|z_1 - f|}{f} D \quad I = \frac{P}{\pi (D/2)^2} \frac{f^2}{(z_1 - f)^2}$$

- Fails at Edges of Shadow and at Focus
- Focal Region Done in Fraunhofer Diffraction
- Fresnel Diffraction Used at Edges
- Assume Integrals Separable in x and y

$$E(x_1, y_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} \int E_x(x, 0) e^{jk\frac{(x-x_1)^2}{2z_1}} dx \int E_y(y, 0) e^{jk\frac{(y-y_1)^2}{2z_1}} dy$$

Uniformly Illuminated Rectangular Aperture

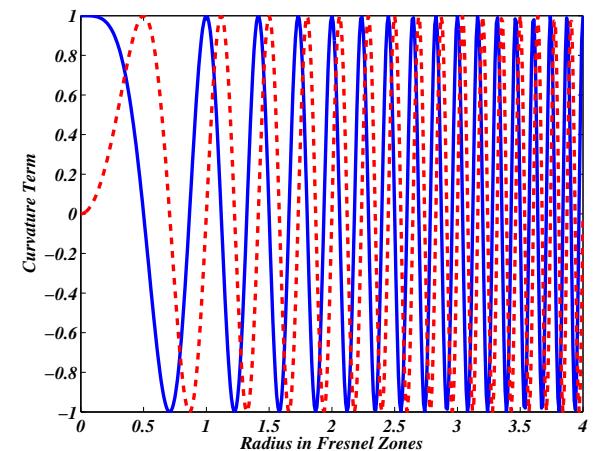
- Aperture Size $2a$ by $2b$

$$E(x_1, y_1, z_1) = \frac{jke^{jxz_1}}{2\pi z_1} \sqrt{\frac{P}{4ab}} I(x_1, a, z_1) I(y_1, b, z_1)$$

$$I(x_1, a, z_1) = \int_{-a}^a e^{jk\frac{(x-x_1)^2}{2z_1}} dx_1$$

$$I(x, a, z_1) = \int_{-a}^a \cos\left(k\frac{(x-x_1)^2}{2z_1}\right) dx + j \int_{-a}^a \sin\left(k\frac{(x-x_1)^2}{2z_1}\right) dx$$

- Fresnel Cosine and Sine Integral
- Cosine Interand Contributes
Near $x = x_1$
- Sine is Zero at $x = x_1$
- Away from Edges, $I(x, y, z_1) = I(x, y, 0)$



Fresnel Sine and Cosine Integrals (1)

- Dimensionless Coordinates

$$u = (x - x_1) \sqrt{\frac{k}{2z_1}} \quad du = dx_1 \sqrt{\frac{k}{2z_1}}$$

- Limits

$$u_1 = (x + a) \sqrt{\frac{k}{2z_1}} \quad u_2 = (x - a) \sqrt{\frac{k}{2z_1}}$$

- Integrals

$$I(x_1, a, z_1) = \sqrt{\frac{2z_1}{k}} \int_{u_1}^{u_2} \cos^2 u \ du + j \sqrt{\frac{2z_1}{k}} \int_{u_1}^{u_2} \sin^2 u \ du$$

$$I(x_1, a, z_1) = \sqrt{\frac{2z_1}{k}} [C(u_2) - C(u_1) + S(u_2) - S(u_1)]$$

Fresnel Sine and Cosine Integrals (2)

$$I(x_1, a, z_1) = \sqrt{\frac{2z_1}{k}} [C(u_2) - C(u_1) + S(u_2) - S(u_1)]$$

$$C(u_1) = \int_0^{u_1} \cos^2 u \ du \quad S(u_1) = \int_0^{u_1} \sin^2 u \ du$$

$$F(u_1) = C(u_1) + jS(u_1)$$

- Above 3 Are Odd Functions

$$C(u) = -C(-u) \quad S(u) = -S(-u) \quad F(u) = -F(-u)$$

- Approximations

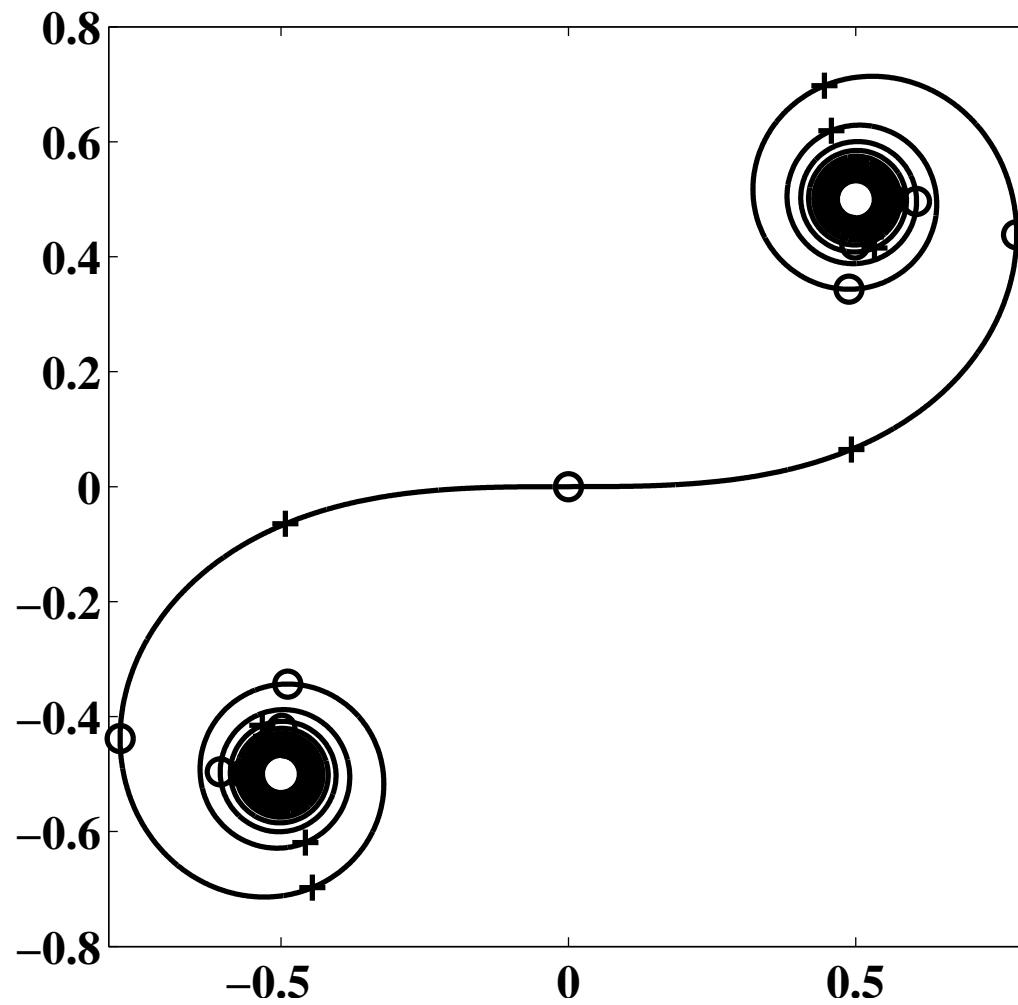
$$\begin{aligned} F(u) &\approx ue^{j\pi u^2/2} & u \approx 0 \\ F(u) &\approx \frac{1+j}{2} - \frac{j}{\pi u} e^{j\pi u^2/2} & u \rightarrow \infty \end{aligned} \quad (1)$$

Fresnel Diffraction Results

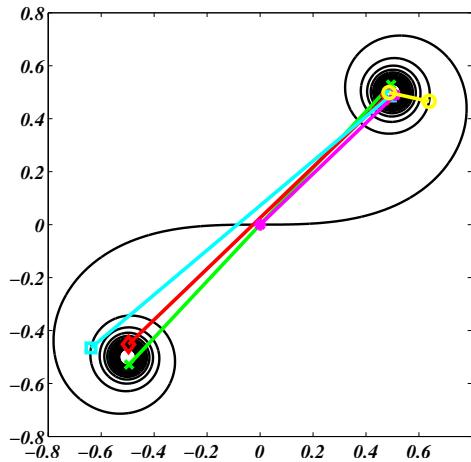
$$E(x_1, y_1, z_1) = \frac{jke^{jkz_1}}{2\pi z_1} \frac{P}{4ab} \frac{2z_1}{k} \times \\ \left\{ F\left[(x_1 - a)\sqrt{\frac{k}{2z_1}}\right] - F\left[(x_1 + a)\sqrt{\frac{k}{2z_1}}\right] \right\} \times \\ \left\{ F\left[(y_1 - b)\sqrt{\frac{k}{2z_1}}\right] - F\left[(y_1 + b)\sqrt{\frac{k}{2z_1}}\right] \right\}$$

- Large x_1 or y_1 : $E \rightarrow 0$: Q: How Close? Use Approximations on Previous Page.
- Near the Center: $E(x_1, y_1, z_1) = E(x_1, y_1, 0)$ (Using the Odd Property of F)

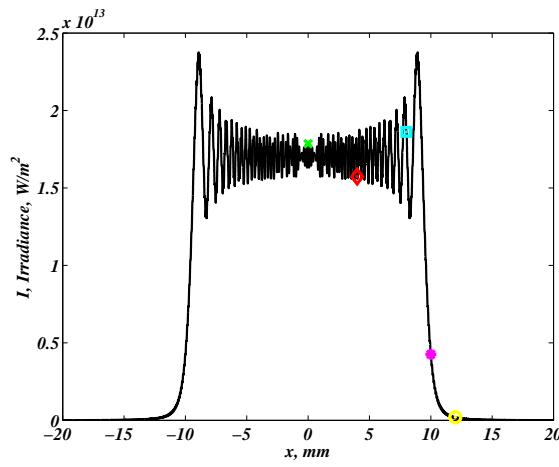
Cornu Spiral



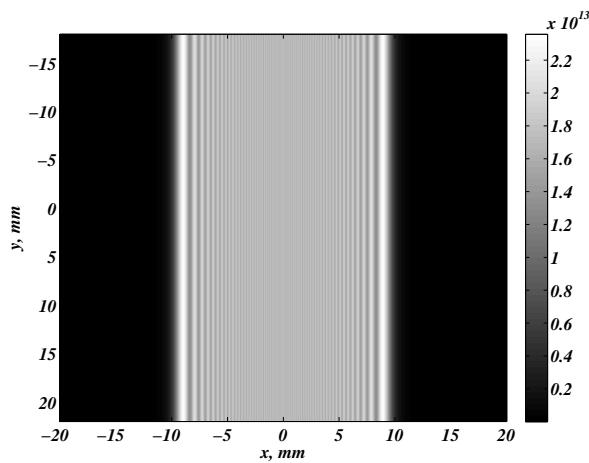
Example of Fresnel Diffraction: Near Field Diffraction



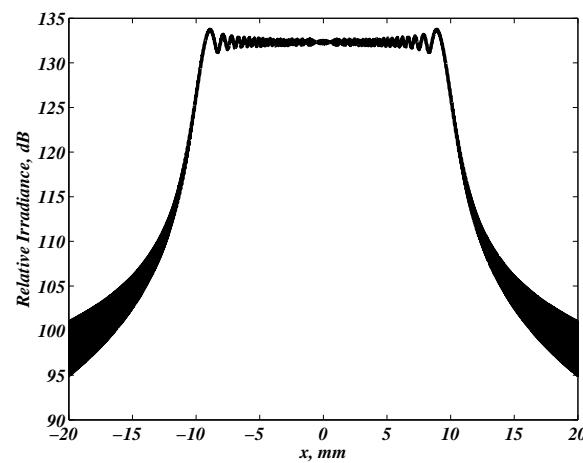
A. Cornu Spiral



B. Irradiance

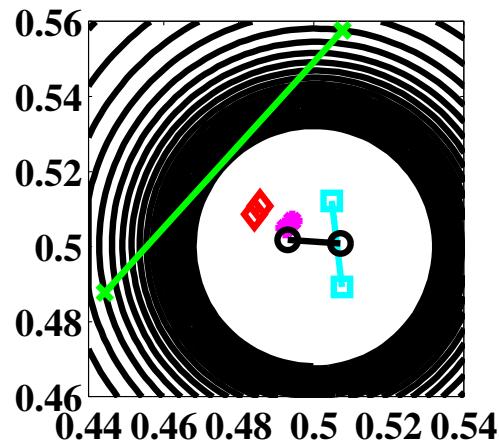


C. Image

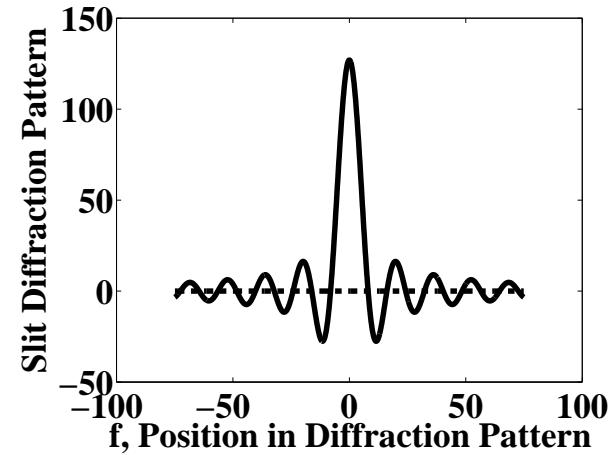


D. Irradiance, dB

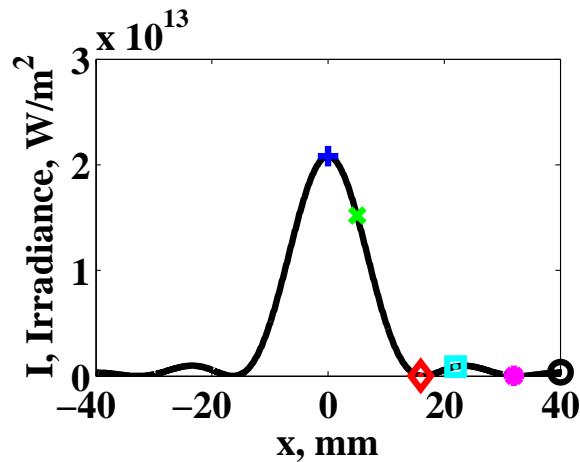
Fresnel Diffraction Includes Fraunhofer Diffraction



A. Cornu Spiral

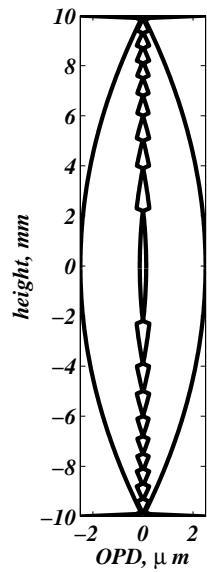


B. Field

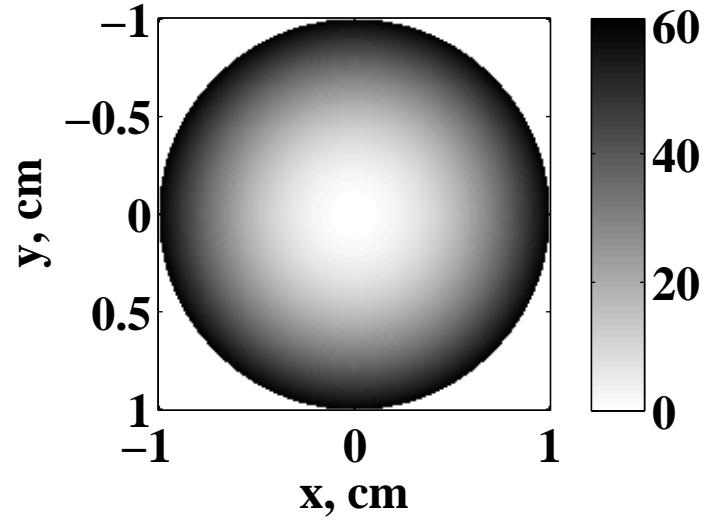


C. Irradiance

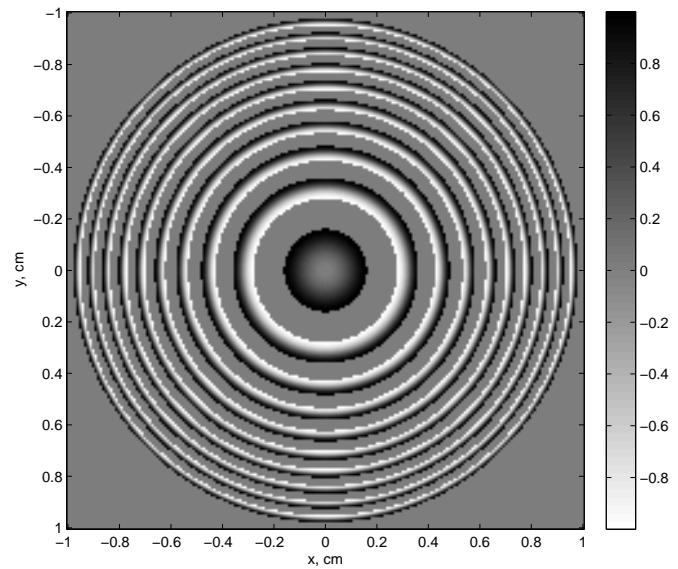
Fresnel Lens



Lens

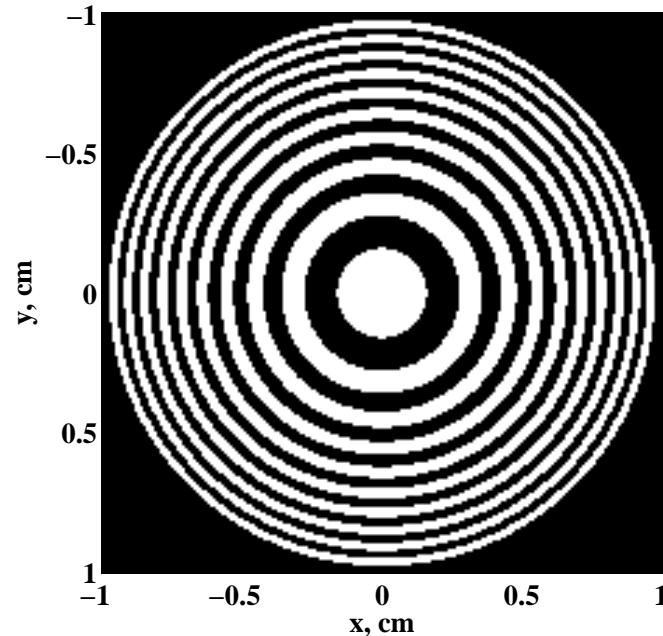


Unwrapped Phase
for Regular Lens

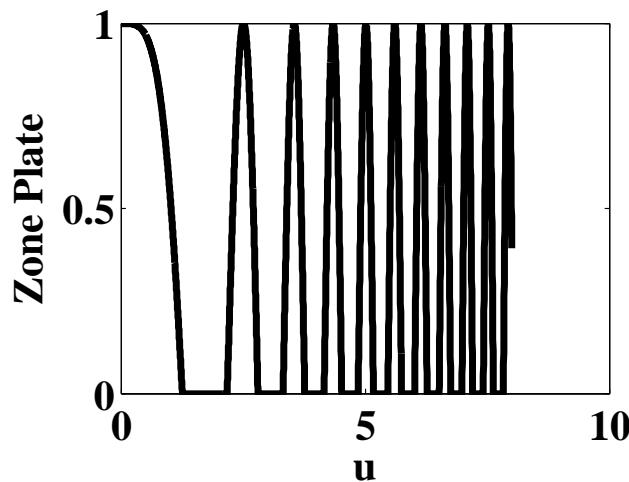


Phase for
Fresnel Lens

Fresnel Zone Plate

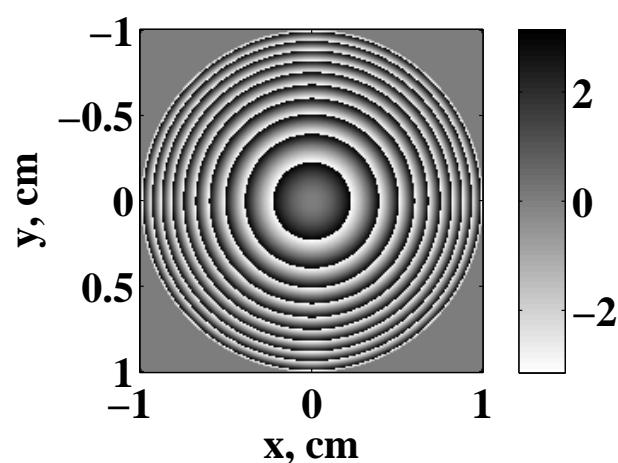


Fresnel zone plate with Diameter 2cm at distance of 10m. The zone plate acts as a lens.

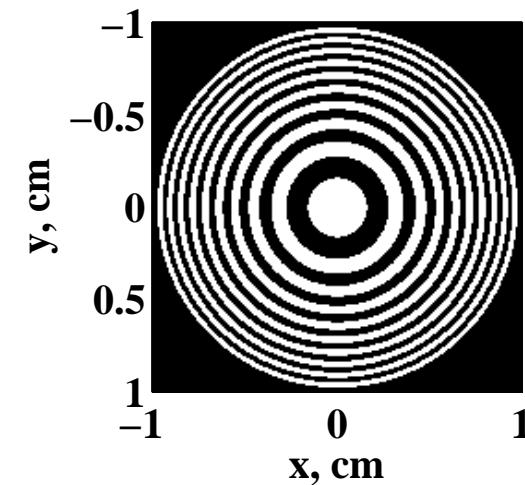


Integrand of Cosine Integral passing through zone plate. All negative contributions are blocked.

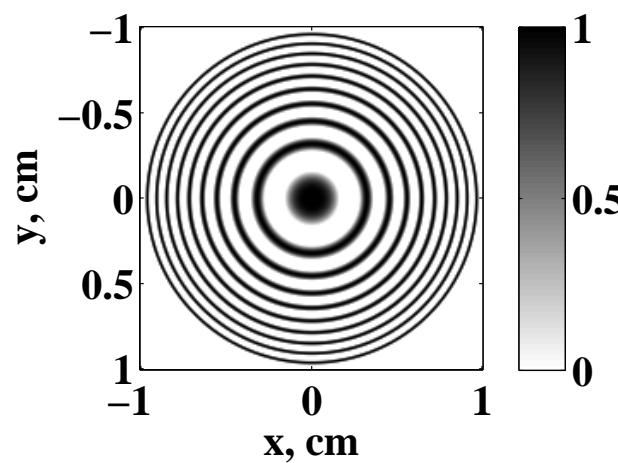
Phase in Fresnel Zone Plate



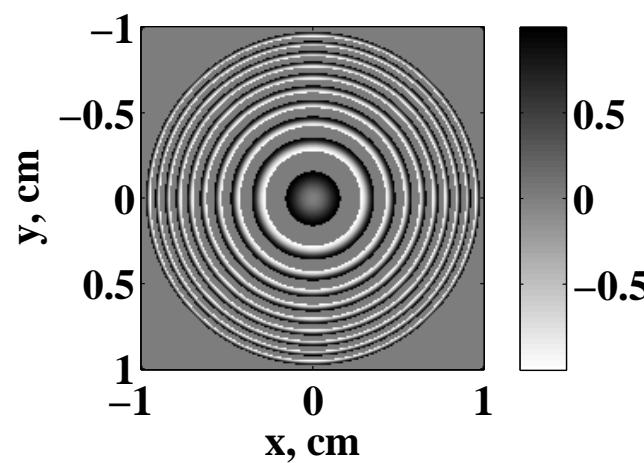
Raw Phase in Radians



Zone Plate Transmission

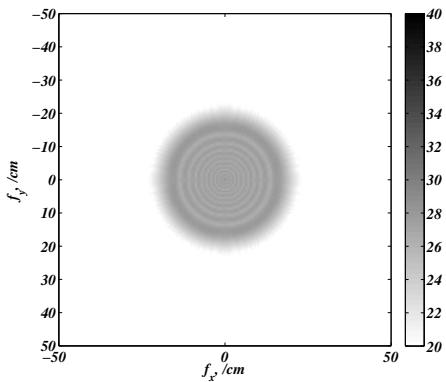


Real Transmitted Field

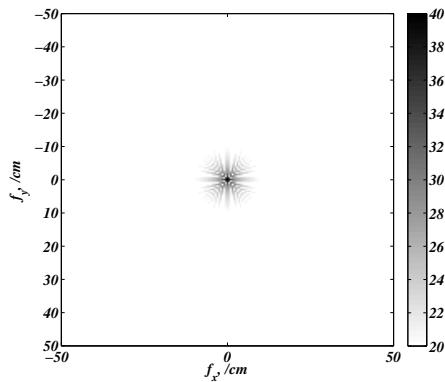


Imag. Transmitted Field

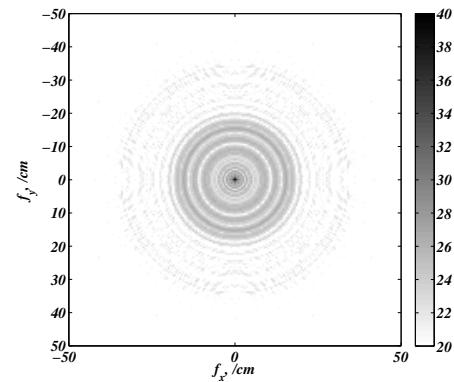
Focusing With Fresnel Zone Plate



Unfocused
Pattern



Focused
With Lens



Focused with
Zone Plate