

Optics for Engineers

Chapter 9

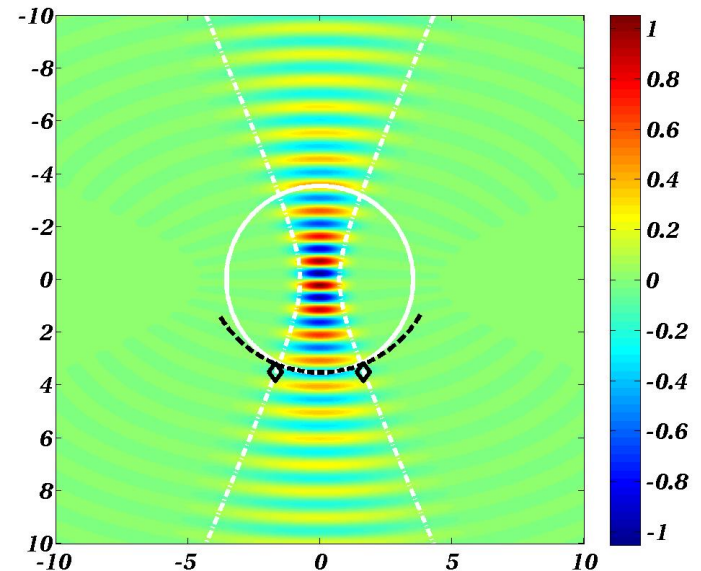
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Gaussian Beams

- Applications
 - Many Laser Beams
 - Minimum–Uncertainty
 - Simple Equations
 - Good Approximation
 - Extensible (e.g. Hermite–Gaussian)
- Equations
 - Solution of Helmholtz Equation
 - Solution to Laser Cavity
 - Kogelnik and Li, 1966
 - Spherical Gaussian Waves
 - “Gaussians Are Forever”

- Imaginary Part of Field
 - Gaussian Profile
 - Spherical Wavefront



- Focusing and Propagation
 - Simple Equations
 - Relation to ABCD Matrix

Derivation (1)

- Spherical Wave and Paraxial Approximation

$$E_{sphere} = \sqrt{\frac{P_{sphere}}{4\pi}} \frac{e^{jk\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}} \approx \sqrt{\frac{P_{sphere}}{4\pi z}} e^{jkz} e^{jk\frac{x^2+y^2}{2z}}$$

- Radius of Curvature = z : Substitute Complex q

$$E = \sqrt{\frac{P_{sphere}}{4\pi}} \frac{e^{jk\sqrt{x^2+y^2+q^2}}}{\sqrt{x^2+y^2+q^2}} \approx \sqrt{\frac{P_{sphere}}{4\pi}} \frac{1}{q} e^{jkq} e^{jk\frac{x^2+y^2}{2q}}$$

- Separate Real and Imaginary

$$E \approx \sqrt{\frac{P_{sphere}}{4\pi}} \frac{1}{q} e^{jkq} e^{jk(x^2+y^2)\Re\frac{1}{2q}} e^{-k(x^2+y^2)\Im\frac{1}{2q}}$$

Derivation (2)

- Curvature and Gaussian Profile

$$e^{jk(x^2+y^2)} \Re \frac{1}{2q} = e^{jk \frac{x^2+y^2}{2\rho}} \quad e^{-k(x^2+y^2)} \Im \frac{1}{2q} = e^{-k \frac{x^2+y^2}{2b'}}$$

- Definitions

$$q = z + jb \qquad \frac{1}{q} = \frac{1}{\rho} - \frac{j}{b'}$$

$$\rho = \frac{z^2 + b^2}{z} \qquad b' = \frac{z^2 + b^2}{b}$$

Gaussian Beam Parameters

- Beam Radius, w

$$w^2 = \frac{2b'}{k} \quad b' = \frac{\pi w^2}{\lambda} = \frac{\pi d^2}{4\lambda}$$

$$e^{-k(x^2+y^2)\Im\frac{1}{2q}} = e^{-\frac{x^2+y^2}{w^2}}$$

- Curvature (Prev. Pg.)

$$e^{jk(x^2+y^2)\Re\frac{1}{2q}} = e^{jk\frac{x^2+y^2}{2\rho}}$$

- Gouy Phase

$$\frac{1}{q} = \frac{1}{z + jb} = \frac{1}{\sqrt{z^2 + b^2}} e^{-\arctan \frac{z}{b}}$$

$$= \frac{\lambda}{b} \frac{1}{\sqrt{\pi w^2}} e^{-\arctan \frac{z}{b}} = e^{-j\psi}$$

- Linear Phase

$$e^{jkq} = e^{jk(z+jb)} = e^{jkz} e^{-kb}$$

- Normalized Complete Equation

$$E \approx \sqrt{\frac{2P}{\pi w^2}} e^{-\arctan \frac{z}{b}} e^{jkz} e^{jk\frac{x^2+y^2}{2\rho}} e^{-\frac{x^2+y^2}{w^2}}$$

Physical Meaning of Parameters

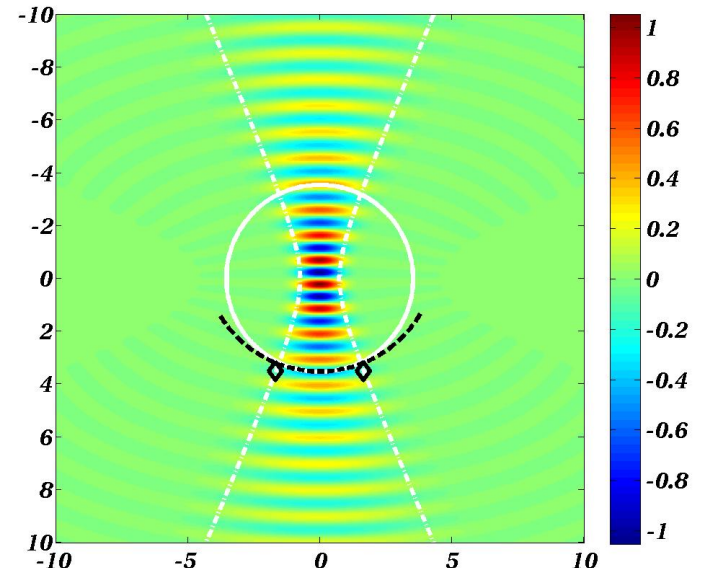
$$E \approx \sqrt{\frac{2P}{\pi w^2}} e^{jkz} e^{jk \frac{x^2+y^2}{2\rho}} e^{-\frac{x^2+y^2}{w^2}} e^{-j\psi}$$

- Radius of Curvature, ρ
Dashed Black Line

$$\frac{1}{\rho} = \Re \frac{1}{q}$$

- Beam Diameter, d
Black Diamonds

$$\frac{1}{b'} = -\Im \frac{1}{q} \quad b' = \frac{\pi w^2}{\lambda} = \frac{\pi d^2}{4\lambda}$$



- Distance from Waist, z

$$z = \Re q$$

- Rayleigh Range, b

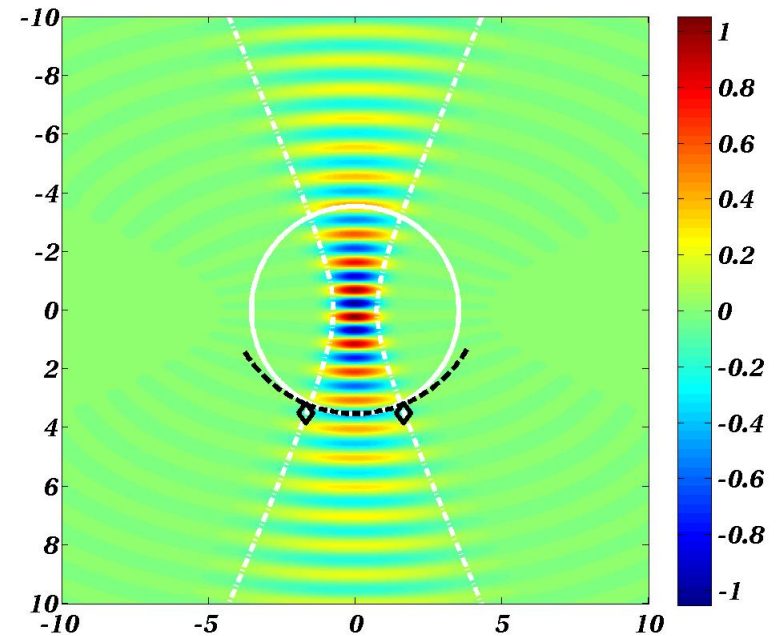
$$b = \Im q \quad b = \frac{\pi w_0^2}{\lambda} = \frac{\pi d_0^2}{4\lambda}$$

Gouy Phase

- Phase Term

$$\psi = \arctan \frac{z}{b}$$

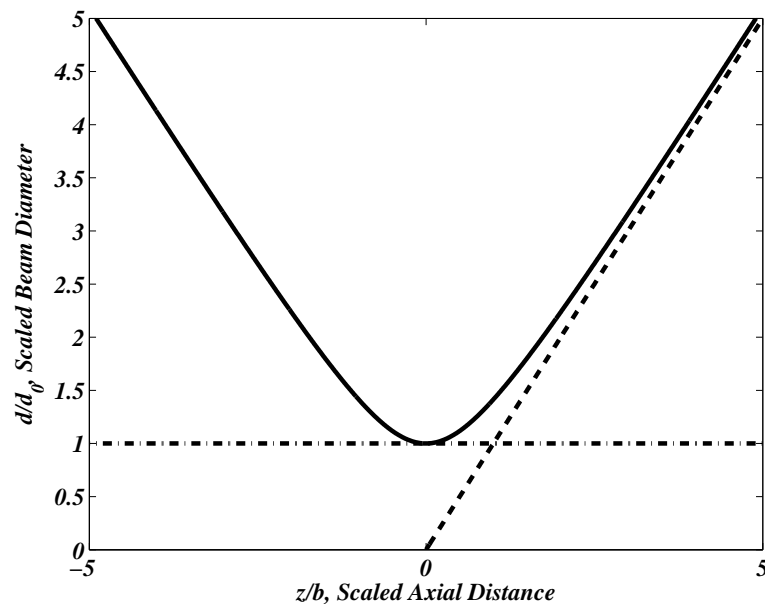
- See White Circle
- Plot is $\Im E$



The Really Useful Equations

- Beam Diameter, d

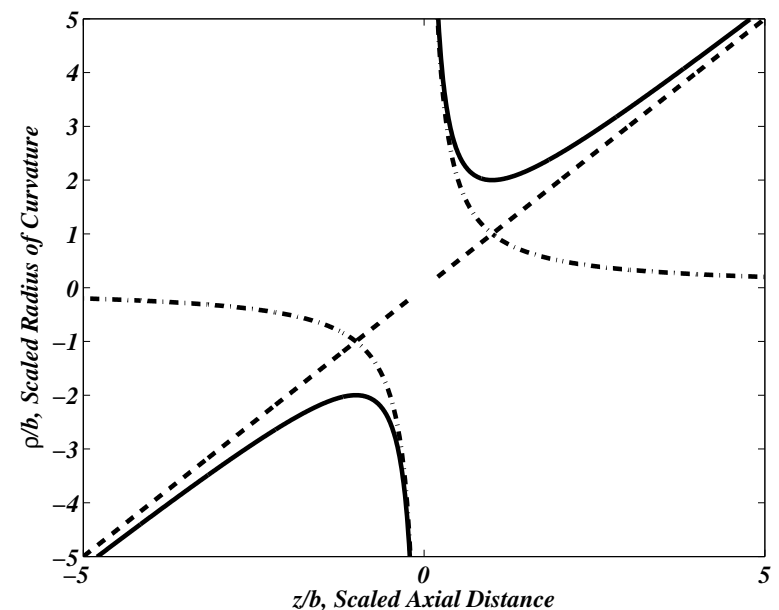
$$d = d_0 \sqrt{1 + \frac{z^2}{b^2}}$$



- Near Field $d_g \approx d_0$
- Far Field $d_d \approx \frac{4}{\pi} \frac{\lambda}{d_0} z$

- Radius of Curvature, ρ

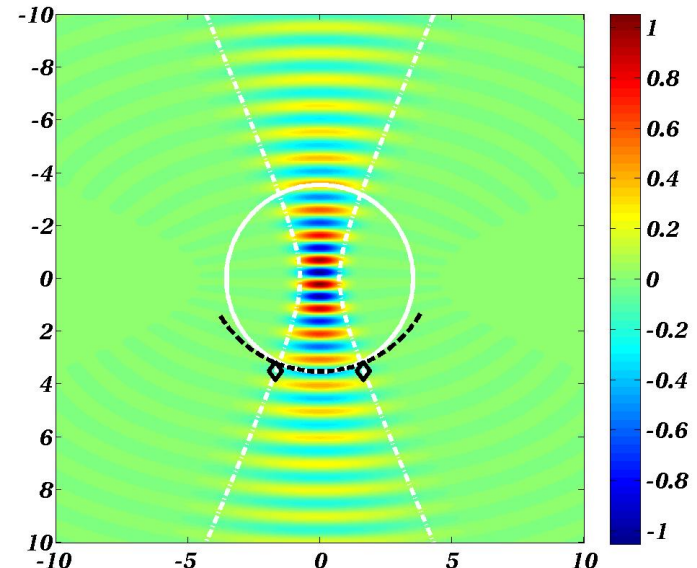
$$\rho = z + \frac{b^2}{z}$$



- Near Field $\rho \approx b^2/z \rightarrow \infty$
- Far Field $\rho \approx z \rightarrow \infty$

Six Questions

- Four Parameters for q
 - z , Distance from Waist
 - b or d_0 , Waist Diameter
 - ρ , Radius of Curvature
 - b' or d Local Diameter (at z)
- Two Independent Numbers for Complex q
- Pick Any Pair
- Solve for the Rest
- Practical Problems for Each Pair



z and b known
 ρ and b'
 z and b'
 ρ and b
 z and ρ
 b and b'

Question 1: Known z and b

- Sample Application: Starting with a collimated (plane-wave) Gaussian, what are d and ρ at distance, z ?
- Example: Green Laser Pointer: $\lambda = 532\text{nm}$.
- Solution

$$b = \frac{\pi d_0^2}{4\lambda} = 5.91\text{m} \quad \rho = z + \frac{b^2}{z} \quad d = d_0 \sqrt{1 + \frac{z^2}{b^2}}$$

$z = 3\text{m}:$	$\rho = 14.6\text{m}$	$d = 2.2\text{mm}$	$\left(\frac{4\lambda}{\pi d_0^2} z\right)$ (1mm)
$z = 30\text{m}:$	$\rho = 31.2\text{m}$	$d = 10.4\text{mm}$	(10.2mm)
$z = 376,000\text{km}:$	$\rho = 376,000\text{km}$	$d = 127\text{km}$	(127km)

Question 2: Known ρ and b'

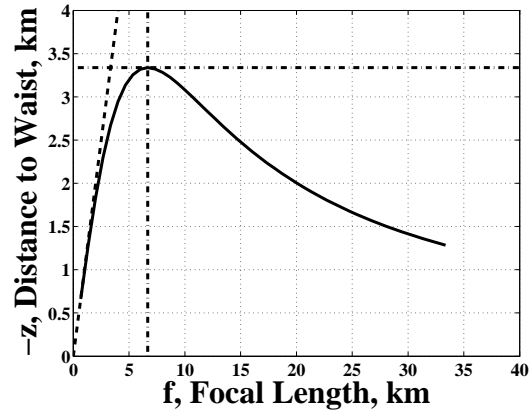
- Sample Application: What is the waist location for a focused beam?
- Example: Laser Radar, $\lambda = 10.59\mu\text{m}$, $d = 30\text{cm}$, $b' = 6.6\text{km}$
- Solution

$$\rho = -f \quad b' = \pi d^2 / (4\lambda) \quad \frac{1}{q} = \frac{1}{\rho} - \frac{j}{b'}$$

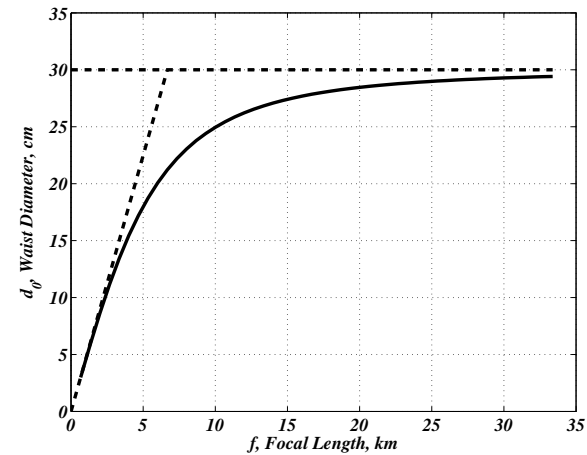
$$\frac{1}{q} = \frac{1}{\rho} - \frac{j}{b'} \quad q = \frac{\frac{1}{\rho}}{\frac{1}{\rho^2} + \frac{1}{b'^2}} + \frac{\frac{j}{b'}}{\frac{1}{\rho^2} + \frac{1}{b'^2}}$$

$$z = \Re q = -\frac{f}{1 + \left(\frac{4\lambda f}{\pi d^2}\right)^2} \quad b = \Im q = \frac{b'}{\frac{b'}{f^2} + 1} \quad d_0^2 = \frac{d^2}{1 + \left(\frac{\pi d^2}{4\lambda f}\right)^2}$$

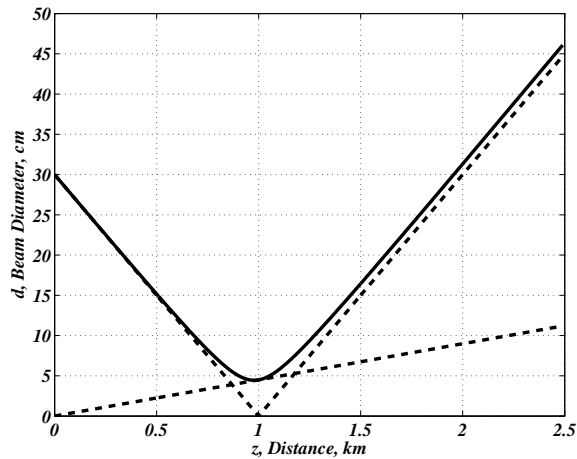
Question 2 Results



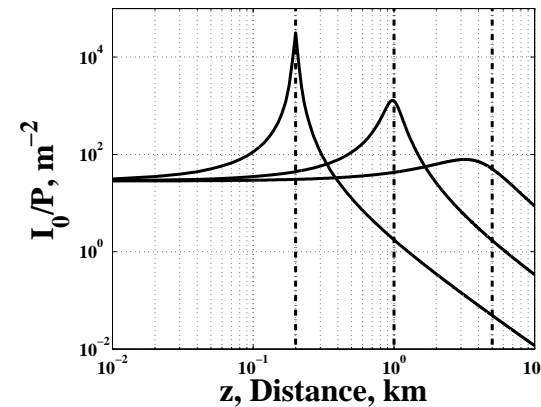
A. Waist Location



B. Waist Diameter



C. Beam Diameter ($f = 1\text{km}$)



Axial Irradiance

Question 2 Summary

- Maximum Distance to Waist: $(-z)_{max} = b/2$ at $f = b$
- Two Ways to Make a Waist at any $(-z) < b/2$
- Near Field Strong Focusing:
 - $(-z) \approx -\rho = f$
 - $d_0 = \frac{4\lambda}{\pi d}$
- Depth of Focus: Quadratic in f
$$z_c = 2b_1 = \frac{8f^2\lambda}{\pi d^2} \frac{1}{\left(\frac{4\lambda f}{\pi d^2}\right)^2 + 1}$$
- Collimated Beam as $f \rightarrow \infty$

Geometric and Diffraction Diameters

- Diameter at Any Distance

$$d^2 = d_g^2 + d_d^2$$

- Simple Calculation
- Intuitive
- Valid for Gaussians

- Geometric Optics

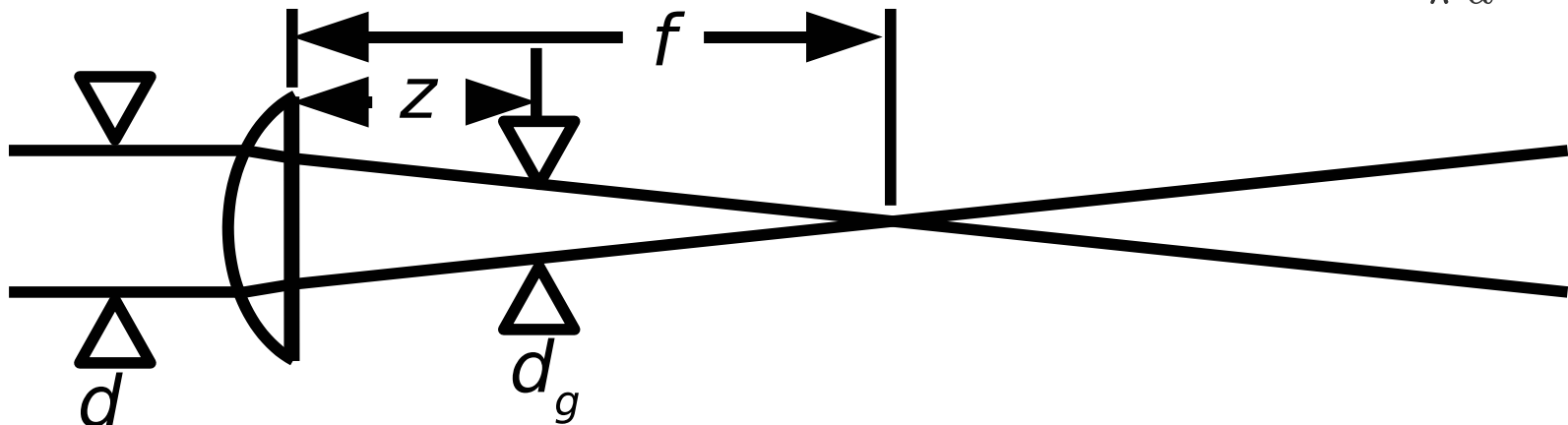
$$d_g = d \frac{|z - f|}{f}$$

- Diffraction Limit

$$d_d = \frac{4\lambda}{\pi d} z$$

- Fraunhofer Zone

$$d_0 \approx d_d = \frac{4\lambda}{\pi d} z$$



Question 3: Known z and b'

- Sample Application: What lens to make a waist at z ? How big is the waist?
- Example: Endoscope: $\lambda = 1.06\mu\text{m}$, waist at $-z = 1\text{mm}$ with d never $> 100\mu\text{m}$
- Solution: $b' = \pi d^2 / (4\lambda)$

$$b' = \frac{b^2 + z^2}{b} \quad \text{Solve for } b \text{ only if } (z < b'/2)$$

$$b = \frac{b' \pm \sqrt{b'^2 - 4z^2}}{2} \quad \rho = z + \frac{b^2}{z}$$

- Typically 2 Solutions

Endoscope Focusing

- Fixed Diameter at Source $d = 100\mu\text{m}$, $z = 1\text{mm}$

$$b' = \frac{\pi d^2}{4\lambda} = 7.4\text{mm} \quad z = -1\text{mm}$$

$$b = \frac{b' \pm \sqrt{b'^2 - 4z^2}}{2} \quad b = 3.7047\text{mm} \pm 3.5672\text{mm}$$

- Options

$$b = 137\mu\text{m} \quad d_0 = 13.6\mu\text{m} \quad \text{or} \quad b = 7.27\text{mm} \quad d_0 = 99\mu\text{m}$$

- Probably Choose the First
 - High Irradiance
 - Small Depth of Field

Question 4: Known ρ and b

- Sample Application: How long must the laser be to get a given d_0 with a mirror of curvature ρ ?
- Example: Later
- Solution: Solve for z if d_0 Small Enough

$$\rho = \frac{z^2 + b^2}{z} \quad (2b < \rho)$$

$$z = \frac{\rho \pm \sqrt{\rho^2 - 4b^2}}{2}$$

$$d = d_0 \sqrt{1 + \frac{z^2}{b^2}}$$

Question 5: Known z and ρ

- Sample Application: How large is the beam from a laser cavity?
- Example: Later
- Solution: Solve for b

$$\rho = \frac{z^2 + b^2}{z} \quad (\rho > z)$$

$$b = \sqrt{\rho z - z^2} \quad b = z \sqrt{\frac{\rho}{z} - 1} \quad \text{or} \quad q = z + jb$$

$$d_0 = \sqrt{\frac{4\lambda b}{\pi}} \quad d = d_0 \sqrt{1 + \frac{z^2}{b^2}}$$

Question 6: Known b and b'

- Sample Application: How do we launch a beam of diameter, d , into a fiber with diameter, d_0 ?
- Example: Beam diameter, $d = 15\mu\text{m}$ and desired diameter, $d_0 = 5\mu\text{m}$
- Solution: Solve for z

$$b' = \frac{z^2 + b^2}{b} \quad (b' > b \text{ of course: } d > d_0)$$

$$z = \pm \sqrt{bb' - b^2} \quad \rho = z + \frac{b^2}{z}$$

Fiber Launch

- Equations

$$z = \pm\sqrt{bb' - b^2} \quad \rho = z + \frac{b^2}{z}$$

- Inputs for $d = 15\mu\text{m}$ and $d_0 = 5\mu\text{m}$

$$b = \frac{\pi d_0^2}{4\lambda} = 13\mu\text{m} \quad b' = \frac{\pi d^2}{4\lambda} = 118\mu\text{m}$$

- Results

$$z = -\sqrt{bb' - b^2} = -37\mu\text{m} \quad \rho = z + \frac{b^2}{z} = -41.7\mu\text{m}$$

- Lens: $f = -\rho = 41.7\mu\text{m}$, with $D_{lens} > 15\mu\text{m}$, ($< f/3$)
- Distance: $-z = 37\mu\text{m}$ (A Little Shorter than f)

Gaussian Beam Characteristics

- The complex radius of curvature can be manipulated to solve problems of Gaussian beam propagation.
- The complex radius of curvature and its inverse contain four terms, which are related in such a way that only two of them are independent.
- Given two parameters it is possible to characterize the Gaussian beam completely. Six examples have been discussed.
- Solutions are at worst quadratic, producing zero, one, or two solutions.
- Other, more complicated problems can also be posed and solved using this formulation.

Gaussian Beam Propagation

- Propagation through Free Space

$$q(z_2) = q(z_1) + z_2 - z_1$$

- Propagation through a Lens

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

$$\frac{1}{\rho} + \frac{1}{-\rho'} = \frac{1}{f}$$

$$\frac{1}{\rho'} = \frac{1}{\rho} - \frac{1}{f}$$

$$\frac{1}{q'} = \frac{1}{q} - \frac{1}{f} \quad \text{or} \quad \frac{1}{q'} = \frac{1}{q} - P$$

Matrix Optics Again

- General

$$q_{out} = \frac{Aq_{in} + B}{Cq_{in} + D}$$

- Translation

$$q_2 = \frac{q_1 + z_{12}}{0 + 1} = q_1 + z_{12}$$

- Lens

$$q' = \frac{q + 0}{-\frac{P}{n'}q + \frac{n}{n'}} \quad \frac{1}{q'} = \frac{n}{qn'} - \frac{P}{n'}$$

– In Air

$$\frac{1}{q'} = \frac{1}{q} - P = \frac{1}{q} - \frac{1}{f}$$

Refraction at a Surface

- General

$$q' = \frac{q + 0}{\frac{n-n'}{n'R}q + \frac{n}{n'}}$$

$$\frac{1}{q'} = \frac{n - n'}{n'R} + \frac{n}{qn'}$$

- Planar Surface

$$q' = q \frac{n'}{n}$$

- Rayleigh Range in a Medium

$$b = \frac{\pi d_0^2}{4\lambda/n}$$

Propagation Example with a Lens: F to F'

- Waist at Front Focal Plane, F

$$q_0 = jb_0 \quad b_0 = \frac{\pi d_0^2}{4\lambda}$$

- Three Steps

$$q_1 = q_0 + f_1 \quad \frac{1}{q'_1} = \frac{1}{q_1} - \frac{1}{f_1} \quad q_2 = q'_1 + f_1$$

- Result: $\Re z_2 = 0$ (Waist)

$$q_1 = jb_0 + f_1 \quad q'_1 = -f_1 + j\frac{f_1^2}{b_0} \quad q_2 = \frac{jf_1^2}{b_0}$$

$$d_2 = \frac{4\lambda}{\pi d_0} f_1 \quad (\text{Fourier Transform})$$

Propagation Example

Continued: A Second Lens (f_3)

- Relay Telescope

$$d_4 = \frac{4 \lambda}{\pi d_2} f_3 = \frac{f_3}{f_1} d_0$$

- Matrix Optics: First Lens

$$\mathcal{M}_{02} = \mathcal{T}_{12} \mathcal{L}_1 \mathcal{T}_{01} = \begin{pmatrix} 1 & f_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & f_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & f_1 \\ -\frac{1}{f_1} & 0 \end{pmatrix}$$

$$q_2 = \frac{Aq_0 + B}{Cq_0 + D} = \frac{0 + f}{-\frac{1}{f}q_0 + 0} = -\frac{f^2}{q_0} \quad d_2 = \frac{4 \lambda}{\pi d_0} f_1$$

- Both Lenses ($f_3 = f_1$): Inverting 1:1 Telecentric System

$$\mathcal{M}_{04} = \mathcal{M}_{24} \mathcal{M}_{02} = \begin{pmatrix} 0 & f_3 \\ -\frac{1}{f_3} & 0 \end{pmatrix} \begin{pmatrix} 0 & f_1 \\ -\frac{1}{f_1} & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$q_4 = \frac{Aq_0 + B}{Cq_0 + D} = \frac{-q_0 + 0}{0 - 1} = q_0$$

Two 1:1 Relays

- Telecentric (Previous Page)

$$\mathcal{M}_{04} = \mathcal{M}_{24}\mathcal{M}_{02} =$$

$$\begin{pmatrix} 0 & f_3 \\ -\frac{1}{f_3} & 0 \end{pmatrix} \begin{pmatrix} 0 & f_1 \\ -\frac{1}{f_1} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Afocal

$$q_4 = q_0 = jb_0$$

- Plane Wave
- No Field Curvature
- $\rho \rightarrow \infty$

- Single Lens
(4-f Configuration)

$$\mathcal{M}_{ac} = \mathcal{T}_{bc}\mathcal{L}_b\mathcal{T}_{ab} =$$

$$\begin{pmatrix} 1 & 2f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2f \\ 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} -1 & 0 \\ -\frac{1}{f} & -1 \end{pmatrix}$$

$$q_c = \frac{Aq_a + B}{Cq_a + D} =$$

$$\frac{-1q_a + 0}{-\frac{1}{f}q_a - 1} = \frac{1}{\frac{1}{f} + \frac{1}{q_a}}$$

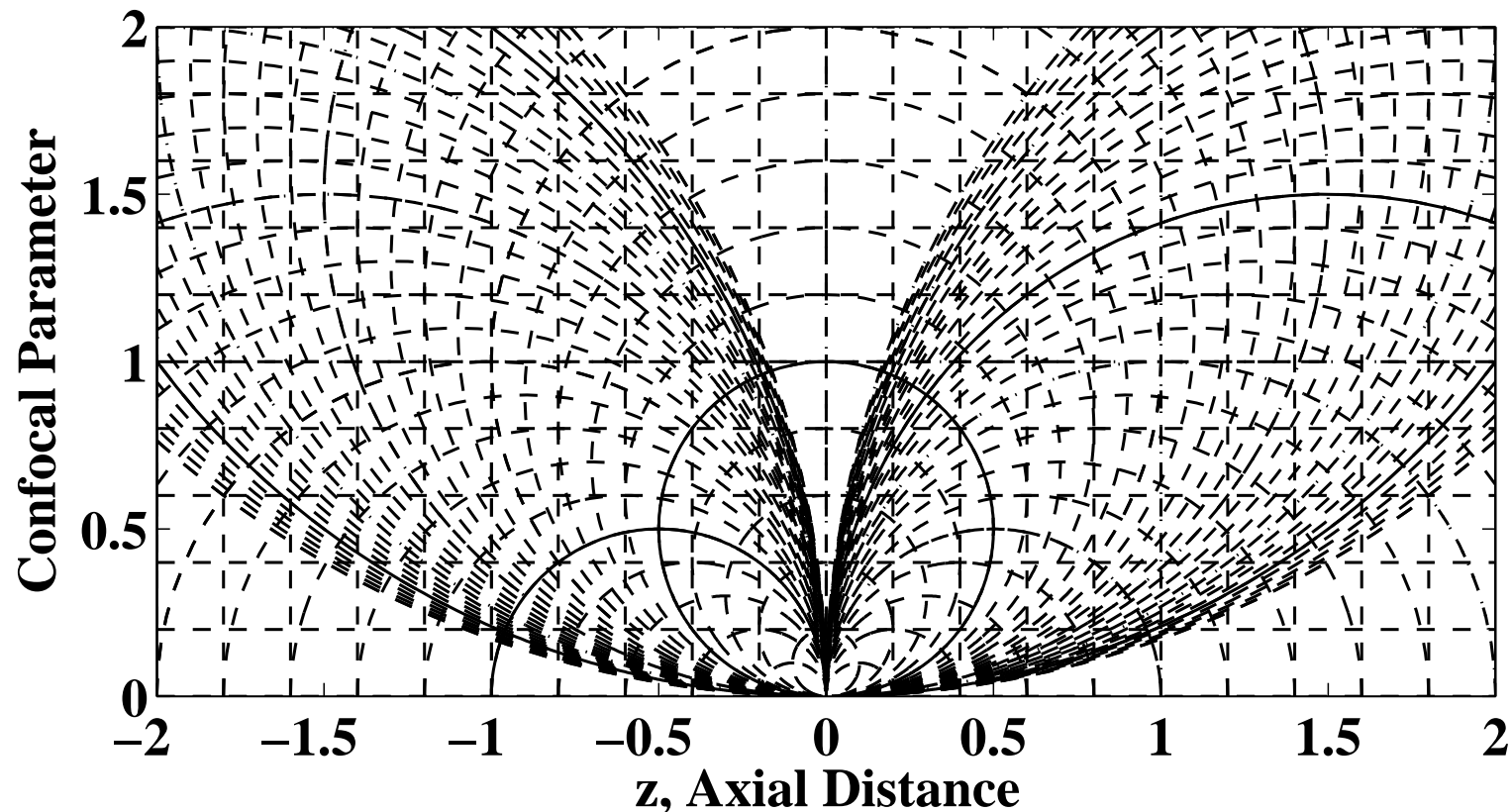
$$- \rho = f$$

Summary of Gaussian Beam Propagation

- Once two parameters of a Gaussian beam are known, then its propagation can be analyzed using simple equations for translation, refraction, and focusing.
- Translation changes the beam size and curvature. It is represented by adding the distance to q , as in Equation .
- Refraction through a lens changes the curvature, but keeps the beam diameter constant. It is represented by subtracting the optical power of the lens from the inverse of q as in Equation .
- Refraction through a dielectric interface changes the curvature and scale factor of q .
- These equations can be generalized using the ABCD matrices developed for geometric optics, as in Equation .

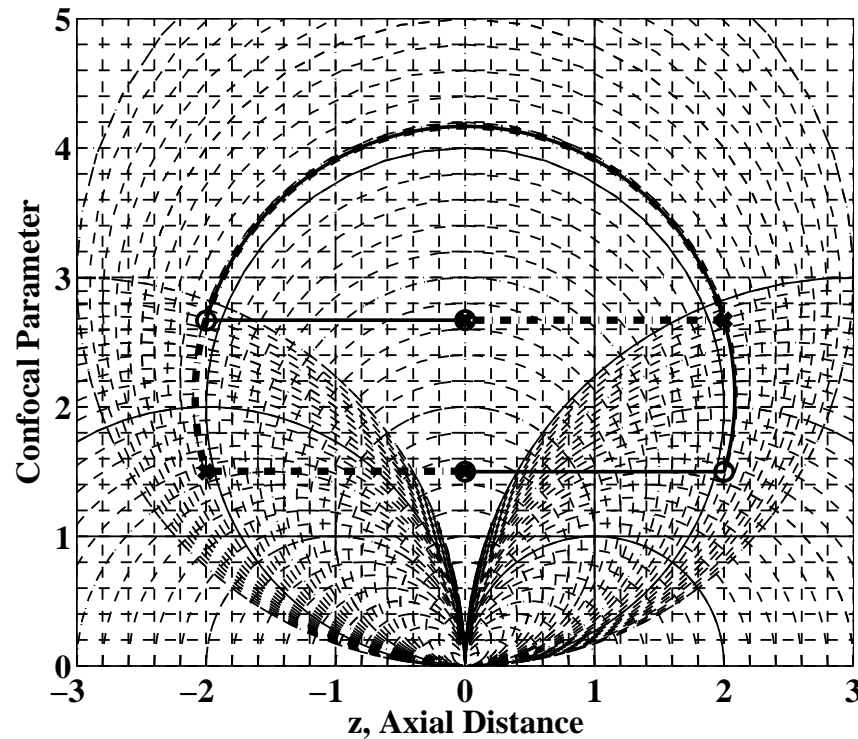
The Collins Chart

Lines of Constant z (vertical) and Constant b (horizontal)
Curves of Constant b' (circles) and Constant ρ (semicircles)
Now Mostly a Visualization Aid

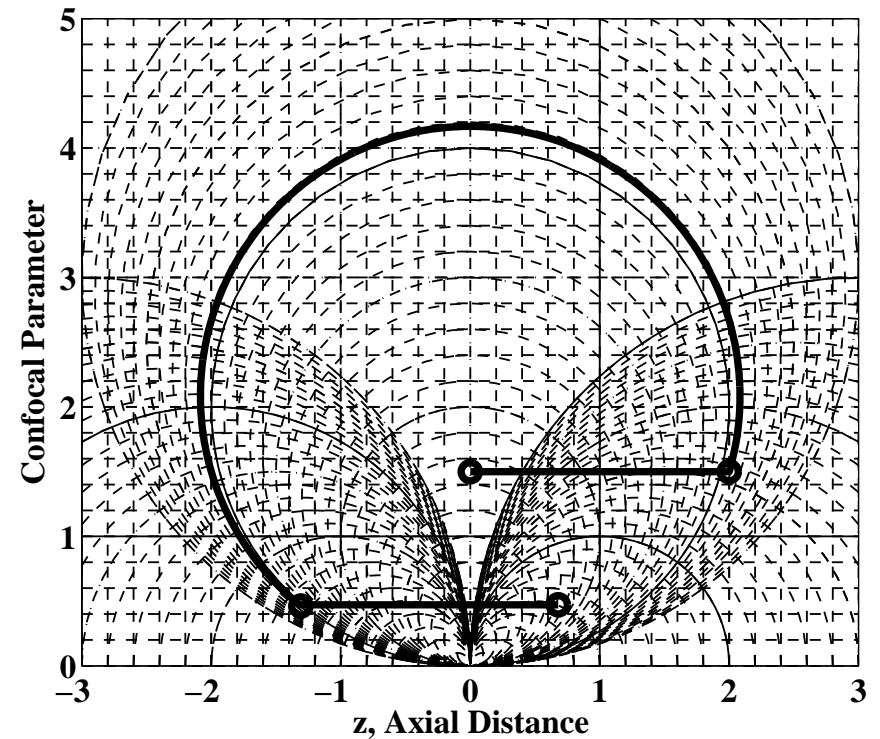


- Translation (Horizontal)
- z Changes, b Doesn't
- Focusing (Along Circles)
- ρ Changes, d & b' Don't

Two 1:1 Relays on Collins Chart

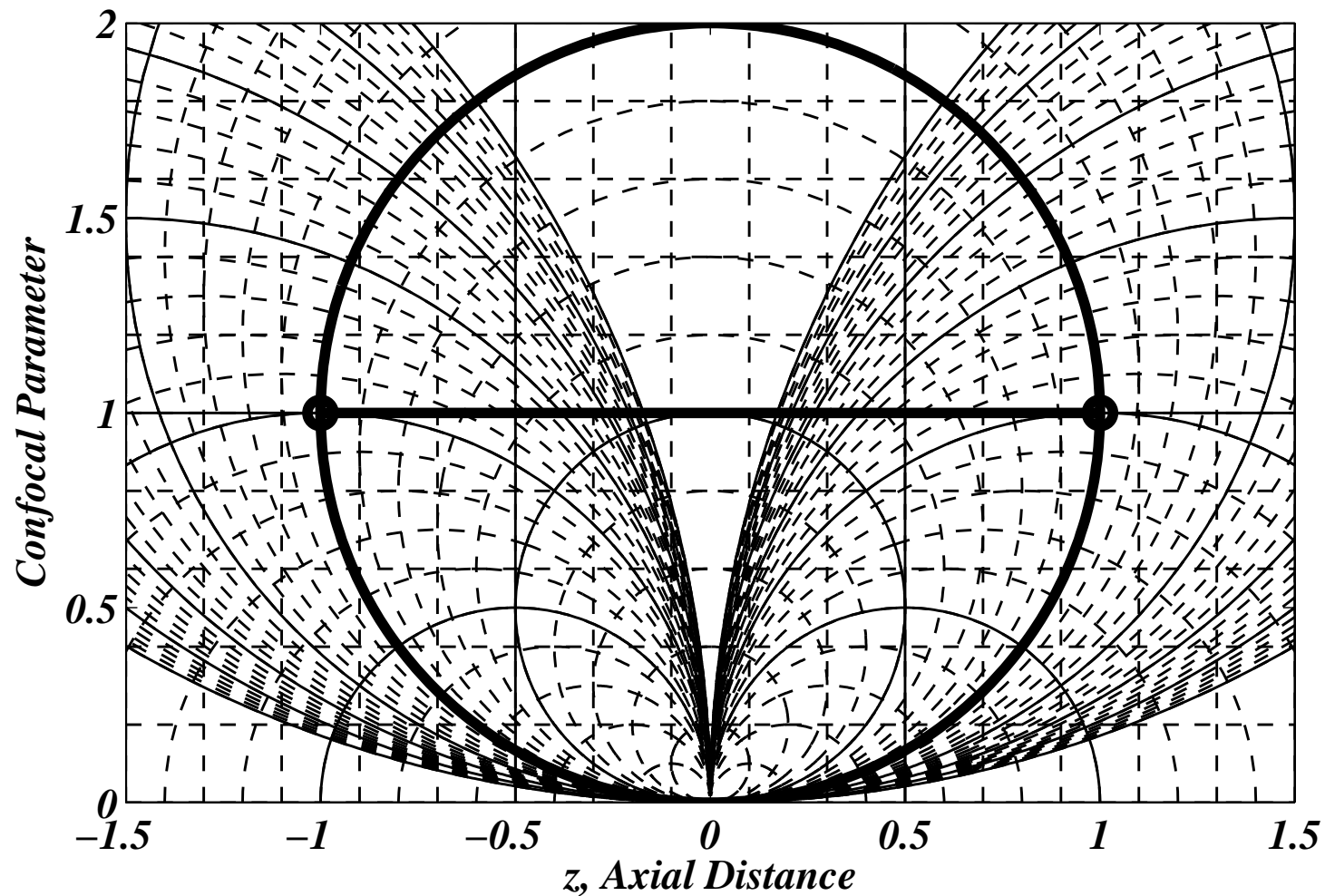


A. Afocal Telescope

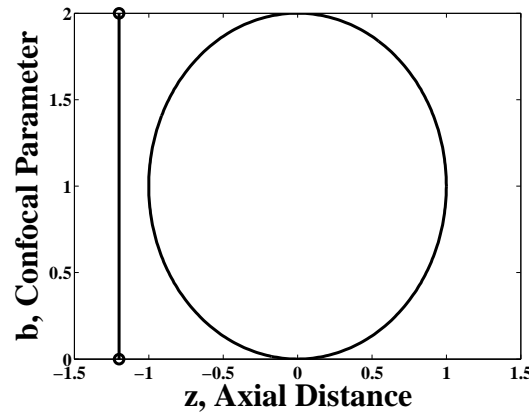


B. Single Lens

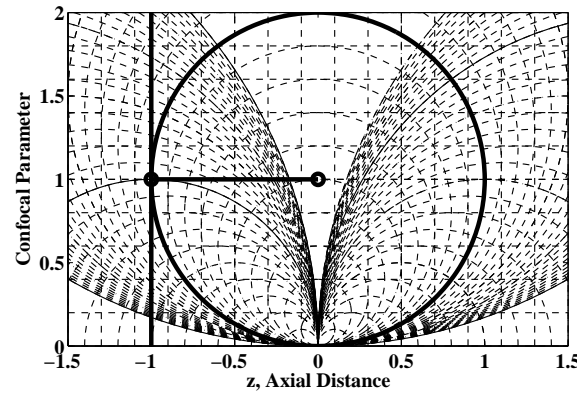
Minimum Beam Diameter for Distance z_{12}



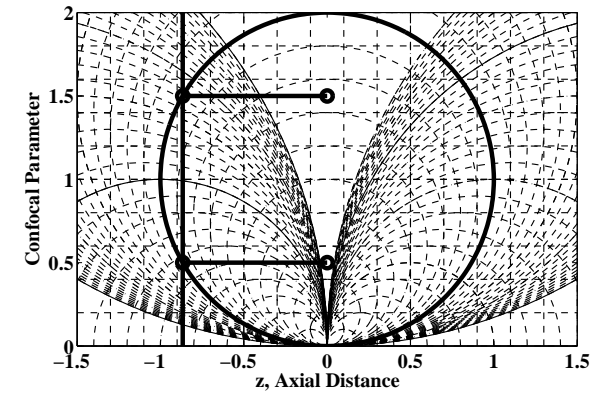
Waist at $-z$ for diameter, d



B. No Solution
 $|z| > b'/2$



C. One Solution
 $|z| = b'/2$



D. Two Solutions
 $|z| < b'/2$

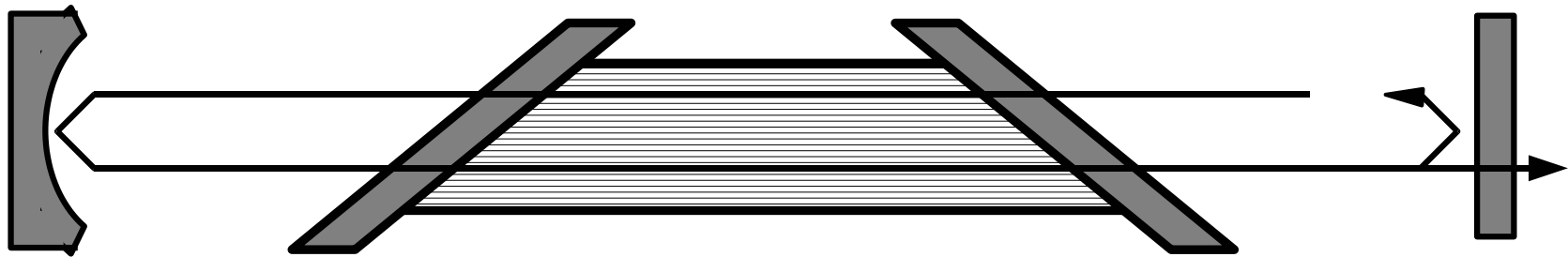
- Question 3

$$b' = \frac{b^2 + z^2}{b}$$

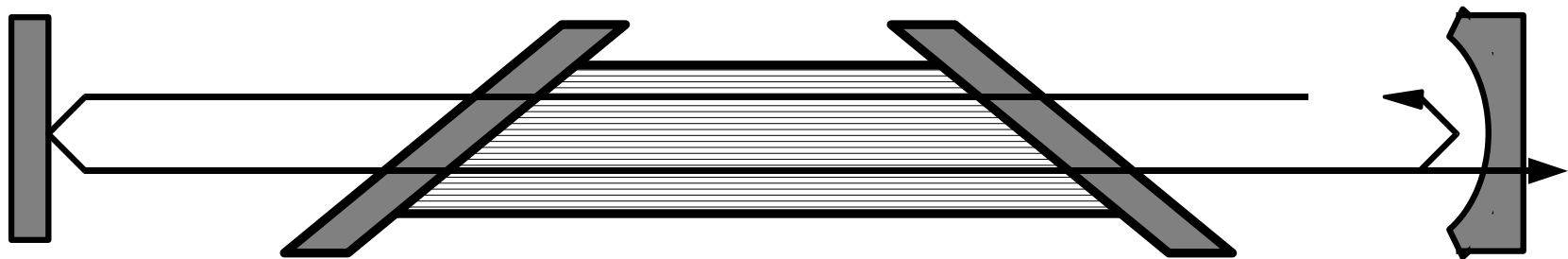
Solve for b only if $(z < b'/2)$

$$b = \frac{b' \pm \sqrt{b'^2 - 4z^2}}{2} \quad \rho = z + \frac{b^2}{z}$$

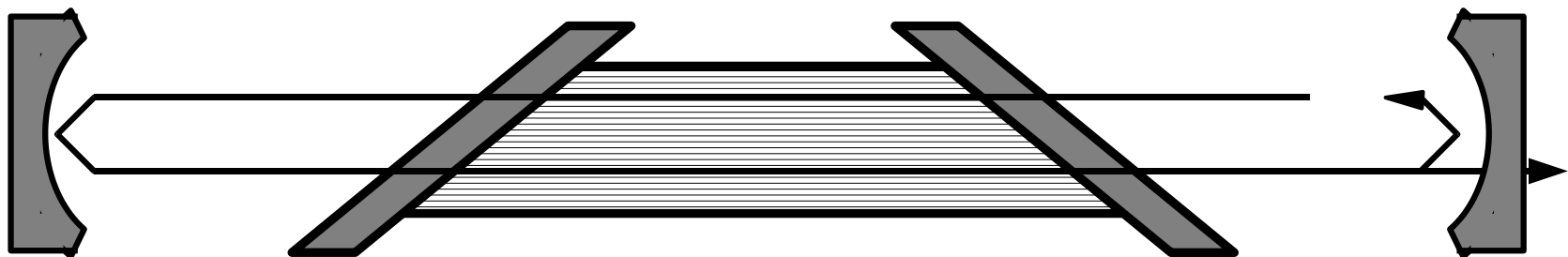
Stable Laser Cavity Design



A. Flat Output Coupler



B. Flat Rear Mirror



C. Confocal Resonator

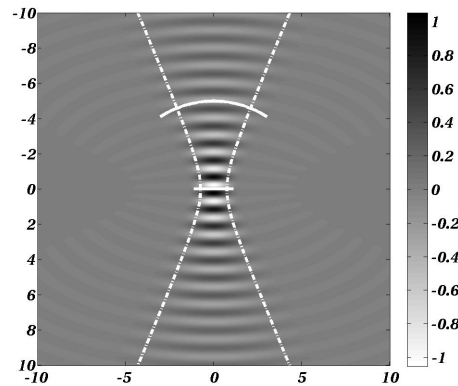
Steady State in Laser Cavity

- The amplitude after a round trip is unchanged. This means that any loss (including power released as output) must be offset by corresponding gain. (Gain Saturation)
- The phase after a round trip must be unchanged. We discussed, in our study of interference, how this requirement on the axial phase change, e^{jkz} , affects the laser frequency.
- The beam shape must be unchanged, so that the phase and amplitude is the same for all x and y . This is the subject to be considered in this section.

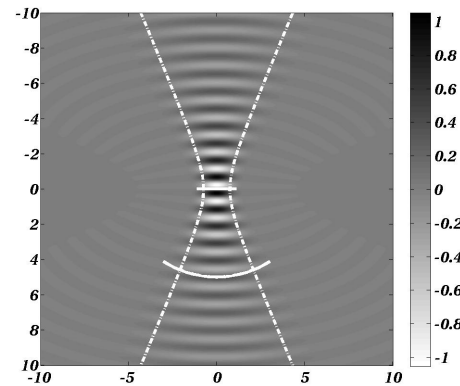
Design Problem

- Carbon Dioxide Laser: P(20), $\lambda = 10.59\mu\text{m}$
- Beam Output: Collimated, 5mm Diameter
- Cavity Length: 1m (Probably because of Gain)
- Solution
 - Collimated Output: Flat Output Coupler
 - Rear Mirror to Match Curvature at $z = -1\text{m}$
$$b = 1.85\text{m} \quad \rho = -4.44\text{m} \quad d = 5.7\text{mm}$$
 - Rear Mirror Concave, $\rho = -4.44\text{m}$
 - Diameter Larger than $d = 5.7\text{mm}$ (Typically 1.5X)

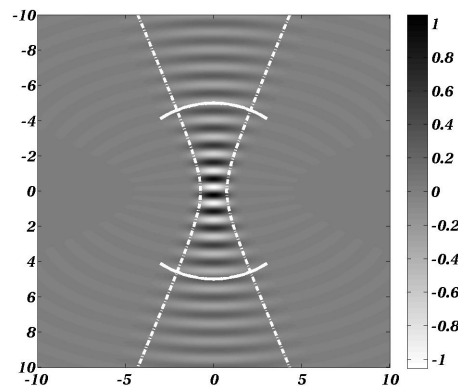
Stable Cavity Examples



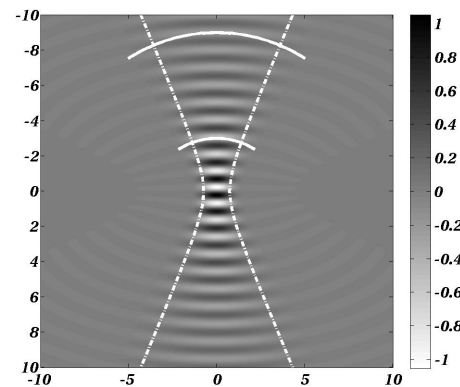
A. Flat Output Coupler



B. Flat Rear Mirror



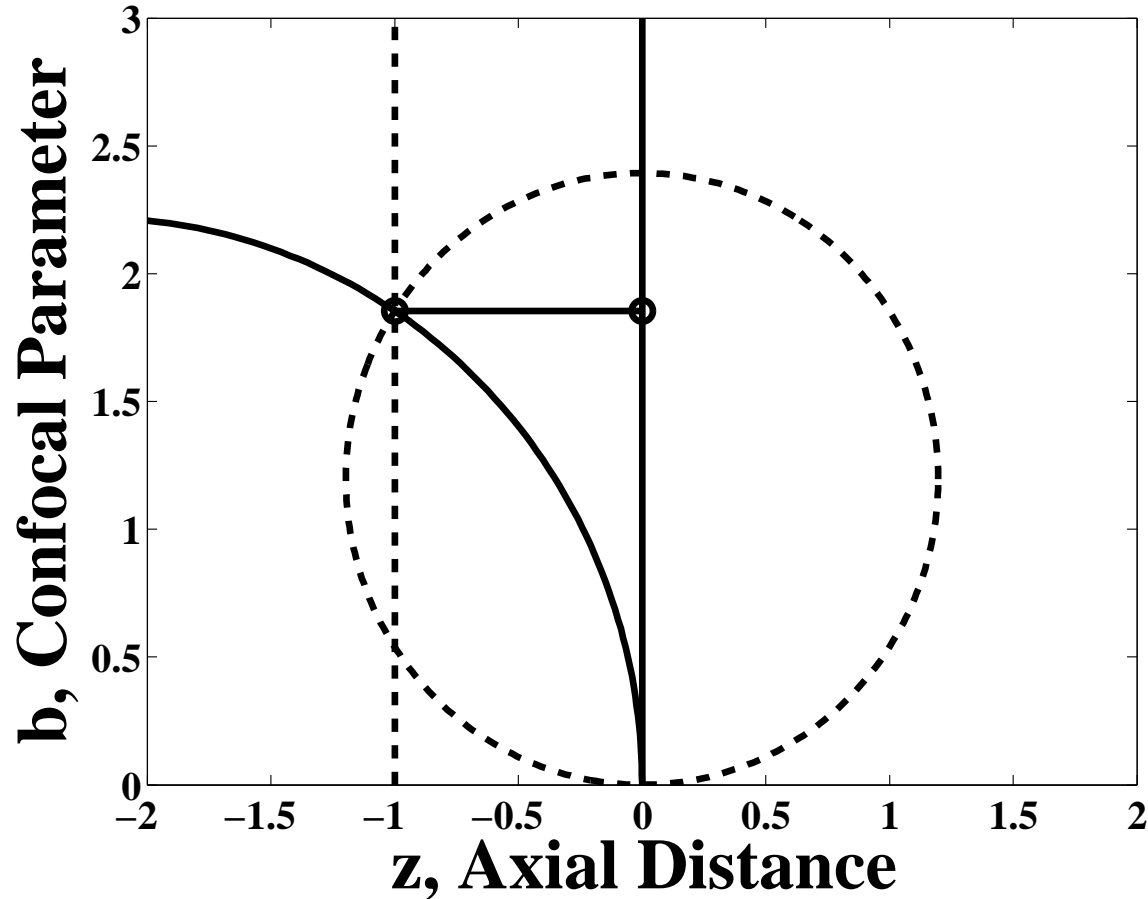
C. Confocal Resonator



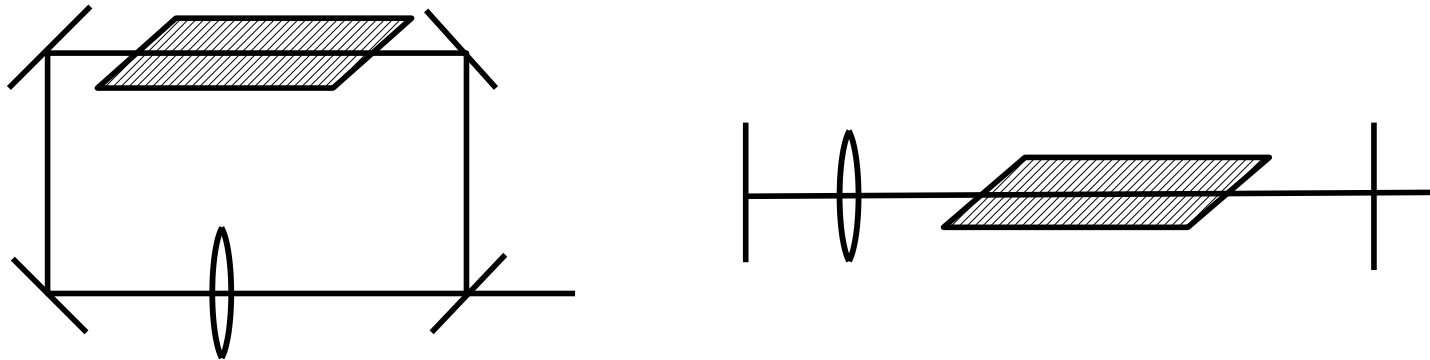
D. Focusing Cavity

Laser Cavity on the Collins Chart

Given ρ_1 ρ_2 and z_{12}



More Complicated Cavities



- Ring Laser with Focusing Lens
- Intracavity Lens (Potential Loss)
- Many Other Configurations

Matrix Optics for Stable Cavities

- Round-Trip Equation

$$q_{out} = q_{in}$$

- Need to Solve This:

$$q = \frac{Aq + B}{Cq + D} \quad Cq^2 + (D - A)q - B = 0$$

- Result (Need Imaginary Solutions)

$$q = \frac{A - D \pm \sqrt{(A - D)^2 + 4CB}}{2C}$$

$$(A - D)^2 + 4CB < 0$$

Stability Condition

- Argument of Square Root Must be Negative

$$(A - D)^2 + 4CB < 0$$

- Determinant Condition ($n = n'$ on Round Trip)

$$AD - CB = 1$$

- Result

$$(D - A)^2 + 4DA < 4 \quad (D + A)^2 < 4$$

Summary of Stable Cavity Design

- Stable cavities or stable optical resonators have mirrors and possibly other optical elements that cause a Gaussian beam to replicate itself upon one round trip.
- Resonator design can be accomplished using Gaussian beam propagation equations.
- Most cavities will support only one Gaussian beam, and the beam parameters are determined by the curvature of the mirrors and their spacing along with the behavior of any intervening optics.
- Matrix optics can be used to determine the round-trip matrix, which can be used to find the Gaussian beam parameters.
- The Collins chart can help to evaluate the stability of a laser cavity.

Hermite–Gaussian Modes

- Mode Definitions (Solutions of Helmholtz Equation)

$$h_{mn}(x, y, z) = h_m(x, z) h_n(y, z)$$

- One–Dimensional Functions with Hermite Polynomials, H

$$h_m(x, z) = \left(\frac{2}{\pi}\right)^{1/4} \sqrt{\frac{1}{2^m m! w}} H_m\left(\frac{x}{w}\right) e^{\frac{x^2}{w^2}} e^{\frac{jkx^2}{2\rho}} e^{j\psi_m}$$

- Hermite Polynomials

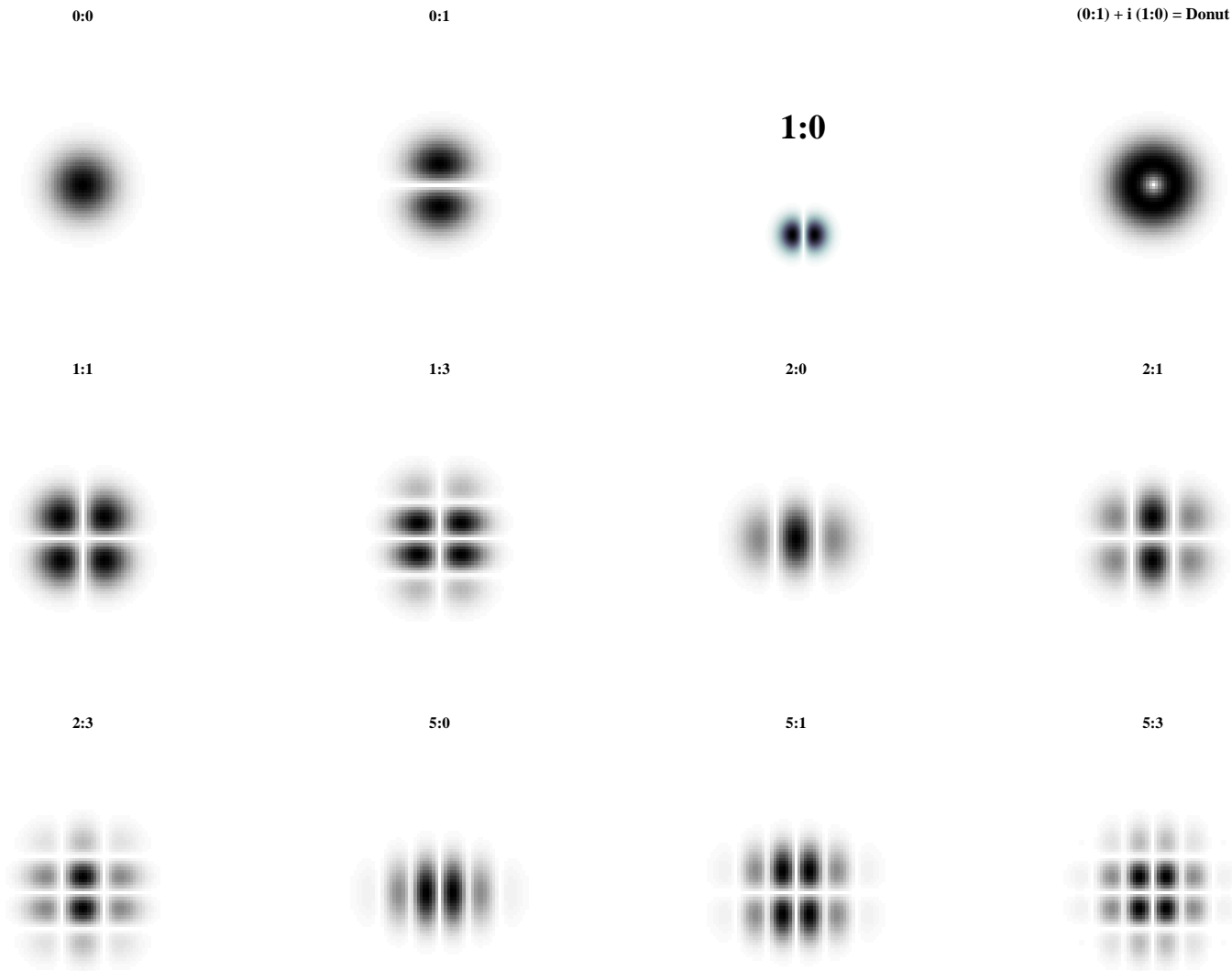
$$H_1(x) = 1 \quad H_2(x) = 2x$$

$$H_{m+1}(x) = 2xH_m(x) - 2(m-1)H_{m-1}(x)$$

- Gouy Phase for Hermite–Gaussian Modes

$$\psi_m = \left(\frac{1}{2} + m\right) \arctan \frac{z}{b}$$

Some Hermite–Gaussian Modes



Expansion in Hermite–Gaussian Modes

- Expand and Field in Hermite–Gaussian Modes
- Ortho–Normal Basis Set

$$E(x, y, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn} h_{mn}(x, y, z)$$

- May Take a Lot of Terms
- Power in Each Mode (and Phase)

$$P_{mn} = C_{mn} C_{mn}^* \quad \phi = \angle C_{mn}$$

Finding the Coefficients, C

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x, y, z) h_{m'n'}^*(x, y, z) dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn} h_{mn}(x, y, z) h_{m'n'}^*(x, y, z) dx dy$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x, y, z) h^*(x, y, z) dx dy = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{mn}(x, y, z) h_{m'n'}^*(x, y, z) dx dy$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{mn}(x, y, z) h_{m'n'}^*(x, y, z) dx dy = \delta_{m,m'} \delta_{n,n'} \quad \text{Ortho-Normality}$$

$$C_{m'n'} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x, y, z) h_{m'n'}^*(x, y, z) dx dy.$$

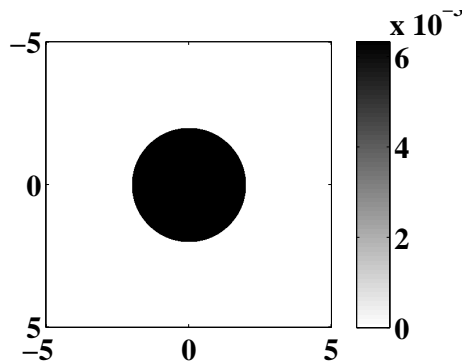
Picking the Best Waist Size

- H–G Expansion Works for Any w
- But it Might Not Converge Nicely
- Pick the Best w to Maximize Power in TEM_{00} Term

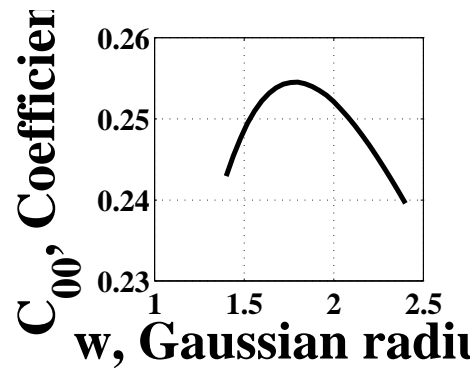
$$C_{00} = \min [C_{00}(w)]$$

$$C_{00}(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x, y, z) h_{00}^*(x, y, z) dx dy$$

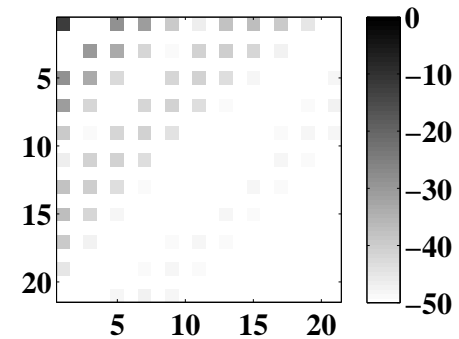
Expanding Uniform Circular Wave in H-G Modes



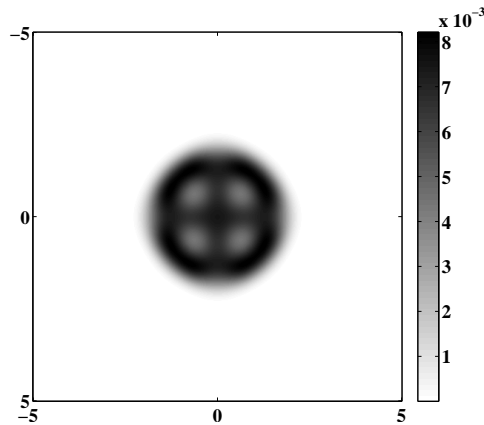
A.



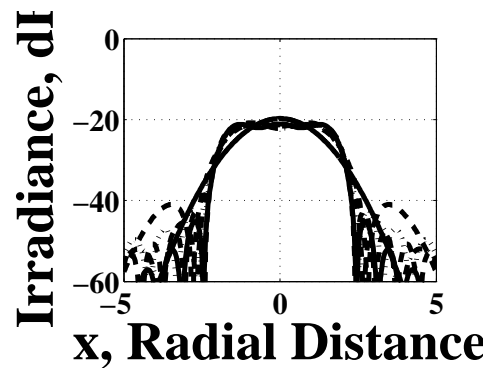
B.



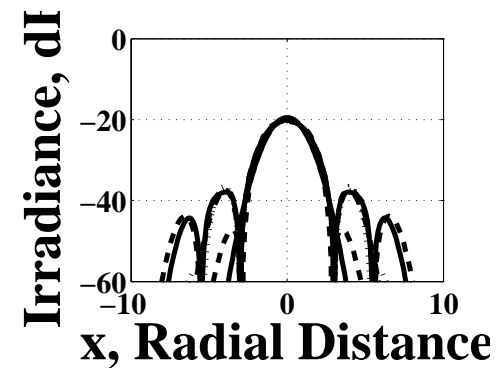
C.



D.



E.



F.

Coupling Equations and Mode Losses

- Integration over Finite Aperture

$$K_{mnm'n'} = \int \int_{\text{aperture}} h_{mn}(x, y, z) h_{m'n'}^*(x, y, z) dx dy = \delta_{m,m'} \delta_{n,n'}$$

$$\begin{pmatrix} C'_{00} \\ C'_{01} \\ C'_{10} \\ \vdots \end{pmatrix} = \begin{pmatrix} K_{0000} & K_{0001} & K_{0010} & \dots \\ K_{0100} & K_{0101} & K_{0110} & \dots \\ K_{1000} & K_{1001} & K_{1010} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} C_{00} \\ C_{01} \\ C_{10} \\ \vdots \end{pmatrix}$$

Transverse Mode Beating

- Round Trip

$$2k\ell + 2(1 + m + n) \left(\arctan \frac{z_2}{b} - \arctan \frac{z_1}{b} \right) = 2\pi N$$

$$2 \frac{f(m, n, N)}{c} \ell + \frac{1}{\pi} (1 + m + n) \left(\arctan \frac{z_2}{b} - \arctan \frac{z_1}{b} \right) = N$$

$$f(m, n, N) = N \frac{c}{2\ell} - \frac{c}{2\ell} (1 + m + n) \frac{\arctan \frac{z_2}{b} - \arctan \frac{z_1}{b}}{\pi}$$

- Transverse Mode Beat Frequency

$$f(1, n, N) - f(0, n, N) = \frac{c}{2\ell} \frac{\arctan \frac{z_2}{b} - \arctan \frac{z_1}{b}}{\pi}$$

- Carbon Dioxide Laser $b = 1.85\text{m}$, $z = 1\text{m}$

$$f(1, n, N) - f(0, n, N) = \frac{c}{2\text{m}} \frac{\left(\arctan \frac{1\text{m}}{1.85\text{m}} - \arctan \frac{0\text{m}}{1.85\text{m}} \right)}{\pi} = 24\text{MHz}$$

Summary of H–G Modes

- Hermite–Gaussian modes (Equation) form an orthogonal basis set of functions which can be used to describe arbitrary beams.
- The beam can be expanded in a series using Equation .
- Coefficients of the expansion can be calculated using Equation .
- Each mode maintains its own profile with a size w and radius of curvature, ρ given by the usual Gaussian–beam equations.
- The choice of w in Equation is arbitrary, but a value chosen badly can lead to poor convergence of the series expansion.
- Higher order modes have increased Gouy phase shifts as shown in Equation .
- Beam propagation problems can be solved by manipulating the coefficients with appropriate Gouy phase shifts.
- Apertures can be treated using coupling coefficients which can be computed by Equation .
- Higher–order transverse modes in a laser can mix with each other to produce unwanted mixing, or beat, signals.
- The higher–order modes can be eliminated by placing an appropriate aperture inside the laser cavity.