Optics for Engineers
Chapter 10

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Coherent Sum

- Adding Fields (Chapter 7)

$$I = E_1^* E_1 + E_2^* E_2 + E_1^* E_2 + E_1 E_2^*$$

- Mach–Zehnder Interferometer

$$E_1(t) = E_s\left(t - \frac{z_1}{c}\right) \qquad E_2(t) = E_s\left(t - \frac{z_2}{c}\right)$$

- Mixing Term: (Source, $E_s(t) = E_0 e^{j\omega t}$, with $k = 2\pi/\lambda = \omega/c$)

$$E_2 E_1^* = E_s(t - z_2/c)\, E_s^*(t - z_1/c) =$$

$$E_0(t - z_2/c)\, E_0^*(t - z_1/c)\, e^{j(\phi_2 - \phi_1)}$$

$$\phi_1 = k z_1 \qquad \phi_2 = k z_2$$

Irradiance of Coherent Sum

- Coherent Light as in Chapter 7

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\left(\phi_2 - \phi_1\right)$$

- Visibility for Equal Irradiances

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{4I_1 - 0}{4I + 0} = 1$$

- Random Source (E_0 Varies Randomly)

$$\langle E_2 E_1^* \rangle = \langle E_0\left(t - z_2/c\right) E_0^*\left(t - z_1/c\right)\rangle\, e^{j(\phi_2 - \phi_1)}$$

$$\langle E_2 E_1^* \rangle = \langle E_0\left(t\right) E_0^*\left(t - \tau\right)\rangle\, e^{j(\phi_2 - \phi_1)} \qquad \tau = \frac{z_2 - z_1}{c}$$

Coherence

- Previous Result

$$\langle E_2 E_1^* \rangle = \langle E_0(t)\, E_0^*(t-\tau) \rangle\, e^{j(\phi_2-\phi_1)} \qquad \tau = \frac{z_2 - z_1}{c}$$

- Temporal Autocorrelation Function, Γ

$$I = I_1 + I_2 + 2\Re\left[\Gamma e^{j(\phi_2-\phi_1)}\right] \qquad \Gamma = \langle E_0^*(t-\tau)\, E_0(t) \rangle$$

- Visibility with $I_1 = I_2$

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{2I_1(1+|\Gamma|) - 2I_1(1-|\Gamma|)}{2I_1(1+|\Gamma|) + 2I_1(1-|\Gamma|)}$$

$$V = \gamma = \frac{4\,|\Gamma|}{4I} = \frac{|\Gamma|}{I}$$

Temporal Autocorrelation Function

- General Definition (See Previous Page)

$$\Gamma = \langle E_0^*(t-\tau)\, E_0(t)\rangle \qquad \gamma(\tau) = \frac{\Gamma(\tau)}{\left|E_1\right|\left|E_2^*\right|}$$

- Short Times: Equal (Or at Least Correlated) Fields

$$\Gamma = \langle E_1^*(t)\, E_1(t)\rangle$$

$$\gamma(\tau) \to 1 \qquad \text{as} \qquad \tau \to 0$$

- Long Time: Uncorrelated Fields

$$\Gamma = \langle E_1^*(t-\tau)\, E_1(t)\rangle = \langle E_1^*(t-\tau)\rangle \langle E_1(t)\rangle$$

$$\gamma(\tau) \to 0 \qquad \text{as} \qquad \tau \to \infty$$

Coherent and Incoherent Addition

- Coherent Addition (*e.g.* Short Times)

$$E = E_1 + E_2 \qquad I = |E|^2$$

- Incoherent Addition (*e.g.* Long Times)

$$\langle I \rangle = \langle I_1 \rangle + \langle I_2 \rangle \qquad \text{(Incoherent Addition)}$$

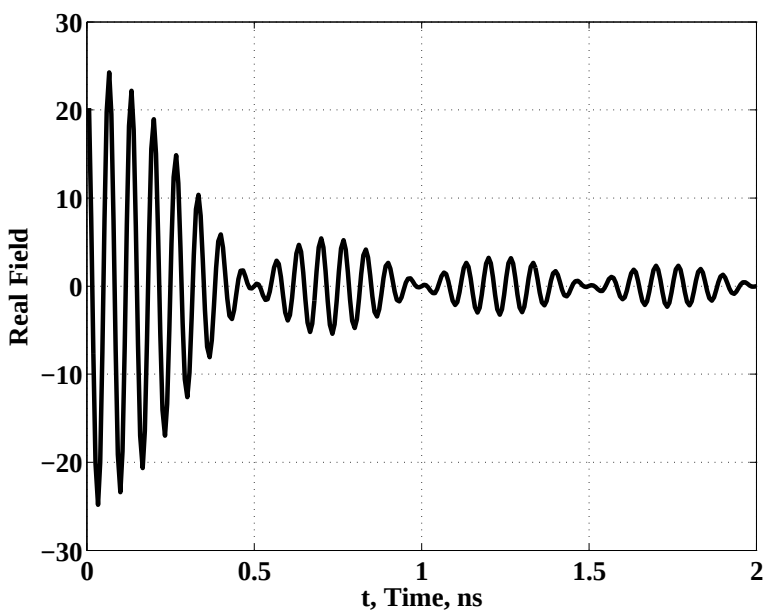
- Need to Quantify Times

Digression: Mode–Locked Lasers

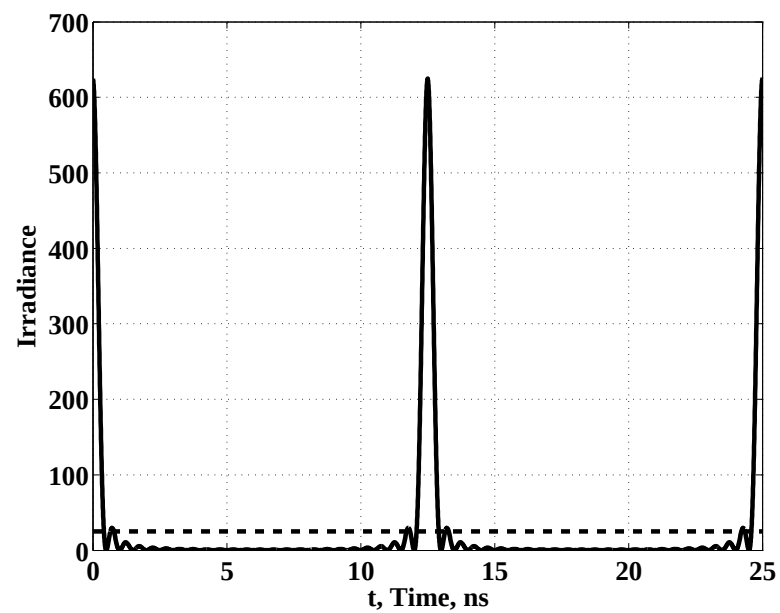
- Sum of Longitudinal Modes: *e.g.* 25 Modes

$$E = \sum_{m=-12}^{12} E_m = \sum_{m=-12}^{12} e^{i2\pi[f_{center}+m\times FSR]t}$$

- Transform–Limited Pulses Carrier Frequency Reduced for Clarity

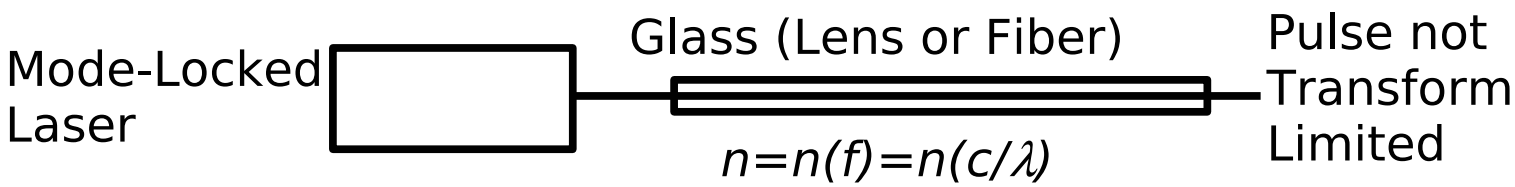


A. Mode–Locked Field

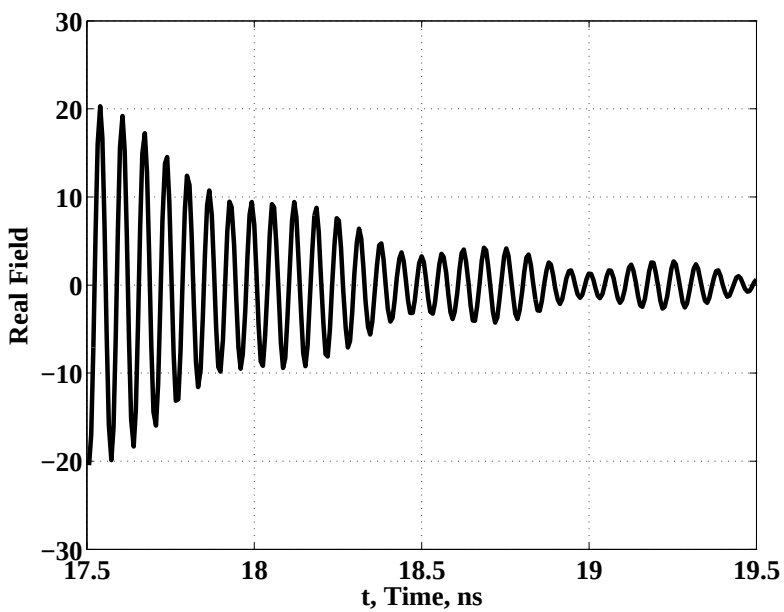


B. Mode–Locked Irradiance

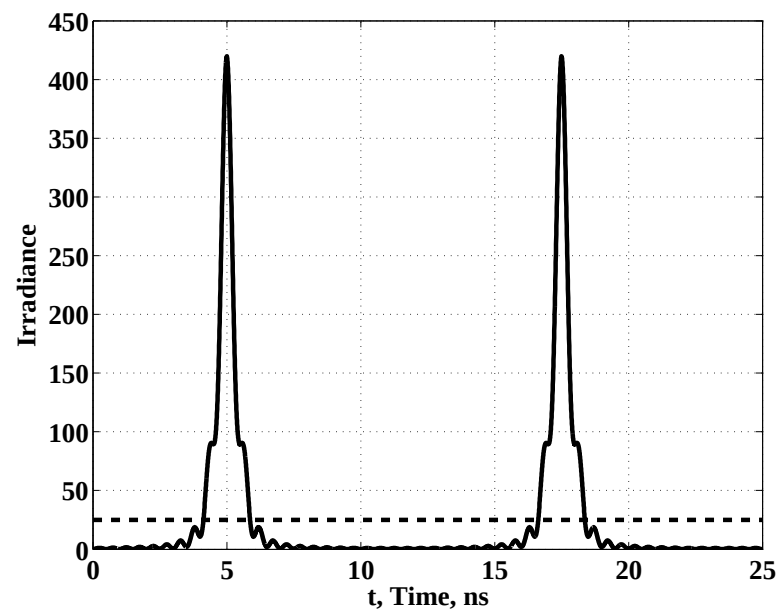
Pulse Dispersion



- Mode–Locked Laser Passing Through Glass
- "Chirped Pulses:" *e.g.* $\frac{dt}{df} = -(1/3) \times 10^{-18}$sec/Hz
- Pulses Broadened in Time (Not Transform–Limited)



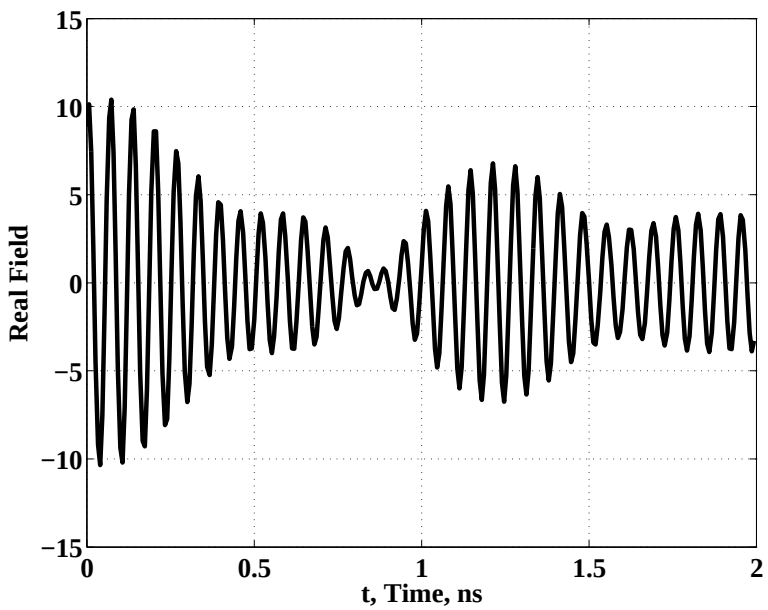
C. Chirped Field



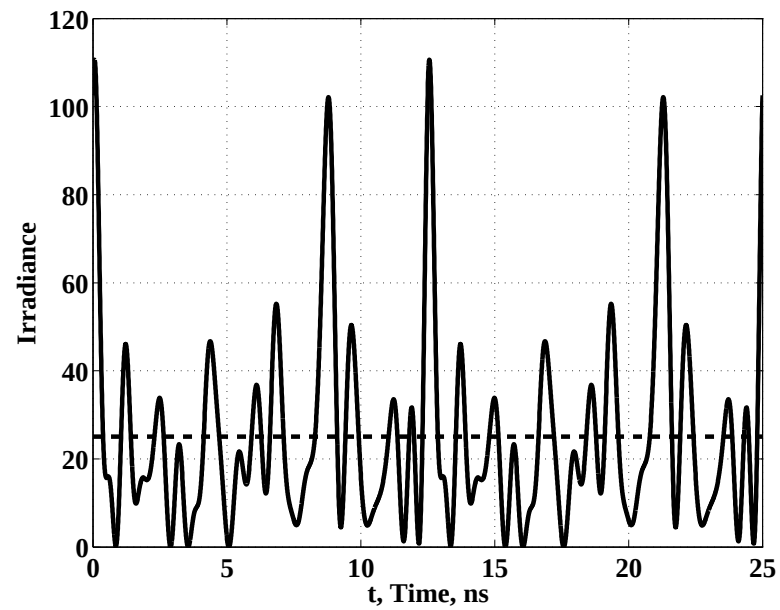
D. Chirped Irradiance

Free–Running Multi–Mode Laser

- "Unlocked" Laser
- Random Output with No Predictable Pulses
- Occasional "Hot Pulse"



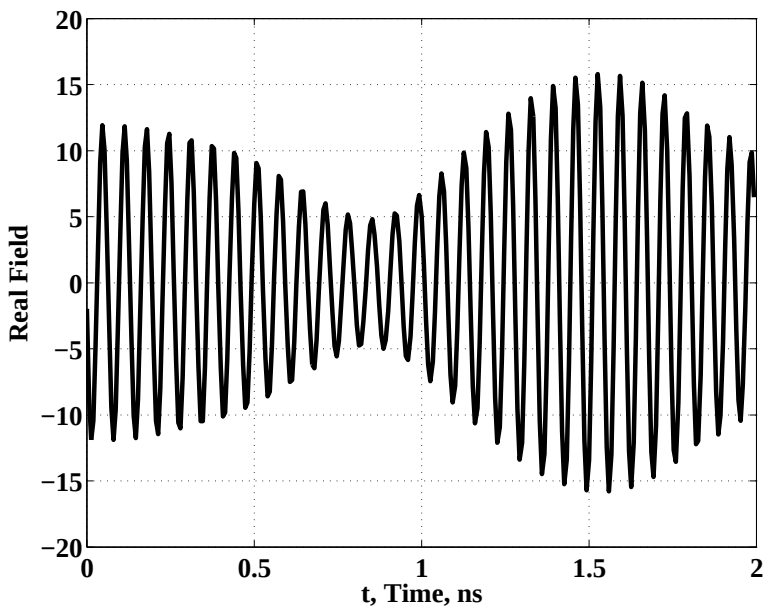
E. Unlocked Field



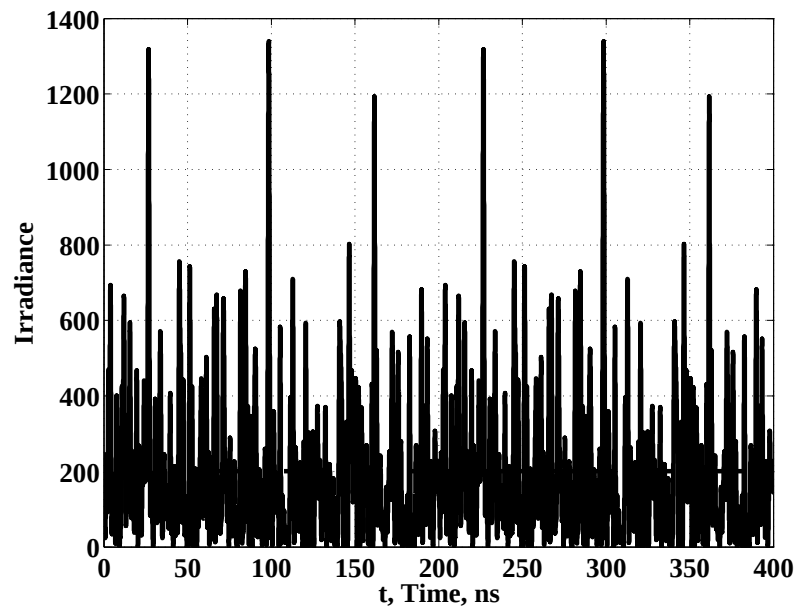
F. Unlocked Irradiance

Multi–Mode Laser: Incoherent Addition

- 201 Modes 5MHz apart with Random Phases
- Short–Time Coherence ($\ll$ 1/Bandwidth)



A. Field (Short Time)



B. Irradiance (Long Time)

Temporal Coherence (1)

- Fourier Transforms

$$\tilde{E}(\omega) = \int_{-\infty}^{\infty} E(t)\, e^{-j\omega t} dt$$

$$E(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}(\omega)\, e^{j\omega t} d\omega$$

- Autocorrelation

$$\Gamma(\tau) = \langle E^*(t-\tau)\, E(t) \rangle$$

$$E(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}(\omega)\, e^{j\omega t} d\omega$$

$$E^*(t-\tau) =$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}^*\left(\omega'\right) e^{-j\omega'(t-\tau)} d\omega'$$

- Combine and Simplify

$$\Gamma(\tau) = \left\langle \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}^*\left(\omega'\right) e^{-j\omega'(t-\tau)} d\omega' \ldots \right.$$

$$\left. \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}(\omega)\, e^{j\omega t} d\omega \right\rangle$$

$$\Gamma(\tau) = \left\langle \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \right.$$

$$\left. \tilde{E}^*\left(\omega'\right) \tilde{E}(\omega)\, e^{j\omega'\tau} e^{j(\omega-\omega')t} d\omega' d\omega \right\rangle$$

- Exchange Average and Integrals

$$\Gamma(\tau) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \ldots$$

$$\left[\int_{-\infty}^{\infty} \left\langle \tilde{E}^*\left(\omega'\right) \tilde{E}(\omega) \right\rangle e^{j\omega'\tau} e^{j(\omega-\omega')t} d\omega' \right] d\omega$$

Temporal Coherence (2)

- Previous Result

$$\Gamma(\tau) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left\langle \tilde{E}^*\left(\omega'\right)\tilde{E}(\omega)\right\rangle e^{j\omega'\tau}e^{j(\omega-\omega')t}d\omega'\right]d\omega$$

- Independent Sources with $\left\langle \tilde{E}(\omega)\right\rangle = 0$

$\langle E_m E_n^* \rangle = I_m \delta_{m-n}$ (Discrete Spectrum; *e.g.* Multi-Mode Laser)

$\left\langle \tilde{E}^*\left(\omega'\right)\tilde{E}(\omega)\right\rangle d\omega' = \tilde{I}\left(\omega'\right)\delta\left(\omega-\omega'\right)d\omega'$ (Continuous Spectrum *e.g.* LED)

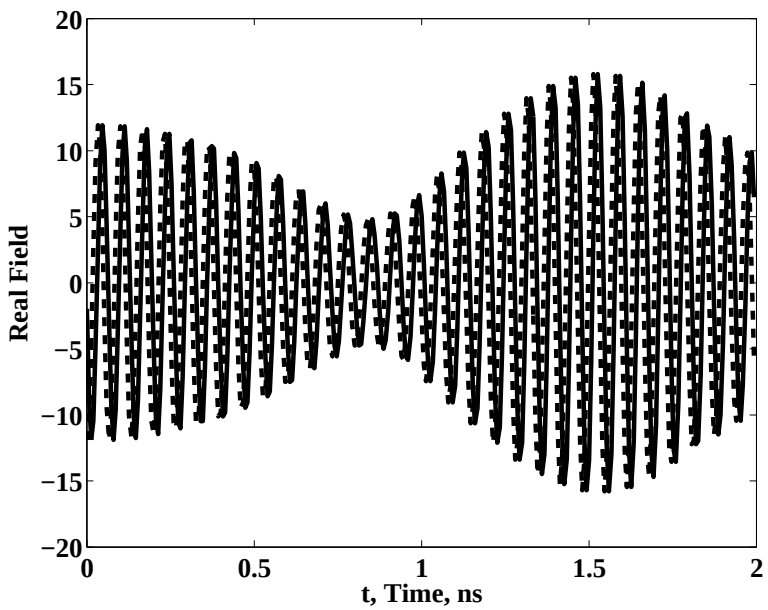
- Result

$$\left[\int_{-\infty}^{\infty}\left\langle \tilde{E}^*\left(\omega'\right)\tilde{E}(\omega)\right\rangle e^{j\omega'\tau}e^{j(\omega-\omega')t}d\omega'\right] = \tilde{I}(\omega)\,e^{j\omega\tau}$$

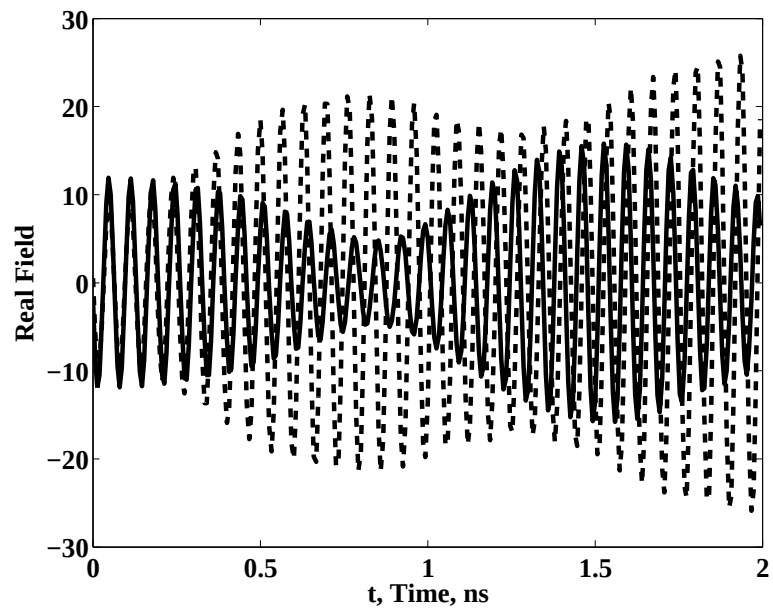
$$\boxed{\Gamma(\tau) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\tilde{I}(\omega)\,e^{j\omega\tau}d\omega \qquad \text{(Weiner–Khintchine)}}$$

Coherence in Interfermetry

- Two Fields Shown (Solid and Dashed)
- Same Source with Different Time Delays
- Linewidth $\approx$ 10GHz



A. 10ps differential delay



B. 2ns differential delay

Example: LED

- Wavelength Spectrum (Gaussian)

$$\lambda_0 = 635\text{nm} \qquad \lambda_{FWHM} = 20\text{nm}$$

$$\left|\frac{\delta\lambda}{\lambda}\right| = \left|\frac{\delta f}{f}\right|$$

- Gaussian Fourier Transform Pair

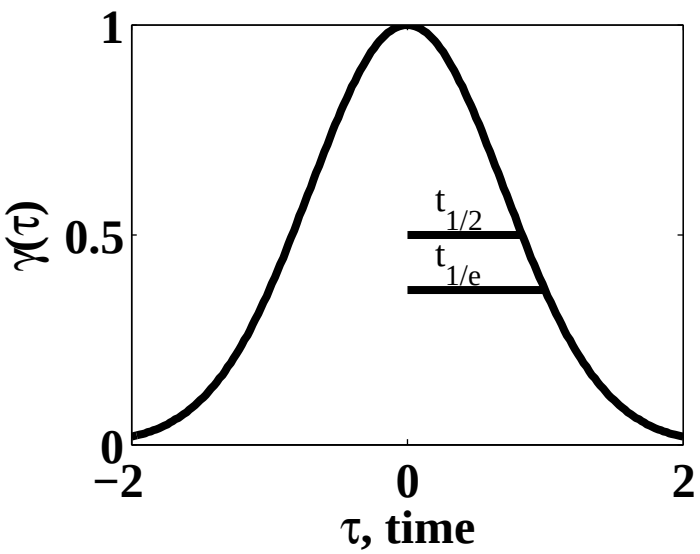
$$\gamma(\tau) = e^{-\left(t/t_{1/e}\right)^2} \qquad \frac{2\sqrt{\pi}}{\omega_{1/e}}\tilde{\gamma}(\omega) = e^{-\left(\omega/\omega_{1/e}\right)^2}$$

- Convert Widths and Complete Calculation

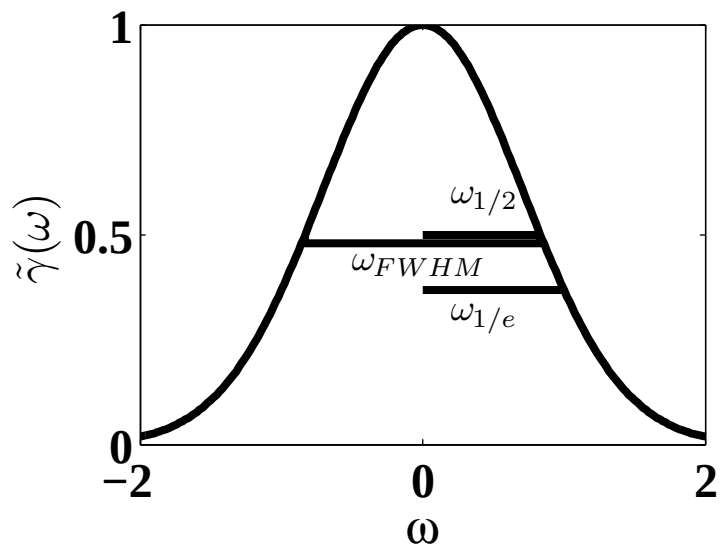
Gaussian Widths Defined

$$e^{-\left(\omega_{1/2}/\omega_{1/e}\right)^2} = \frac{1}{2} \qquad \text{at} \qquad \omega_{1/2} = \sqrt{\ln 2}\,\omega_{1/e}$$

$$2\pi f_{1/2} = \omega_{1/2} = \sqrt{\ln 2}\,\omega_{1/e} \qquad \omega_{1/e} = \frac{2\pi f_{1/2}}{\sqrt{\ln 2}}$$



A. Time Domain



B. Angular Frequency Domain

LED Example Conclusion

$$t_{1/2} = \sqrt{\ln 2}\frac{2\sqrt{2}}{\omega_{1/e}} = \ln 2\frac{2\sqrt{2}}{2\pi f_{1/2}} \approx \frac{0.31}{f_{1/2}} = \frac{0.62}{f_{FWHM}}$$

$$z_{1/2} = ct_{1/2}$$

$$\frac{z_{1/2}}{\lambda} = ft_{1/2} = \ln 2 \times \frac{2\sqrt{2}}{\pi}\frac{f}{f_{FWHM}} \approx 0.62\frac{f}{f_{FWHM}}$$

$$\frac{z_{1/2}}{\lambda} \approx 0.62\frac{f}{\delta f_{FWHM}} = 0.62\frac{\lambda}{\lambda_{FWHM}}, \qquad \text{(Gaussian Spectrum)}$$

$$\frac{z_{1/2}}{\lambda} \approx 0.62 \times \frac{635}{20} \approx 20 \text{ Wavelengths} \qquad z_{1/2} \approx 12.5\mu\text{m}$$

Laser *vs.* LED

- LED (Previous Page: Linewidth = 20nm)

$$\frac{z_{1/2}}{\lambda} \approx 0.62 \times \frac{635}{20} \approx 20\text{Wavelengths} \qquad z_{1/2} \approx 12.5\mu\text{m}$$

- HeNe Laser (30cm Coherence Length)

$$\frac{30 \times 10^{-2}\text{m}}{633 \times 10^{-9}\text{m}} \approx 4.7 \times 10^5 \approx 0.62\frac{f}{f_{FWHM}} = 0.62\frac{\lambda}{\lambda_{FWHM}}$$

$$f_{FWHM} = \frac{0.62f}{4.7 \times 10^5} = \frac{0.62c}{47 \times 10^6\lambda} = 620 \times 10^6\text{Hz}$$

$$\lambda_{FWHM} = \frac{0.62\lambda}{47 \times 10^5} = 830 \times 10^{-15}\text{m}$$

  - Linewidth 620MHz or 830fm.

Coherence in Beamsplitters

- Beamsplitter 5mm Thick ($n = 1.5$: OPD 75mm One Way)

$$\frac{z_{1/2}}{\lambda} = \frac{15 \times 10^{-3}\text{m}}{633 \times 10^{-9}\text{m}} = 23700 \approx 0.62\frac{f}{f_{FWHM}} = 0.62\frac{\lambda}{\lambda_{FWHM}}$$

$$\lambda_{FWHM} = \frac{0.62 \times 633\text{nm}}{23700} = 16.6\text{pm}$$

$$f_{FWHM} = \frac{0.62c}{23700\lambda} = 12.4\text{GHz}$$

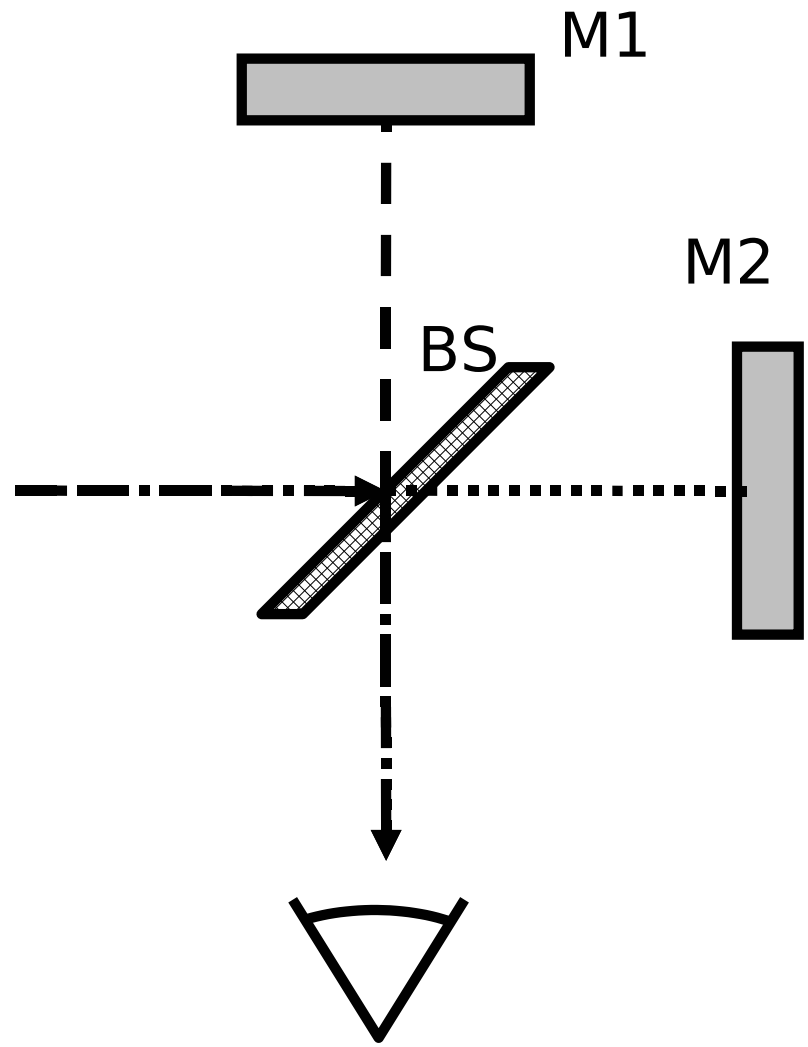
- Laser: Coherent Effects Produce Fringes

- LED: $z_{1/2}$ Much Less than Thickness: No Fringes

Optical Coherence Tomography

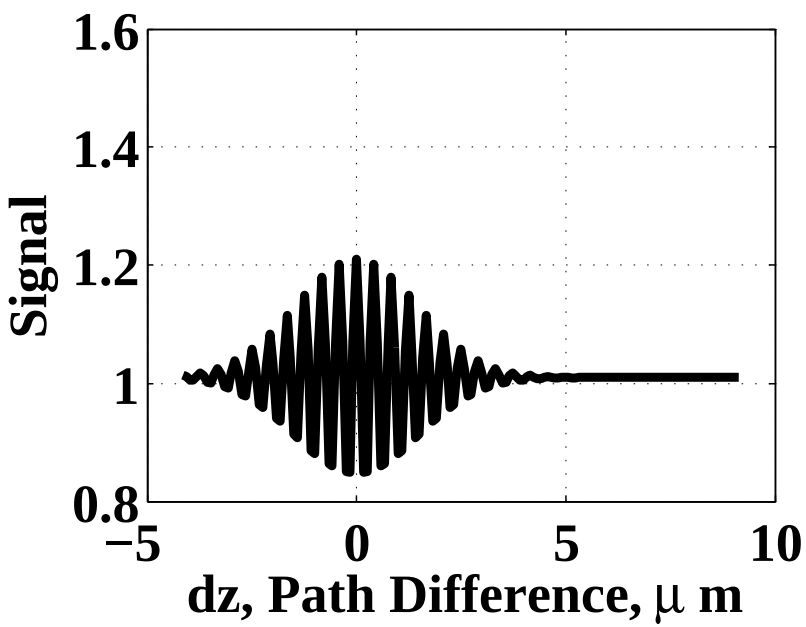
- Short Coherence Source
  - Super–Luminescent Diode
  - Ti:Sap Laser
  - Other
- $M1$ is Reference
- Moving Reference Mirror
- $M2$ is Target
- Interference? Compare. . .
  - Path Difference
  - Coherence Length

- Michaelson Interfereometer

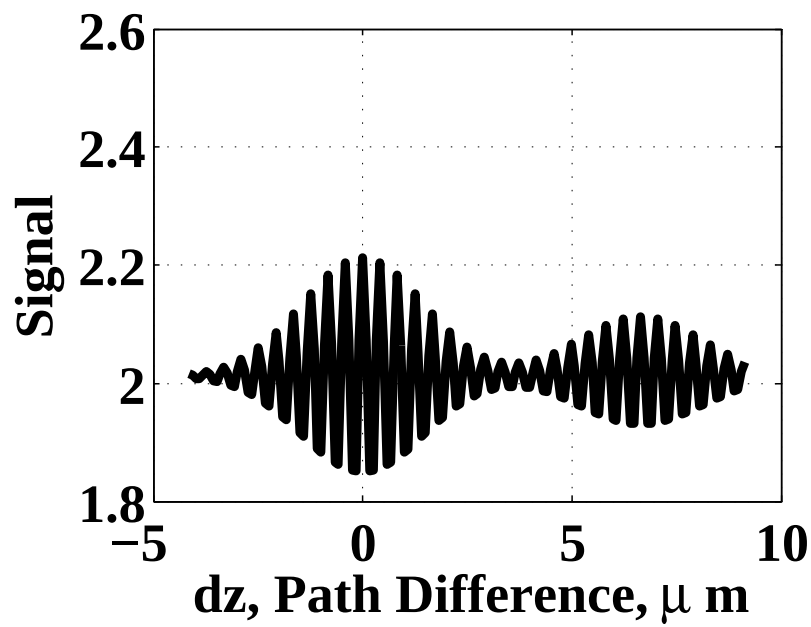


OCT Signals

- Examples with Partial Reflectors
- Air–Glass Interfaces (Simulated Signals)
- Idea Extends to Thick "Distributed" Targets



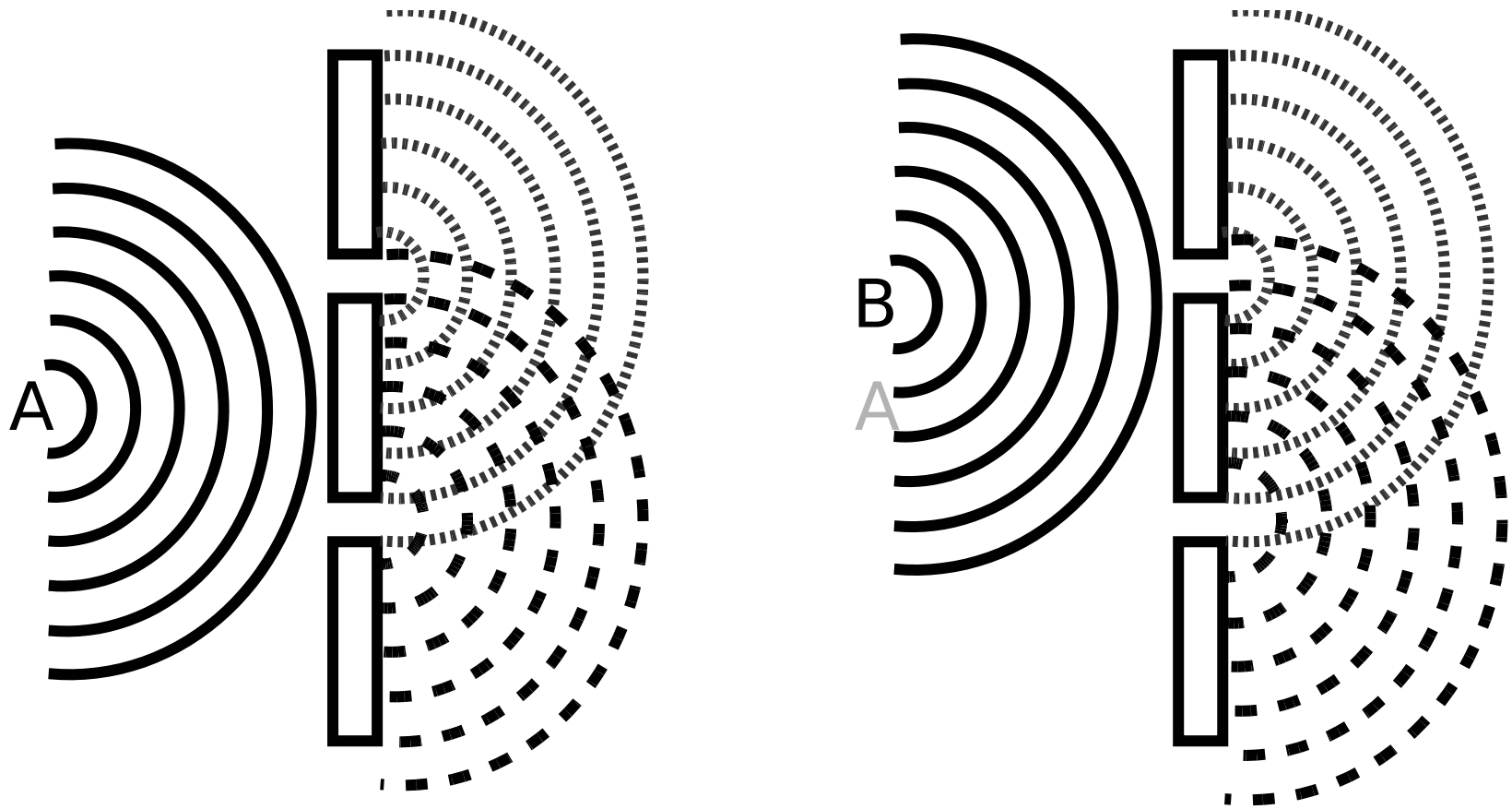
A. Target at Zero



B. Added target at 8μm

Spatial Coherence

- Double Slit with Source at A Produces Diffraction Pattern
- Source at B Produces a Different Diffraction Pattern
- What about Both Together?



Double Slit with Different Sources

- Sources Mutually Coherent

$$\langle |E_1^* E_2| \rangle = \sqrt{|E_1|^2 \, |E_2|^2}$$

  – Some Well–Defined Pattern with High Visiblity

- Incoherent Sources

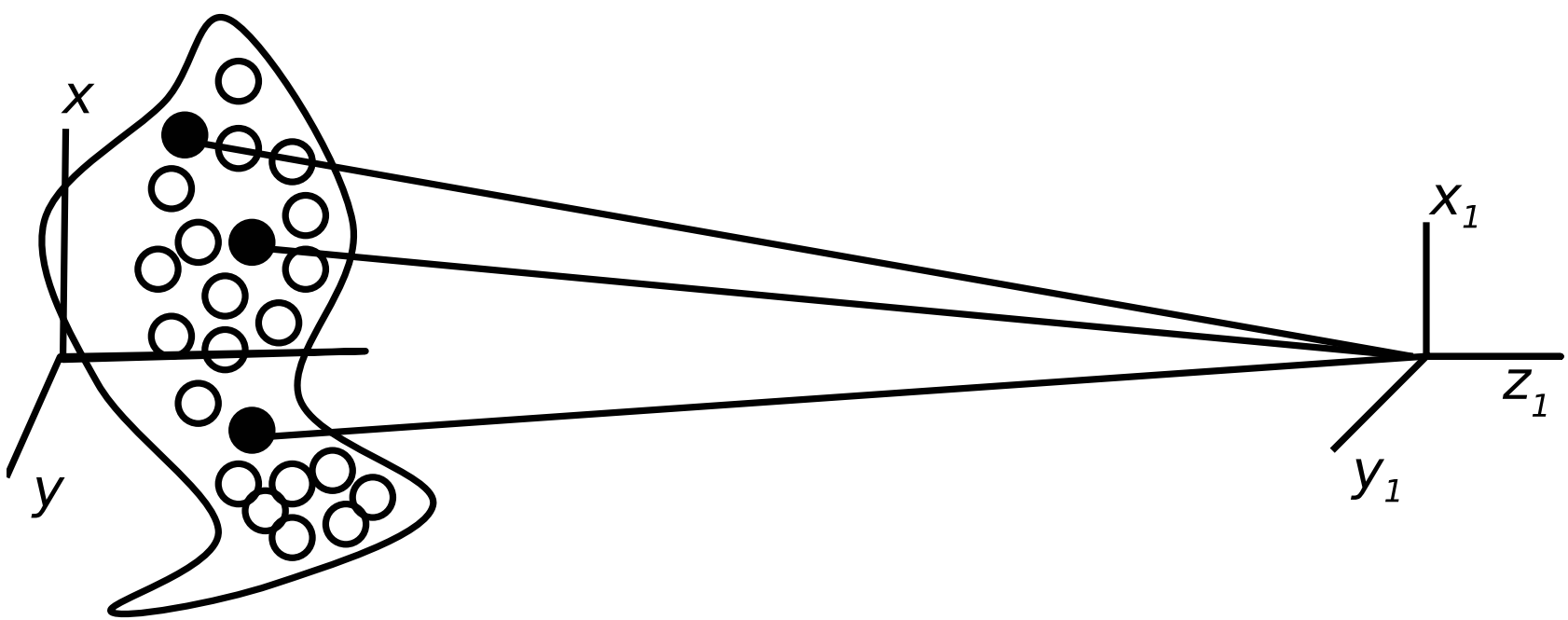
  – Each Produces its Own Diffraction Pattern

  – Peaks and Nulls in Different Places

  – Ultimately Incoherent Addition

Model for Spatial Coherence

- Multiple Point Sources
- Each Independent of the Others
- Compute Coherence Between (x_1, y_1, z_1), and $\left(x_1', y_1', z_1\right)$



Van Cittert–Zernicke Theorem (1)

- Correlation

$$\Gamma(x_1, y_1, x_1', y_1') = \left\langle E^*\left(x_1', y_1', z_1\right) E\left(x_1, y_1, z_1\right)\right\rangle$$

- Fraunhofer Diffraction

$$E\left(x_1, y_1, z_1\right) = \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{x_1^2+y_1^2}{2z_1}} \int\int E\left(x, y, 0\right) e^{jk\frac{x^2+y^2}{2z_1}} e^{-jk\frac{xx_1+yy_1}{z_1}} dxdy$$

$$E^*\left(x_1', y_1', z_1\right) = \frac{-jke^{-jkz_1}}{2\pi z_1} e^{-jk\frac{(x_1')^2+(y_1')^2}{2z_1}} \times$$

$$\int\int E^*\left(x', y', 0\right) e^{-jk\frac{(x')^2+(y')^2}{2z_1}} e^{+jk\frac{x'x_1'+y'y_1'}{z_1}} dx'dy'$$

Van Cittert–Zernicke Theorem (2)

- Combine Previous Equations

$$\Gamma(x_1, y_1, x_1', y_1') = \frac{-jke^{-jkz_1}}{2\pi z_1} e^{-jk\frac{(x_1')^2+(y_1')^2}{2z_1}} \frac{jke^{jkz_1}}{2\pi z_1} e^{jk\frac{x_1^2+y_1^2}{2z_1}} \times$$

$$\int\int\Big[\int\int \left\langle E^*\left(x', y', 0\right) E\left(x, y, 0\right)\right\rangle \times$$

$$e^{-jk\frac{(x')^2+(y')^2}{2z_1}} e^{+jk\frac{x^2+y^2}{2z_1}} e^{+jk\frac{x'x_1'+y'y_1'}{z_1}} e^{-jk\frac{xx_1+yy_1}{z_1}} dx'dy'\Big] dxdy$$

- Simplification: $x^2 - (x')^2 = (x + x')(x - x')$

$$\Gamma(x_1, y_1, x_1', y_1') = \frac{-jke^{-jkz_1}}{2\pi z_1} e^{jk\frac{\left(x_1'\right)^2+\left(y_1'\right)^2}{2z_1}} \frac{jke^{jkz_1}}{2\pi z_1} e^{-jk\frac{x_1^2+y_1^2}{2z_1}} \int\int \times$$

$$\Big[\int\int \left\langle E^*\left(x', y', 0\right) E\left(x, y, 0\right)\right\rangle e^{jk\frac{(x'+x)(x'-x)+(y'+y)(y'-y)}{2z_1}} e^{jk\frac{\left(xx_1-x'x_1'\right)+\left(yy_1-y'y_1'\right)}{z_1}} dx'dy'\Big] dxdy$$

Van Cittert–Zernicke Theorem (3)

- Random Phases: $\langle E \rangle = 0$

$$\left\langle E^* \left(x', y', 0\right) E\left(x, y, 0\right) \right\rangle = 0$$

- Analogy with Wiener–Khintchine Theorem Earlier

$$\left\langle E^* \left(x', y', 0\right) E\left(x, y, 0\right) \right\rangle dx'dy' = I\left(x, y, 0\right) \delta\left(x - x'\right) \delta\left(y - y'\right) dx'dy'$$

$$\Gamma(x_1, y_1, x_1', y_1') = \frac{-jke^{-jkz_1}}{2\pi z_1} e^{jk\frac{(x_1')^2 + (y_1')^2}{2z_1}} \frac{jke^{jkz_1}}{2\pi z_1} e^{-jk\frac{x_1^2 + y_1^2}{2z_1}} \int\int \times$$

$$\left[\int\int I\left(x, y, 0\right) \delta\left(x - x'\right) \delta\left(y - y'\right) e^{jk\frac{\left(xx_1 - x'x_1'\right) + \left(yy_1 - y'y_1'\right)}{z_1}} dx'dy'\right] dxdy$$

Van Cittert–Zernicke Theorem (4)

- Outer Integral

$$\Gamma(x_1, y_1, x_1', y_1') = \frac{-jke^{-jkz_1}}{2\pi z_1} e^{jk\frac{(x_1')^2+(y_1')^2}{2z_1}} \frac{jke^{jkz_1}}{2\pi z_1} e^{-jk\frac{x_1^2+y_1^2}{2z_1}} \times$$

$$\int\int I(x, y, 0)\, e^{jk\frac{x(x_1-x_1')+y(y_1-y_1')}{z_1}} dxdy$$

- Simplify

$$\Gamma(x_1, y_1, x_1', y_1') = \left(\frac{k}{2\pi z_1}\right)^2 e^{jk\frac{(x_1')^2+(y_1')^2}{2z_1}} e^{-jk\frac{x_1^2+y_1^2}{2z_1}} \times$$

$$\int\int I(x, y, 0)\, e^{jk\frac{x(x_1-x_1')+y(y_1-y_1')}{z_1}} dxdy$$

Van Cittert–Zernicke Theorem (5)

- Define Coordinate Differences

$$\xi = x_1' - x_1 \qquad \eta = y_1' - y_1$$

- Van Cittert–Zernicke Theorem

$$\Gamma(\xi, \eta) = \left(\frac{k}{2\pi z_1}\right)^2 \int\int I\left(x, y, 0\right) e^{jk\frac{x\xi + y\eta}{z_1}} dxdy$$

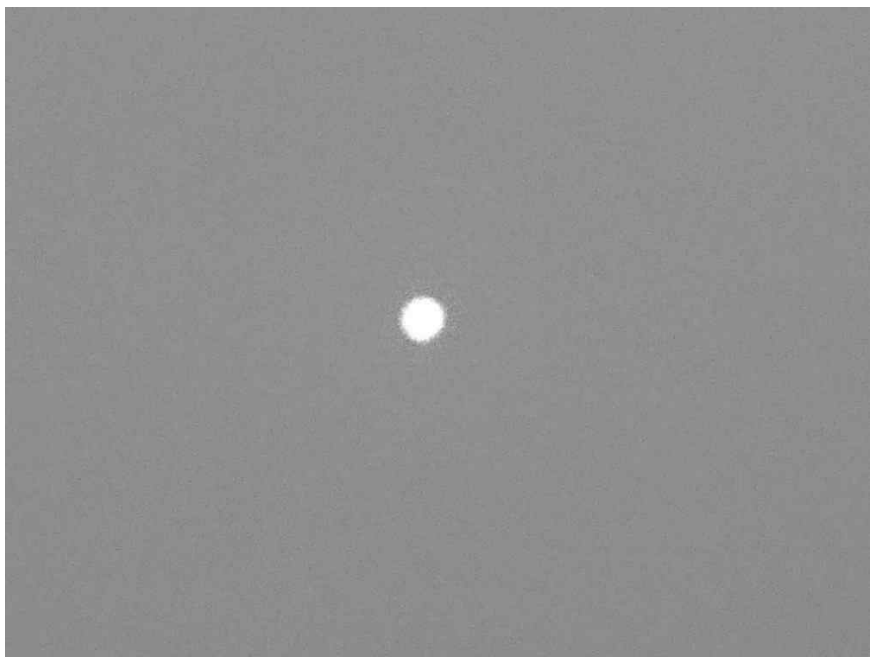
- Equation Similar to Fraunhofer Diffraction. . .

$$E\left(x_1, y_1, z_1\right) = \frac{jke^{jkz_1}}{2\pi z_1} \int\int E\left(x, y, 0\right) e^{-jk\frac{(xx_1 + yy_1)}{z_1}} dxdy$$

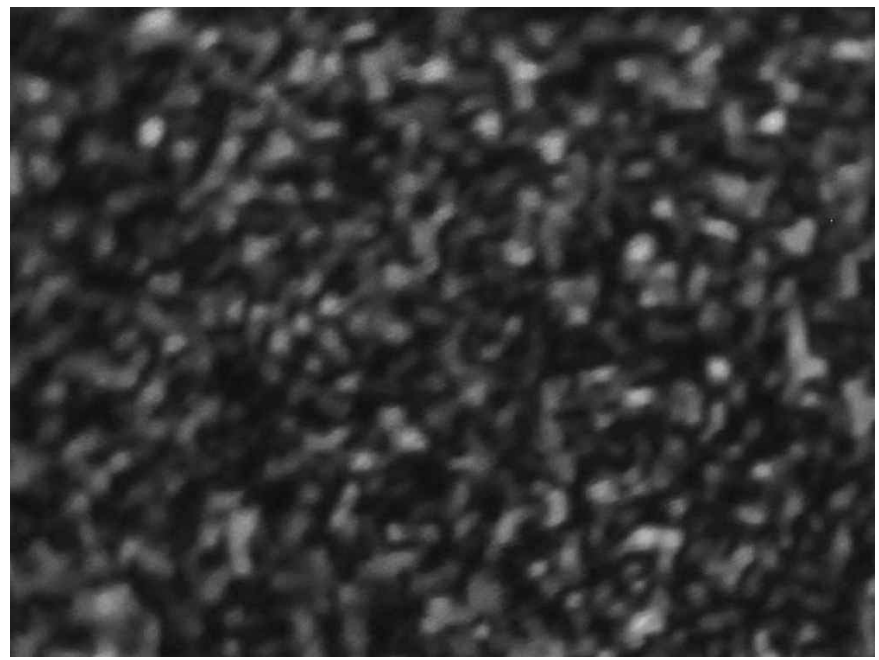
- . . . but New Meaning (Coherence rather than Field)

Spatially Coherent and Incoherent Illumination

$E(x, y, 0)$	$\rightarrow$	Fraunhofer	$\rightarrow$	$E(x_1, y_1, z_1)$
$I(x, y, 0)$	$\rightarrow$	vC–Z	$\rightarrow$	$\Gamma\left(x_1 - x_1', y_1 - y_1'\right)$ at z_1



A. Spatially Coherent
(Laser Source Direct)



B. Spatially Incoherent
(Same Size Laser Source
on Ground Glass)

Coherent and Incoherent Source: Calculation

- Coherent Source (1mm HeNe Viewed at 5m)
  - Beam Size d at Receiver
- Spatially Incoherent Source (Same Laser on Ground Glass)
  - Whole Field Illuminated
  - Bright and Dark Regions
  - Diameter of Γ is d
  - "Typical Spot Size:" d

- Spatially and Temporally Incoherent Source: Filtered Tungsten 400 to 800nm
  - Whole Field Illuminated
  - Bright and Dark Regions Only for Short Times

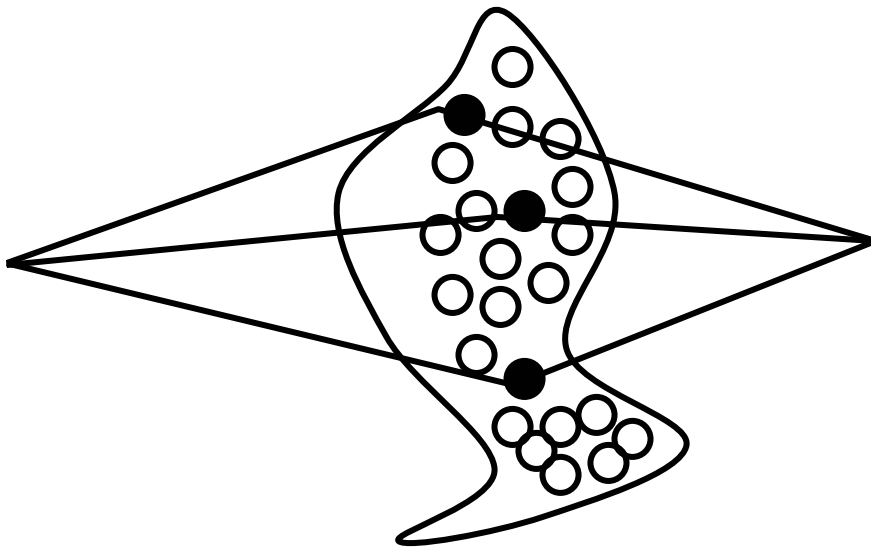
$$\frac{\delta\lambda}{\lambda} = \frac{\delta f}{f} \to 1$$

  - Uniform when Measured for Longer than a Cycle (femtoseconds)

$$d = \frac{4}{\pi}\frac{\lambda}{D}z = \frac{4}{\pi}\,\frac{633\times 10^{-9}\text{m}}{10^{-3}\text{m}}\,5\text{m} = 4\times 10^{-3}\text{m}$$

Speckle in Scattered Light

- Coherent Source
  - Focused or Diverging
  - Spatially Coherent
  - Temporally Coherent
- Scattering Particles
  - Secondary Sources
  - Randomly Distributed
  - Illuminated by Source



- Large Illumination Pattern: Small Speckles

- Small Illumination Pattern: Large Speckles

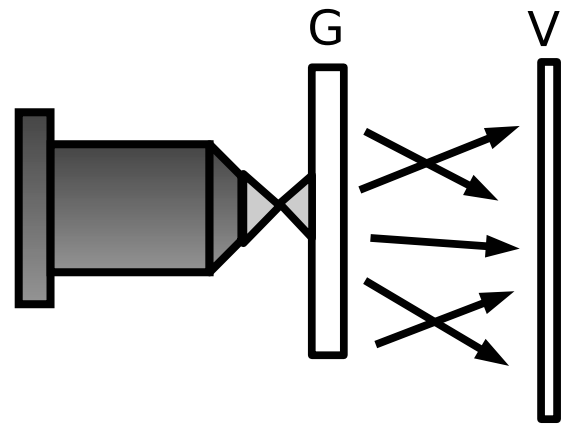
- Infer Size of Illumination by Size of Speckles?
  - See Next Page

Example: Finding the Focus

- Example ($NA = 0.25$): Waist

$$d_0 = \frac{4}{\pi}\frac{\lambda}{d}z \qquad d_0 = \frac{2}{\pi}\frac{\lambda}{NA}$$

$$d_0 = \frac{2}{\pi}\frac{532 \times 10^{-9}m}{0.25} = 1.4\times10^{-6}\text{m}$$



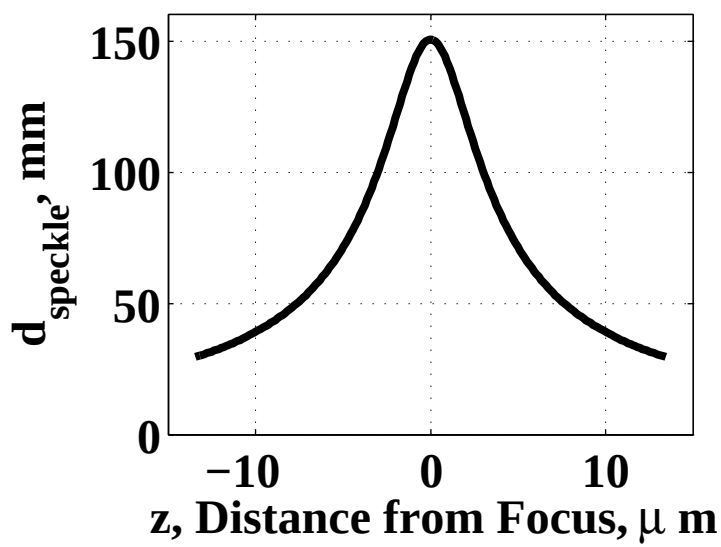
- At Glass

$$d_{glass} = d_0\sqrt{1 + \frac{z^2}{b^2}} \qquad b = \frac{\pi d_0^2}{4\lambda}$$

- Speckle Size

$$d_{speckle} = \frac{4}{\pi}\frac{\lambda}{d_{glass}}z$$

  – Same as Field of Coherent Source of Size d_{glass}



Example: Speckle in Laser Radar

- Carbon Dioxide Laser
  - $\lambda = 10.59\mu$m
  - $d_0 = 30$cm
  - Rayleigh Range

$$b = \pi d_0^2/(4\lambda) = 6.7\text{km}$$

- Diameter at Target

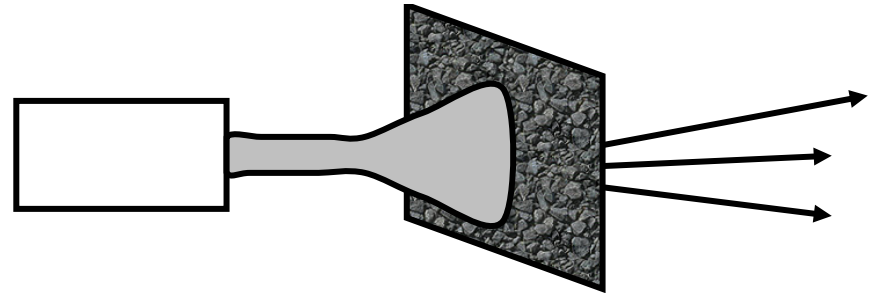
$$d_{target} = d_0\sqrt{1 + \frac{z_t^2}{b^2}}$$

  - Far Field $z_t = 20$km

$$d_{target} \approx 90\text{cm}$$

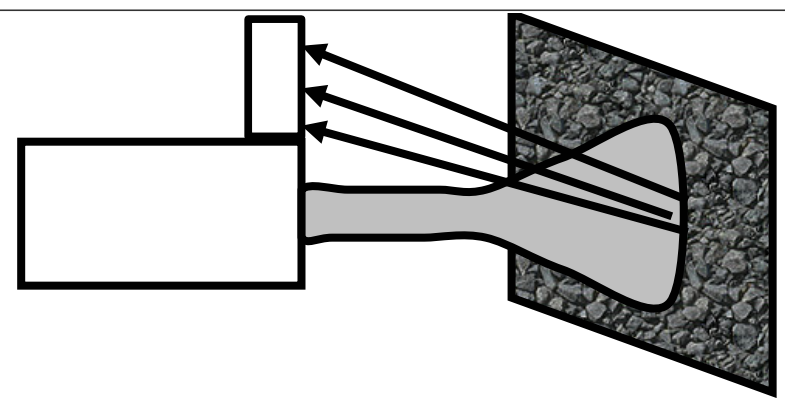
- Speckle Size at Receiver

$$\frac{4}{\pi}\frac{\lambda}{d_{target}}z_r = 15\times10^{-6}\text{rad}\times z_r$$



- Coaxial System $z_r = z_t$ and Target in Far–Field

$$d_{speckle} = \frac{4}{\pi}\frac{\lambda}{\frac{4}{\pi}\frac{\lambda}{d_0}z}z = d_0$$



Speckle Decorrelation

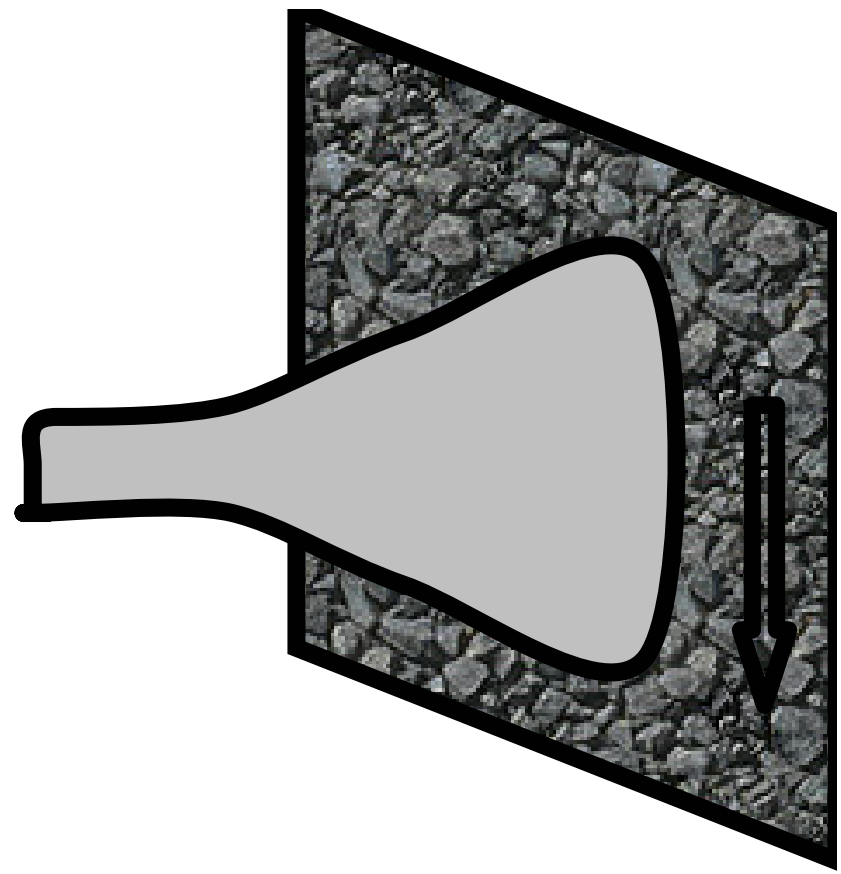
- Transverse Motion
  - New Scatterers
  - New Speckle Pattern
- Time on Target (ToT)

$$t_{target} = \frac{d_{target}}{v_\perp}$$

$$B_{tot} = \frac{v_\perp}{d_{target}}$$

- Recall Doppler Shift

$$f_{DR} = \frac{2v_\parallel}{\lambda}$$



Speckle Reduction

- Average n Measurements
  - Improvement with $1/\sqrt{n}$
  - Stationary Process
- Temporal Averaging

$$\sigma = \sqrt{\frac{t_{target}}{T_{integration}}}$$

- Long Time
  - Reduced Fluctuation but . . .
  - . . . More Target Variation
- "Matched Filtering"
- Area Averaging
  - Incoherent Averaging
  - Space–Invariant Process
- Aperture Averaging

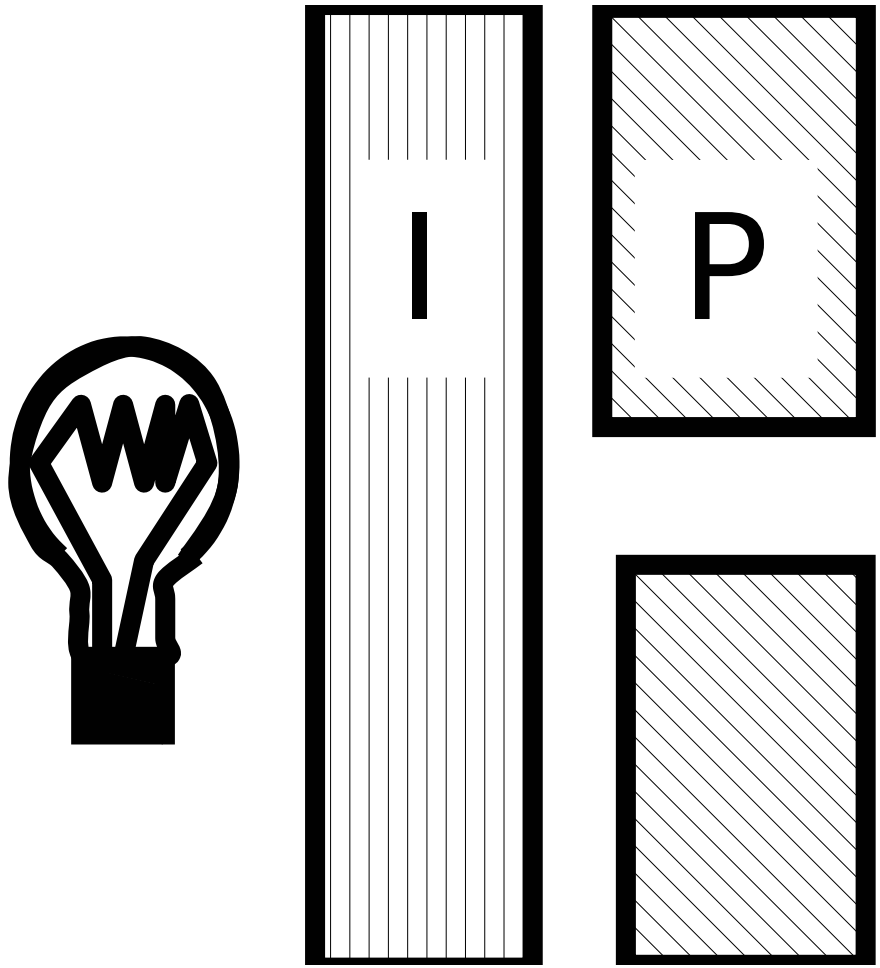
$$\sigma = \sqrt{\frac{A_{tx}}{A_{rx}}} = \frac{D_{tx}}{D_{rx}}$$

- Coherent Averaging Not Effective

$$\langle E \rangle = 0$$

- Other Diversity
  - Wavelength
  - Polarization

Making Light Coherent



- Interference Filter
  - Dielectric Stack
  - Narrow Bandpass
  - Temporal Coherence

$$\frac{\lambda}{\delta\lambda} = \frac{\delta z}{\lambda}$$

- Pinhole
  - Unresolved
  - Spatial Filter
  - Narrow Spatial Bandpass
  - Spatial Coherence

Making Light Incoherent

- Ground Glass Reduces Spatial Coherence
- Moving it Reduces Temporal Coherence