Vibrational Feedback Control of Time Delay Systems

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***Abstract*— This paper applies vibrational feedback control to time lag systems. Averaging theory for differential delay systems is presented and then applied to aid in the design of periodic controllers. Both stabilization and transient issues are discussed. An illustrative example is given which demonstrates that the proposed controller: 1) provides superior robustness properties in comparison to time-invariant controllers when applied correctly; 2) is not robust with respect to unmodeled delays; and 3) does not have zero placement capabilities, in the sense defined in the paper.**

***Index Terms*—Averaging, delays, nonminimum phase, periodic controllers.**

I. Introduction

RECENTLY, there has been a great deal of research interest in showing that systems with time-varying periodic controllers can have superior robustness properties in comparison to time-invariant controllers [1], [12], [13], [16], [21]–[25], [35], [36], [38]. In particular, periodic controllers, for both continuous and discrete systems, have demonstrated capabilities of arbitrarily improving the gain margins for classes of linear time-invariant (LTI) plants [12], [16], [21], [25], [38]. Likewise, periodic controllers have been shown to stabilize systems with decentralized fixed modes [1], [22], [35], [36].

There are several approaches that can be used in the design of periodic controllers. For discrete systems, the technique of lifting is commonly used [13], [21], [22]. For continuous systems, much of the literature centers around hybrid periodic controllers, i.e., a system composed of both a continuous-time component and a discrete-time component [21], [36], [38]. By utilizing hybrid controllers, it is often possible to apply discrete mathematical techniques, which simplifies the analysis. In fact, there are a limited number of constructive techniques that can be used as tools to fully understand the dynamic behavior of general classes of continuous-time periodic systems. Lyapunov functions can be applied, but this is by no means a simple, constructive method when applied to time-varying systems. One can apply Floquet analysis, but this usually requires the computation of a Monodromy matrix that usually cannot be obtained explicitly. (It is sometimes possible to estimate the Monodromy matrix, making this approach more tractable for linear periodic ODE's as in [33]. However, the approach of [33] is limited to stability analysis and has not yet been extended to delay differential equations, which is the focus of this paper.)

This paper proposes to use the techniques of vibrational feedback control, introduced in [25], in the analysis. Specifically, high-gain high-frequency periodic controllers are applied to continuous LTI systems with delay. Then, coordinate transformations are made so that the method of averaging can be applied. In this way, it is possible to relate the dynamic behavior of a time-varying system to the dynamic behavior of a time-invariant system. This type of analysis was first introduced to aid in the design of open-loop vibrational controllers for ODE's [1], [3], [4], [25], [37] (see [2] for tutorial) and most recently, to design open-loop vibrational controllers for parabolic partial differential equations with Neumann boundary conditions [7], [8] and for delay differential equations [5], [9], [27], [28], [31]. In this paper, however, appropriate adjustments are made to apply the methods of vibrational *feedback* controllers to time lag systems, which is a closed-loop problem.

It should be noted that much of the literature for periodic controllers centers around stabilization issues for finite-dimensional plants [1], [12], [13], [16], [21]–[25], [35], [36], [38]. We differ from these types of results in that the plant considered includes a measurement and/or actuator delay. Hence, the problem becomes infinite dimensional.

One of the first results published on closed-loop periodic controllers for infinite-dimensional systems is given in [34]. In [34], sufficient conditions are given for the existence of stabilizing periodic output feedback controllers from an abstract operator point of view. However, since [34] proves only existence theorems, the results do not lead to constructive control algorithms. In the present paper, our approach is to restrict analysis to a specific class of infinite dimensional systems—those with time lags. Because analysis is only focused on time lag systems, an abstract operator approach is not needed, and constructive control algorithms can be presented utilizing design criteria such as phase and gain margin. To our knowledge, the results of this paper present the first step-by-step design algorithm that can be used to design periodic output feedback controllers for continuous-time time lag systems.

Another difference between the results in this paper and others found in the literature is that techniques for controlling transient behavior are presented. For example, using the proposed techniques, it is now possible to control the rise time (in the sense described below) of a periodically controlled delayed plant with a step input. Of course, all results in this paper can

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be applied to the ODE case by setting the delay equal to zero. Hence, the results of this paper represent new techniques in the control of ODE's as well.

A main difficulty in extending vibrational feedback control to time lag systems is the lack of mathematical theory for time-varying delay differential equations. As noted above, the primary mathematical tool in the analysis of vibrational feedback control is the method of averaging, which is well known for ODE's [19]. For fast oscillating systems with delay, however, the method of averaging is an emerging area of research [20], [26], [28], [29]. Therefore, to extend vibrational feedback control to time lag systems, a new averaging theory for delay equations is presented. These mathematical results are given in Section II.

The formulation and the design of vibrational feedback control for time lag systems is presented in Sections III and IV. The results obtained are, at times, surprising. For example, much of the research on periodic controllers centers around apparent zero placement capabilities [12], [16], [21], [25]. However, this research demonstrates that vibrational feedback controllers do not have zero placement capabilities, in the sense discussed in this paper, even for the ODE case. Instead of interpreting the effects of vibrational feedback control as a type of zero placement, this paper suggests that a system under vibrational feedback control has the stability and transient properties similar to those of a corresponding new time-invariant system subject to state feedback. This is further explained in Section IV.

Another important, and perhaps surprising, result is the lack of robustness that the vibrational feedback controllers proposed in [25] have with respect to time delays. In Section V, an example from [21] and [25] is reworked and shown to be unstable with delay of $\tau = 0.02$, even though the system has gain margin over 20 for zero delay. Perhaps more unusual is the fact that for larger delay, e.g., $\tau = 0.125\,66$, the system becomes stable again. These results appear to be consistent with the results obtained in [5], [9], [27], and [28], which demonstrate a similar lack of robustness for open-loop vibrational control. Similarly, [6] discovered that fast, high-amplitude periodic systems admit extreme sensitivity to unmodeled delays. New control algorithms are presented in this paper to compensate for the delay, and applications to robustness problems are also introduced in Sections IV and V. An example is given to demonstrate that the algorithms in this paper can have superior performance over known finite-dimensional time-invariant controllers. Conclusions are presented in Section VI.

II. Averaging Theory

In this section, the mathematical foundations of averaging differential delay equations are presented. These techniques will be used in subsequent sections to develop the theory of vibrational feedback control of time lag systems. The proofs of the averaging theorems combine methods of [20], [26], [28], and [29] and are presented in the Appendix.

Let $F_i(t)$ be continuous T-periodic $n \times n$ matrices for $i = 1, 2, 3$, and let $\varepsilon > 0$ and $\mu \geq 0$ be constants. Suppose $\varphi(t)$ is a continuous function for $t \in [-\tau, 0]$, where τ is a fixed positive constant. Let $b(t)$ be a continuous T-periodic $n \times 1$ vector. Consider the linear delay differential equations given by

$$\begin{aligned}\dot{x}(t) &= F_1(t/\varepsilon)x(t) + F_2(t/\varepsilon)F_3\left(\frac{t-\mu}{\varepsilon}\right)x(t-\tau) + b(t/\varepsilon)\\ x(t) &= \varphi(t), \qquad \text{for } t \in [-\tau, 0]\end{aligned} \tag{1}$$

along with

$$\begin{aligned}\dot{y}(t) &= G_1 y(t) + G_2(\mu/\varepsilon)y(t-\tau) + v\\ y(t) &= \varphi(t), \qquad \text{for } t \in [-\tau, 0]\end{aligned} \tag{2}$$

where

$$\begin{aligned}G_1 &= \overline{F_1(t/\varepsilon)} \equiv \frac{1}{T}\int_0^T F_1(s)\,ds\\ G_2(\mu/\varepsilon) &= \overline{F_2(t/\varepsilon)F_3\left(\frac{t-\mu}{\varepsilon}\right)}\\ &\equiv \frac{1}{T}\int_0^T F_2(s)F_3(s-\mu/\varepsilon)\,ds\\ v &= \overline{b(t/\varepsilon)} \equiv \frac{1}{T}\int_0^T b(s)\,ds.\end{aligned}$$

The principle of averaging is to determine conditions under which solutions of (2) can be used to approximate solutions of the more complicated time-varying differential equation with constant delay given by (1). Both stability issues (infinite-time behavior) and transient issues (finite-time behavior) will be discussed in this section.

Notice that the average value of $F_2(s)F_3(s - \mu/\varepsilon)$, given by $G_2(\mu/\varepsilon)$, depends explicitly on ε. This indicates that (1) cannot be placed in the so-called "standard form," and known averaging theory cannot be applied [20], [26]. In order to avoid this difficulty, several "tricks" have been previously introduced in the literature [5], [9], [27], [28]. These tricks (see the example below) are no longer needed when the following averaging theory is applied. Additionally, it is no longer necessary to have ε be a fixed constant, as previously indicated in [5], [9], [27], and [28]. The following theorems are used to develop a concept, termed δ-equivalence in the next section.

Assume $x(t) = y(t) = \varphi(t)$ on $t \in [-\tau, 0]$ in (1) and (2). Denote the solutions of (1) and (2) as $x(t; \varphi)$ and $y(t; \varphi)$, respectively.

Theorem 2.1: Assume that $F_i(t)$ in (1) are continuous T-periodic $n \times n$ matrices on $\Re$, for $i = 1, 2, 3$. Assume further, that $b(t)$ in (1) is also a continuous T-periodic $n \times 1$ vector on $\Re$.

Then, for any $L > 0$ and $\sigma > 0$, there exists an $\varepsilon_0 = \varepsilon_0(\sigma, L)$ such that, for $0 < \varepsilon \leq \varepsilon_0$

$$|x(t; \varphi) - y(t; \varphi)| \leq \sigma$$

for any $t \in [0, L]$.

Proof: See the Appendix.

Remark 2.1: A primary contribution of Theorem 2.1 is that it lifts the strong restriction, introduced in [5], [9], [27], and [28], that requires ε to be fixed. The fact that Theorem 2.1 is true for any constant $\varepsilon \in (0, \varepsilon_0]$ helps the controller design in Section III of this paper. Additionally, an immediate consequence of Theorem 2.1 is that the notion of global dynamic equivalence for open-loop vibrational control of time delay systems (see [28, Th. 4.1]) can be strengthened by: 1) lifting the restriction that the delay be exactly known; 2) eliminating the need to perform an iterative search for ε_0; and 3) allowing ε to vary from $0 < \varepsilon \leq \varepsilon_0$. Further details of these items are explained in Example 2.1 below. □

Remark 2.2: Although Theorem 2.1 (and Theorem 2.2 below) is stated for linear systems, the results can be extended to periodic nonlinear delay differential equations as well. Future research will address these and other issues. □

Theorem 2.1 addresses local time behavior, i.e., closeness of trajectories on a finite-time interval. It is also of interest to study stability issues and trajectories on an infinite time interval.

Theorem 2.2: Let $F_i(t)$ and $b(t)$ satisfy the assumptions of Theorem 2.1, and let y_s denote the equilibrium point of (2). Suppose that there exist constants $\eta_1 > 0$ and $\eta_2 > 0$, $0 < \eta_1 \leq \eta_2$, such that

$$\det [sI - G_1 - G_2(\mu/\varepsilon)e^{-\tau s}] = 0$$

has all solutions with real parts less than zero for all $\varepsilon \in [\eta_1, \eta_2]$.

Then, for any $\rho > 0$ and $\sigma > 0$, there exists an $\varepsilon_0 = \varepsilon_0(\rho, \sigma)$ such that, when $\varepsilon \in [\eta_1, \eta_2]$ and $\varepsilon \leq \varepsilon_0$:

1) $|x(t; \varphi) - y(t; \varphi)| < \sigma$ for all $t \geq 0$;
2) there is a unique asymptotically stable periodic solution $x^*(t, \varepsilon)$ of (1) satisfying $|x^*(t, \varepsilon) - y_s| < \rho$. Furthermore, $\lim_{t\to\infty} x(t, \varepsilon) = x^*(t, \varepsilon)$.

Proof: See the Appendix.

Remark 2.3: As written in Theorem 2.2, it is possible that $(0, \varepsilon_0] \cap [\eta_1, \eta_2] = \{\emptyset\}$. However, this problem can be eliminated by taking η_1 and η_2 sufficiently small. This is due to the fact that $G_2(\mu/\varepsilon)$ is T-periodic in μ/ε, i.e., $G_2(\mu/\varepsilon + T) = G_2(\mu/\varepsilon)$. As a result $\det [sI - G_1 - G_2(\mu/\varepsilon)e^{-s\tau}]$ has coefficients periodic in μ/ε. The following corollary utilizes this fact and gives conditions for the stability of (1) which guarantee that $\varepsilon \in (0, \varepsilon_0]$. □

Corollary 2.1: Let $\eta_1, \eta_2, \mu, \rho, \sigma$, and ε_0 be as in Theorem 2.2, and let the assumptions of Theorem 2.2 be true. Define constants $\lambda_1 \equiv \mu\eta_1/(\mu + nT\eta_1)$ and $\lambda_2 \equiv \mu\eta_2/(\mu + nT\eta_2)$, where n is an integer and T is the period of $F_i(t)$ and $b(t)$.

Then there exists an integer $n_0 = n_0(\rho, \sigma)$ such that when $n \geq n_0$ and $\varepsilon \in [\lambda_1, \lambda_2]$, Conclusions 1) and 2) of Theorem 2.2 remain valid.

Proof: See the Appendix.

Example 2.1: The later sections of this paper examine delay differential equations similar to

$$\dot{x}(t) = -\cos(t/\varepsilon)\cos\left(\frac{t-\mu}{\varepsilon}\right)x(t) + 0.1x(t-\tau) + 1/2 \quad (3)$$

where μ is an arbitrary constant phase shift. Similar equations also appear in vibrational control of time delay systems with $\mu = \tau$ [5], [9], [27], [28].

The corresponding average is given by

$$\dot{y}(t) = -\frac{1}{2}\cos\left(\frac{\mu}{\varepsilon}\right)y(t) + 0.1y(t-\tau) + \frac{1}{2}. \quad (4)$$

Assuming the same initial functions $\varphi(t)$, Theorem 2.1 guarantees that for any $\sigma > 0$ there exists a sufficiently small range in ε, $0 < \varepsilon \leq \varepsilon_0$, such that the solutions of (3) and (4) satisfy

$$|x(t; \varphi) - y(t; \varphi)| \leq \sigma, \qquad t \in [0, L]. \quad (5)$$

Notice that for different values of ε, the stability properties and the location of the equilibrium point of (4), y_s, change. Despite this fact, Theorem 2.1 can still be applied.

Suppose that there exists a range, not necessarily maximal, of ε such that for $\varepsilon \in [\eta_1, \eta_2]$

$$\frac{1}{2}\cos\left(\frac{\mu}{\varepsilon}\right) > 0.1. \quad (6)$$

Under these conditions (4) has an asymptotically stable equilibrium point $y_s = 1/(\cos(\mu/\varepsilon) - 0.2)$. Therefore, by Theorem 2.2 there exists an $\varepsilon_0 > 0$, sufficiently small, such that if $\eta_2 \leq \varepsilon_0$, then there will be an asymptotically stable periodic orbit $x^*(t, \varepsilon)$ in the vicinity of y_s, and $x(t; \varphi) \to x^*(t, \varepsilon)$ as $t \to \infty$. Furthermore, for any $\sigma > 0$, $|x(t; \varphi) - y(t; \varphi)| \leq \sigma$ for all $t \geq 0$.

It is interesting to explain, via this example, why Theorems 2.1 and 2.2 will strengthen the results of [5], [9], [27], and [28]. In these works, both $\mu = \tau$ and $\varepsilon = \varepsilon_1$ are fixed constants, and a new constant is defined as $\gamma \equiv \tau/\varepsilon_1$. Then the averaged equation becomes $\dot{y} = -\frac{1}{2}\cos(\gamma)y(t) + 0.1y(t-\tau) + 1/2$, which no longer explicitly depends on ε. A verification procedure is proposed to ensure that the fixed ε_1 is sufficiently small. If it is not, the technique is repeated by defining a new ε_1 and a new γ. The averaging results of this paper do not necessarily need for ε or μ to be fixed when discussing (arbitrarily large) finite-time intervals. For sufficiently small ε, (5) is true for any finite $\mu = \tau > 0$. Restrictions on ε are only needed on the infinite-time interval, as given in Theorem 2.2.

III. Controllers and Problem Formulation

Consider a SISO time-invariant plant with time delay having open-loop transfer function $G_p(s)e^{-s\tau}$. In state-space form, this system can be written as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t-\tau) + du(t-\tau) \end{aligned} \quad (7)$$

where $x \in \Re^n$ is the state, $u \in \Re$ is the input, and $y \in \Re$ is the output. For simplicity, assume that $x(t) = u(t) = 0$ on $t \in [-\tau, 0]$ (the results of the paper can be extended to nonzero initial functions). In this research, all delays are lumped together in the output equation as above. That is, the sum of the measurement, computation, and actuator delay is equal to τ in (7). It is important to note that (7) might not be the true state-space representation of the plant. For example, a system with actuator delay is modeled as $\dot{x}(t) = Ax(t) + Bu(t-\tau)$, and $y(t) = Cx(t) + du(t-\tau)$, which also has transfer function $G_p(s)e^{-s\tau}$. However, this paper will model all plants with transfer function $G_p(s)e^{-s\tau}$ by (7). For the coordinate transformations used in this paper, it is necessary that the delay not appear in the open-loop state equation. Since this paper is concerned only with the input–output behavior of a system, there is no loss of generality in always using (7) as the plant's state-space representation.

A. Controller

The proposed controller takes on a similar form to the one proposed in [25], although the parameter values of the controller are often quite different. Additionally, the possibility of a reference input is introduced. Because of the inclusion of a reference input, an input–output analysis can be taken. Concepts of transfer functions, zeros, and transient response can now be discussed for periodically controlled systems. All results can be applied to ODE's by setting the delay equal to zero.

Consider a periodic controller with unity feedback in the form of

$$\begin{aligned}\dot{x}_c(t) &= \left[F + \frac{1}{\varepsilon} F_0(t/\varepsilon)\right] x_c(t) + Ge(t)\\ u(t) &= \left[K + \frac{1}{\varepsilon^r} K_r(t/\varepsilon)\right] x_c(t)\\ e(t) &= l(t) - y(t)\end{aligned} \tag{8}$$

where $l \in \Re$ is the reference input, $F_0(t)$ and $K_r(t)$ are T-periodic zero average matrices, $x_c \in \Re^n$, $0 < \varepsilon \ll 1$, and r is the relative degree of system (7) defined as

$$r = \begin{cases} 0, & \text{if } d \neq 0\\ \min\{k: CA^{k-1}B \neq 0, \\ \quad k = 1, \cdots, n\}, & \text{if } d = 0.\end{cases} \tag{9}$$

Additionally

$$\begin{aligned}F &= \begin{bmatrix} 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\\ f_1 & f_2 & \cdots & f_n\end{bmatrix}\\ F_0(t/\varepsilon) &= \begin{bmatrix} 0 & \cdots & 0 & 0\\ \vdots & \cdots & \vdots & \vdots\\ 0 & \cdots & 0 & 0\\ 0 & \cdots & 0 & \alpha\left(\frac{t}{\varepsilon}\right)\end{bmatrix}\\ G &= [0 \quad \cdots \quad 0 \quad 1]^T\end{aligned} \tag{10}$$

and

$$\begin{aligned}K &= [k_1 \; k_2 \; \cdots \; k_n]\\ K_r\left(\frac{t}{\varepsilon}\right) &= [\beta_1(t/\varepsilon) \; \cdots \; \beta_{n-r}(t/\varepsilon) \; 0 \; \cdots \; 0].\end{aligned} \tag{11}$$

Note that ε is proportional to the period of $F_0(t/\varepsilon)$ and $K_r(t/\varepsilon)$. When $r = n$, then $K_r(\cdot) = 0$. (However, this case is of limited interest since there are no zeros of the plant, and time-invariant controllers are known to provide robust control of the system. Instead, this paper focuses on when the plant has zeros in undesirable locations and limit performance.)

It is assumed that $\alpha(t)$ and $\beta(t)$ as well as $\gamma_i(t) \equiv \int \cdots \int \beta_i(t)(dt)^r$ and $p(t) \equiv \exp\{\int \alpha(t)\, dt\}$ are all scalar functions satisfying the following conditions:

$$\begin{aligned}\overline{\alpha(t)} &= \overline{\beta_i(t)} = \overline{\gamma_i(t)} = 0, \; \overline{p(t)} = \overline{p^{-1}(t)} \neq 0,\\ \overline{p(t)\beta_i(t)} &= \overline{-p^{-1}(t)\beta_i(t)} \neq 0\\ \overline{p(t)\gamma_i(t)} &= \overline{-p^{-1}(t)\gamma_i(t)} \neq 0, \qquad i = 1, \cdots, n-r\end{aligned}$$

where $\overline{q(t)} \equiv 1/T \int_0^T q(s)\, ds$. One set of $\alpha(t)$ and $\beta_i(t)$ that satisfies the conditions follows [25]:

$$\begin{aligned}\alpha(t) &= \cos(t)\\ \beta_i(t) &= \begin{cases} k_i^{(r)} \sin(t), & \text{if } r \text{ is even}\\ k_i^{(r)} \cos(t), & \text{if } r \text{ is odd.}\end{cases}\end{aligned}$$

The above conditions are used for technical reasons in order to simplify the presentation. In particular, the controller structure in (10) and (11) is assumed because it allows us to explicitly solve for state transition matrices of time-varying ODE's in Theorems 4.1 and 4.2.

In this case, the closed-loop equation of (7) and controller (8) is given as

$$\begin{aligned}\begin{bmatrix}\dot{x}(t)\\ \dot{x}_c(t)\end{bmatrix} &= \begin{bmatrix} A & B\left[K + \frac{1}{\varepsilon^r} K_r(t/\varepsilon)\right]\\ 0 & F + \frac{1}{\varepsilon} F_0(t/\varepsilon)\end{bmatrix} \begin{bmatrix} x(t)\\ x_c(t)\end{bmatrix}\\ &\quad + \begin{bmatrix} 0 & 0\\ -GC & -dG\left[K + \frac{1}{\varepsilon^r} K_r\left(\frac{t-\tau}{\varepsilon}\right)\right]\end{bmatrix}\\ &\quad \times \begin{bmatrix} x(t-\tau)\\ x_c(t-\tau)\end{bmatrix} + \begin{bmatrix} 0\\ G\end{bmatrix} l(t)\\ y(t) &= \left[C \quad d\left[K + K_0\left(\frac{t-\tau}{\varepsilon}\right)\right]\right] \begin{bmatrix} x(t-\tau)\\ x_c(t-\tau)\end{bmatrix}.\end{aligned} \tag{12}$$

Definition 3.1: Open-loop system (7) with controller (8) will be referred to as $\sum_1$. The closed-loop dynamics of $\sum_1$ are given by (12). □

B. Time-Invariant System

Along with $\sum_1$ and the closed-loop equation (12), introduce the SISO LTI closed-loop delay system with unity feedback given by

$$\begin{aligned}\dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}\hat{u}(t)\\ \hat{y}(t) &= \hat{C}\hat{x}(t-\tau)\\ \hat{u}(t) &= \hat{l}(t) - \hat{K}\hat{x}(t-\tau)\end{aligned} \tag{13}$$

where $\hat{x} \in \Re^{2n}$, $\hat{u} \in \Re$, $\hat{l} \in \Re$ is the reference, $\hat{y} \in \Re$ is the output, and τ is as defined in (7). Let $\hat{x}(t) = 0$ for $t \in [-\tau, 0]$. The matrices $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{K}$, and $\hat{l}$ are of appropriate dimension and will be defined later.

Definition 3.2: Closed-loop system (13) will be referred to as $\sum_2$. □

Utilizing the averaging theory of Section II, this research demonstrates that the output of $\sum_1$ can be "approximated" by the output of a time-invariant system in the form $\sum_2$. In this manner, robustness properties of $\sum_2$ can be used as a measure of robustness of corresponding time-varying systems.

Definition 3.3: Let $Q(t, \varepsilon)$ be a $2n \times 2n$ T-periodic matrix with bounded inverse, and let, in (12), $[x^T(t)\ x_c^T(t)]^T = Q(t, \varepsilon)\varsigma(t)$. Assume that this change of variables transforms (12) into an algebraically equivalent dynamical system

$$\begin{aligned}\dot{\varsigma}(t) &= A_0(t, \varepsilon)\varsigma(t) + A_1(t, \varepsilon)\varsigma(t-\tau) + G_0(t, \varepsilon)l(t)\\ y(t) &= C_0(t, \varepsilon)\varsigma(t-\tau)\end{aligned}$$

where A_0, A_1, G_0, and C_0 are of appropriate dimensions.

For any given fixed $\delta > 0$ and any $L > 0$, $\sum_1$ and $\sum_2$ are said to be *δ-equivalent* if $|\overline{y}(t) - \hat{y}(t)| < \delta$, $t \in [0, L]$, where $\hat{y}(t)$ is the output of $\sum_2$ and

$$\overline{y}(t) = \frac{1}{T}\int_t^{t+T} C_0(s, \varepsilon)\varsigma(t-\tau)\,ds.$$

When $L = \infty$, $\sum_1$ and $\sum_2$ are said to be *globally δ-equivalent*. □

Remark 3.1: The concept of δ-equivalence gives a measure of how closely the *moving averaged output* of the time-varying system $\sum_1$ is approximated by the output of the time-invariant system $\sum_2$. By approximating the average behavior of a time-varying system by the behavior of a time-invariant system, controller parameters can be designed based on the time-invariant system. This simplifies the problem. Comparisons between $y(t)$ of (7) with $\overline{y}(t)$ and $\hat{y}(t)$ reveal the change of transient behavior due to periodic control, as is further discussed in Section IV-D. □

Remark 3.2: It may be interesting to note that $l(t)$ and $\hat{l}(t)$, the reference inputs to $\sum_1$ and $\sum_2$, respectively, are not necessarily equal. In fact, it is often necessary to set $l(t) = k\hat{l}(t)$ in order to guarantee δ-equivalence. □

Remark 3.3: Ideally, it would be preferable to replace $\overline{y}(t)$ in Definition 3.2 by $y(t)$. However, this is usually not the case since $\sum_1$ is time varying and periodic. For example, in the case of a step input, Theorem 2.2 guarantees that $y(t)$ approaches a periodic orbit and not an equilibrium point when $\sum_1$ is stable. Therefore, the best that can often be hoped for is that $\hat{y}(t)$ approximates the moving average of $y(t)$. As $t \to \infty$, the output, $y(t)$, will often periodically oscillate about an equilibrium point of an averaged time-invariant system. The periodic orbit can have large peak-to-peak values, as estimated in Section IV-D. For the special case when $l(t) = 0$, then both $y(t)$ and $\hat{y}(t)$ tend asymptotically to zero as $t \to \infty$, provided that the system is stable (regulator problem). □

In an effort to develop concepts of performance for time-varying system $\sum_1$, consider the following definitions.

Definition 3.4: $\sum_1$ is said to have *δ-equivalent rise time*, $t_{\delta r}$, if: 1) $\sum_1$ is δ-equivalent to $\sum_2$ and 2) $\sum_2$ has a rise time t_r. □

Definition 3.5: $\sum_1$ is said to have a *δ-equivalent zero* at $z \in C$ if: 1) $\sum_1$ is δ-equivalent to $\sum_2$ and 2) $\sum_2$ has a zero at z. (In this paper, we say that $\sum_2$ has a zero at z when its transfer function $[n(s)/d(s)]e^{-s\tau}$ has $n(z) = 0$.) □

Definitions 3.3–3.5 give additional insight into performance issues for time-varying delay systems and generalize the theory of continuous-time periodic control to include the case of nonzero reference. In a similar manner concepts such as *δ-equivalent gain margin*, *δ-equivalent phase margin*, etc., can be defined. The remaining sections describe conditions when it is possible to design vibrational feedback controllers for systems with step inputs and with delay.

IV. Averaging and δ-Equivalence

This section will demonstrate how it is possible to transform $\sum_1$ into a form in which averaging can be applied. Then, conditions for δ-equivalence to a time-invariant system will be presented. Controller parameters can then be chosen through the analysis of the time-invariant systems.

The asymptotic analysis and expansions used in this section are similar to those introduced in [25]. However, due to the previous lack of mathematical theory for time-varying functional differential equations, it was not possible to design time-varying periodic controllers for time lag systems until

$$\begin{aligned}
\hat{A} &= \begin{bmatrix} A & B[K\overline{\Phi(t/\varepsilon)} + \overline{K_0(t/\varepsilon)\Phi(t/\varepsilon)}] \\ 0 & \overline{\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)} \end{bmatrix}\\
\hat{K} &= \left[\overline{p^{-1}(t/\varepsilon)}C \quad d\left[\overline{Kp^{-1}(t/\varepsilon)\Phi\left(\frac{t-\tau}{\varepsilon}\right)} + \overline{K_0\left(\frac{t-\tau}{\varepsilon}\right)p^{-1}(t/\varepsilon)\Phi\left(\frac{t-\tau}{\varepsilon}\right)}\right]\right]\\
\hat{C} &= \left[C \quad d\left[\overline{K\Phi\left(\frac{t}{\varepsilon}\right)} + \overline{K_0\left(\frac{t}{\varepsilon}\right)\Phi\left(\frac{t}{\varepsilon}\right)}\right]\right]\\
\hat{B} &= \begin{bmatrix} 0 \\ G \end{bmatrix}\\
\hat{l}(t) &= \overline{p^{-1}(t/\varepsilon)}l(t)
\end{aligned} \tag{14}$$

$$\begin{aligned}
\dot{\chi}(t) = &\begin{bmatrix} A & B[K + K_0(t/\varepsilon)]\Phi(t/\varepsilon) \\ 0 & \Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon) \end{bmatrix}\chi(t)\\
&+ \begin{bmatrix} 0 & 0 \\ -\Phi^{-1}(t/\varepsilon)GC & -\Phi^{-1}(t/\varepsilon)\,dG\left[K + K_0\left(\frac{t-\tau}{\varepsilon}\right)\right]\Phi\left(\frac{t-\tau}{\varepsilon}\right) \end{bmatrix}\chi(t-\tau) + \begin{bmatrix} 0 \\ -\Phi^{-1}(t/\varepsilon)G \end{bmatrix}l(t)
\end{aligned} \tag{15}$$

now. Additionally, new techniques are presented that allow for the discussion of robustness issues as well as time response specifications. Another main difference in the proposed techniques is that this research considers periodic controllers for time delay systems, which, as noted earlier, are infinite dimensional.

As mentioned in Section I, an important result of this section is to demonstrate that even small delays must be modeled in order for vibrational feedback controllers to be applied. Virtually every real world control system has a delay of some sort (measurement or actuator) that is normally ignored since it is small. As shown below, for high gain periodic time-varying systems with delay, it is no longer possible to ignore these delays because of the strong influence the delay has on the stability properties of the system. This fact is true, independent of the system time constant or gain margin.

A. Controller with Relative Degree Zero

For simplicity, first consider $\sum_1$, given by (12), when $d \neq 0$ and $r = 0$.

Theorem 4.1: Assume $l(t)$ is a step input and $d \neq 0$. Let $\Phi(t)$ be the fundamental matrix for the ODE $\dot{x}(t) = F_0(t)x(t)$ given by

$$\Phi(t) = \begin{bmatrix} I & 0 \\ 0 & p(t) \end{bmatrix}$$

where $p(t) = \exp\{\int \alpha(t)dt\}$ and $F_0(t)$ and $\alpha(t)$ are defined in (10).

Then, for any $L \geq 0$ and any $\delta > 0$, there exists an $\varepsilon_0 = \varepsilon_0(\delta, L) > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, $\sum_1$ is δ-equivalent to $\sum_2$ as given by (13) with $\hat{A}$, $\hat{K}$, $\hat{C}$, $\hat{B}$, and $\hat{l}$ given in (14), as shown at the bottom of the previous page.

Proof: It is straightforward to verify that $\Phi(t)$ is a fundamental matrix for the ODE $\dot{x}(t) = F_0(t)x(t)$. Let $\chi(t) \in \Re^{2n}$ and introduce into (12) the Lyapunov stability preserving transformation

$$\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \Phi(t/\varepsilon) \end{bmatrix} \chi(t)$$

to obtain (15), also shown at the bottom of the previous page, with output equation

$$y(t) = \left[C \quad d\left[K + K_0\left(\frac{t-\tau}{\varepsilon}\right)\right]\Phi\left(\frac{t-\tau}{\varepsilon}\right)\right]\chi(t-\tau). \tag{16}$$

Notice that (15) is in the form (1) with $\mu = \tau$. Hence, the previously discussed averaging theory can be applied.

The corresponding average of (15) is given by (17), as shown at the bottom of the page, where the relation

$$\overline{\Phi^{-1}(t/\varepsilon)GC} = \overline{p^{-1}(t/\varepsilon)}GC$$

has been used.

By Theorem 2.1 in Section II, it is known that for any $\sigma > 0$ and $L > 0$, there exists an $\varepsilon_0 = \varepsilon_0(\sigma, L)$ such that, for $0 < \varepsilon \leq \varepsilon_0$, $|\chi(t) - \hat{x}(t)| \leq \sigma$ on $t \in [0, L]$. Now, introduce the output equation $\hat{y}(t) = \hat{C}\hat{x}(t-\tau)$, with $\hat{C}$ as defined in (14). Then, noting that $\overline{\Phi(t/\varepsilon - \tau/\varepsilon)} = \overline{\Phi(t/\varepsilon)}$ and $\overline{K_0(t/\varepsilon - \tau/\varepsilon)\Phi(t/\varepsilon - \tau/\varepsilon)} = \overline{K_0(t/\varepsilon)\Phi(t/\varepsilon)}$, (17) can be rewritten as (13), with $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{K}$ as defined in (14).

From (16) and Definition 3.2

$$\overline{y}(t) = \hat{C}\chi(t-\tau).$$

This implies that

$$|\overline{y}(t) - \hat{y}(t)| \leq \|\hat{C}\|\,|\chi(t-\tau) - \hat{x}(t-\tau)|.$$

Defining $\sigma < \delta/\|\hat{C}\|$ completes the proof. Q.E.D.

B. Controller with Nonzero Relative Degree

Theorem 4.2: Assume $l(t)$ is a step input and that $d = 0$. Let $\Phi(t)$ be the fundamental matrix for the ODE $\dot{x}(t) = F_0(t)x(t)$ defined in Theorem 4.1.

Then, for any $L \geq 0$ and any $\delta > 0$, there exists an $\varepsilon_0 = \varepsilon_0(\delta, L) > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, $\sum_1$ is δ-equivalent to $\sum_2$ as given by (13) with $\hat{A}$, $\hat{K}$, $\hat{B}$, $\hat{C}$, and $\hat{l}$ given in (18), as shown at the bottom of the page, where $L_{i+1}(t) = \int L_i(t)\,dt$, $i = 1, 2, \cdots, r-1$, $L_0(t) = K_r(t)$, and $p(t) = \exp\{\int \alpha(t)\}$.

$$\begin{aligned}\dot{\hat{x}}(t) = &\begin{bmatrix} A & B[K\overline{\Phi(t/\varepsilon)} + \overline{K_0(t/\varepsilon)\Phi(t/\varepsilon)}] \\ 0 & \overline{\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)} \end{bmatrix}\hat{x}(t) \\ &+ \begin{bmatrix} 0 & 0 \\ -\overline{\Phi^{-1}(t/\varepsilon)}GC & -\overline{\Phi^{-1}(t/\varepsilon)\, dG\left[K + K_0\left(\frac{t-\tau}{\varepsilon}\right)\right]\Phi\left(\frac{t-\tau}{\varepsilon}\right)} \end{bmatrix}\hat{x}(t-\tau) + \begin{bmatrix} 0 \\ -\overline{p^{-1}(t/\varepsilon)}G \end{bmatrix} l(t)\end{aligned} \tag{17}$$

$$\begin{aligned}\hat{A} &= \begin{bmatrix} A & BK\overline{\Phi(t/\varepsilon)} + (-1)^r B\overline{L_r(t/\varepsilon)F^{r-1}\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)} \\ 0 & \overline{\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)} \end{bmatrix} \\ \hat{K} &= \begin{bmatrix} \overline{p^{-1}(t/\varepsilon)}C & \overline{p^{-1}(t/\varepsilon)CA^{r-1}BL_r\left(\frac{t-\tau}{\varepsilon}\right)} \end{bmatrix} \\ \hat{B} &= \begin{bmatrix} 0 \\ G \end{bmatrix}, \quad \hat{C} = [C \quad 0], \quad \hat{l}(t) = \overline{p^{-1}(t/\varepsilon)}l(t)\end{aligned} \tag{18}$$

Proof: Assume in (7) that $d = 0$ and $r \neq 0$. As in [25], define $\eta^{(0)}(t) = x(t)$. If $r > 1$, introduce successive substitutions, as shown in (18a) at the bottom of the page, in (12) to obtain (19), also shown at the bottom of the page. The following facts have been used:

$$CB = 0, \qquad BL_i(t/\varepsilon)GC = 0.$$

For $r = 1$, (12) reduces to (19); therefore, the remaining discussion is valid for any $r \geq 1$. Let $\chi(t) \in \Re^{2n}$ and define

$$\begin{bmatrix} \eta^{(r-1)}(t) \\ x_c(t) \end{bmatrix} = \begin{bmatrix} I & M(t/\varepsilon) \\ 0 & \Phi(t/\varepsilon) \end{bmatrix} \chi(t)$$

where

$$\begin{bmatrix} I & M(t/\varepsilon) \\ 0 & \Phi(t/\varepsilon) \end{bmatrix}$$

is a fundamental matrix for (19a), as shown at the bottom of the page. Then (19) becomes (20), as shown at the bottom of the page, with output equation

$$y(t) = \left[C \quad CA^{r-1}BL_r\left(\frac{t-\tau}{\varepsilon}\right) \right] \chi(t-\tau) \tag{21}$$

where due to the special structure of the matrices, $\Phi^{-1}(t/\varepsilon)G = p^{-1}(t/\varepsilon)G$, $L_r(t/\varepsilon)F^{r-j-1}\Phi(t/\varepsilon) = L_r(t/\varepsilon)F^{r-j-1}$ for $j = 0, 1, \cdots, r-1$, and we can explicitly solve $M(t/\varepsilon)$ to be

$$M(t/\varepsilon) = \sum_{j=0}^{r-1} (-1)^{r-1-j} \binom{r-1}{j} A^j BL_r(t/\varepsilon)F^{r-j-1}.$$

Additionally, $CM(t/\varepsilon - \tau/\varepsilon) = CA^{r-1}BL_r(t/\varepsilon - \tau/\varepsilon)$ and $M(t/\varepsilon)\Phi^{-1}(t/\varepsilon)G = 0$. Assume that $l(t)$ is a step input and that $d = 0$ with $r \neq 0$. Note that $\overline{L_r(t/\varepsilon - \tau/\varepsilon)} = \overline{L_r(t/\varepsilon)} = \overline{M(t/\varepsilon)} = 0$, and

$$\begin{aligned} &\overline{M(t/\varepsilon)\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)} \\ &\quad = \overline{(-1)^{r-1}BL_r(t/\varepsilon)F^{r-1}\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)} \\ &\qquad + \sum_{j=1}^{r-1} \overline{(-1)^{r-1-j}\binom{r-1}{j}A^jBL_r(t/\varepsilon)F^{r-j-1}} \\ &\quad = \overline{(-1)^{r-1}BL_r(t/\varepsilon)F^{r-1}\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)}. \end{aligned}$$

The proof of Theorem 4.2 now follows using identical arguments as in Theorem 4.1 following equation (16). Q.E.D.

C. δ-Equivalent Zeros

In the results of [25], it is proposed to use vibrational feedback control to move the open-loop zeros of a corresponding averaged equation. In this sense, vibrational feedback control (when $\tau = 0$) was demonstrated to be helpful in problems of finite-gain margin and decentralized fixed modes. However, in this subsection it is shown that the δ-equivalent zeros of $\sum_1$ always contain the open-loop zeros of (7). This implies that vibrational feedback control does not actually have zero placement capabilities in the sense of δ-equivalence.

Theorem 4.3: The zeros of the open-loop system (7) are contained in the δ-equivalent zeros of the closed-loop time-varying system $\sum_1$.

$$\begin{bmatrix} \eta^{(i)}(t) \\ x_c(t) \end{bmatrix} = \begin{bmatrix} I & \dfrac{1}{\varepsilon^{r-i-1}} \displaystyle\sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} A^j BL_{i+1}(t/\varepsilon)F^{i-j} \\ 0 & I \end{bmatrix} \begin{bmatrix} \eta^{(i+1)}(t) \\ x_c(t) \end{bmatrix} \qquad i = 1, 2, \cdots, r-2 \tag{18a}$$

$$\begin{aligned} \begin{bmatrix} \dot{\eta}^{(r-1)}(t) \\ \dot{x}_c(t) \end{bmatrix} = &\begin{bmatrix} A & BK + \dfrac{1}{\varepsilon} \displaystyle\sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{r-1}{j} A^j BL_{r-1}(t/\varepsilon)F^{r-j-1} \\ 0 & F + \dfrac{1}{\varepsilon} F_0(t/\varepsilon) \end{bmatrix} \begin{bmatrix} \eta^{(r-1)}(t) \\ x_c(t) \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ -GC & 0 \end{bmatrix} \begin{bmatrix} \eta^{(r-1)}(t-\tau) \\ x_c(t-\tau) \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} l(t) \end{aligned} \tag{19}$$

$$\dot{x}(t) = \begin{bmatrix} 0 & \dfrac{1}{\varepsilon} \displaystyle\sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{r-1}{j} A^j BL_{r-1}(t/\varepsilon)F^{r-j-1} \\ 0 & \dfrac{1}{\varepsilon} F_0(t/\varepsilon) \end{bmatrix} x(t) \tag{19a}$$

$$\begin{aligned} \dot{\chi}(t) = &\begin{bmatrix} A & AM(t/\varepsilon) + BK\Phi(t/\varepsilon) - M(t/\varepsilon)\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon) \\ 0 & \Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon) \end{bmatrix} \chi(t) \\ &+ \begin{bmatrix} 0 & 0 \\ -\Phi^{-1}(t/\varepsilon)GC & -GCA^{r-1}Bp^{-1}\left(\dfrac{t-\tau}{\varepsilon}\right)L_r\left(\dfrac{t-\tau}{\varepsilon}\right) \end{bmatrix} \chi(t-\tau) + \begin{bmatrix} 0 \\ \Phi^{-1}(t/\varepsilon)G \end{bmatrix} l(t) \end{aligned} \tag{20}$$

Proof: Consider the closed-loop transfer function for $\sum_2$ given by

$$\hat{H}_c(s, \tau) = \hat{C}(sI - \hat{A} + \hat{B}\hat{K}e^{-\tau s})^{-1}\hat{B}e^{-\tau s}. \quad (22)$$

Using simple matrix identities, (22) can be rearranged as follows:

$$\begin{aligned}\hat{H}_c(s, \tau) &= \hat{C}\{(sI - \hat{A})[I + (sI - \hat{A})^{-1}\hat{B}\hat{K}e^{-\tau s}]\}^{-1}\hat{B}e^{-\tau s} \\ &= \hat{C}[I + (sI - \hat{A})^{-1}\hat{B}\hat{K}e^{-\tau s}]^{-1}(sI - \hat{A})^{-1}\hat{B}e^{-\tau s} \\ &= \frac{[\hat{C}(sI - \hat{A})^{-1}\hat{B}]e^{-\tau s}}{[1 + \hat{K}(sI - \hat{A})^{-1}\hat{B}e^{-\tau s}]}.\end{aligned}$$

Clearly, the zeros of $\hat{H}_c(s, \tau)$ are the zeros of $\hat{C}(sI - \hat{A})^{-1}\hat{B}$. Assume $r \neq 0$. Then using $\hat{A}$, $\hat{B}$, and $\hat{C}$, as given in (18)

$$\begin{aligned}\hat{C}(sI - \hat{A})^{-1}\hat{B} &= [C(sI - A)^{-1}B] \\ &\quad \times [M_0(sI - \overline{\Phi^{-1}F\Phi})^{-1}G] \quad (23)\end{aligned}$$

where

$$M_0 = -[K\overline{\Phi} + (-1)^{r-1}\overline{L_r F^{r-1}\Phi^{-1}F\Phi}]$$

and A, B, C, G, and F are given in $\sum_1$.

Therefore, the zeros of $C(sI - A)^{-1}B$ are also the zeros of $\hat{H}_c(s, \tau)$, independent of the value of τ. By the definition of δ-equivalent zeros, this proves the theorem for $r \neq 0$.

Now assume that $r = 0$. In this case, (23) becomes

$$\begin{aligned}\hat{C}(sI - \hat{A})^{-1}\hat{B} &= [C(sI - A)^{-1}B + d] \\ &\quad \times N(sI - \overline{\Phi^{-1}F\Phi})^{-1}G \quad (24)\end{aligned}$$

where $N = K\overline{\Phi(t/\varepsilon)} + \overline{K_0(t/\varepsilon)\Phi(t/\varepsilon)}$. Notice that, once again, the zeros of (24) contain the zeros of (7). Q.E.D.

Remark 4.1: The results of Theorem 4.3 are surprising. Several authors have claimed that periodic controllers have arbitrary zero placement capabilities [12], [15], [16], [25]. In fact, such a claim was made in [25] for vibrational feedback control when $\tau = 0$. Clearly, such claims rely heavily on what is meant by zeros of a time-varying system.

Almost uniformly, results on periodic controllers assume zero reference input [12], [16], [21]–[25], [32], [35], [36], [38]. In fact, it is often shown that periodic controllers can robustly stabilize the zero equilibrium point of a zero input system whose plant has right half-plane zeros. However, when there is a reference input (step), the problem becomes more difficult. Stabilization is not the only critical issue. Hence, the notion of δ-equivalence has been introduced to include transient response performance measures. We suggest that the averaged output of the periodic controlled system should be closely approximated by the output of a time-invariant system in order to introduce a notion of a "zero" of a time-varying periodic system. In this manner, this research is proposing a new definition for zeros of periodic systems.

In [25], coordinate transformations are made in order to convert (7) (with $\tau = 0$) to a system in standard form so that averaging can be applied to zero reference input systems. The corresponding averaged equation can then be rewritten in the form $\dot{x} = Ax + Bu$, $w = Kx$; $w = u$. However, w is actually a dummy variable that is not δ-equivalent to the output of the system. Therefore, the open-loop zeros of the new averaged system in [25] could not be expected to correspond to the open-loop zeros of (7) (with $\tau = 0$). Additionally, for an input–output transfer function relationship, a reference input is needed.

For these reasons, it is proposed to interpret the effects of vibrational feedback control, for both ODE's and delay equations, as being δ-equivalent to a time-invariant system of order $2n$ subject to state feedback. Since state feedback more readily deals with zeros in the right half-plane, it is not surprising that vibrational feedback control has some of the capabilities noted in [25]. While this paper only specifically deals with periodic controllers in the form of (8), the results obviously open new questions/areas of open research to zero placement capabilities of periodic controllers in general. □

D. Controller Design

In this section, techniques are proposed that may be used to select the parameters in (8)–(11) in order to obtain a desired response for $\sum_1$. First, conditions for global δ-equivalence between $\sum_1$ and $\sum_2$ are presented. The controller gains are designed based on the response of $\sum_2$ which is time invariant. In this way it is possible to control δ-equivalent time-domain specifications.

Theorem 4.4: Assume $l(t)$ is a step input. Let $\hat{A}$, $\hat{B}$, $\hat{K}$, and $\hat{l}$ be as given in (14) or (18) [depending on the relative degree of (7)], and assume $\tau > 0$ is fixed. Suppose that there exist constants η_1 and η_2, $0 < \eta_1 < \eta_2$, such that for any $\varepsilon \in [\eta_1, \eta_2]$, $\det[sI - \hat{A} + \hat{B}\hat{K}e^{-\tau s}] = 0$ has all solutions with negative real parts. Define constants $\lambda_i = \tau\eta_i/(\tau + nT\eta_i)$, $i = 1, 2$, where n is an integer and T is the period of $F_0(t)$ and $K_r(t)$ in controller (8).

Then for any $\delta > 0$ there exists an integer $n_0 = n_0(\delta) > 0$ such that, when $n \geq n_0$ and $\varepsilon \in [\lambda_1, \lambda_2]$, $\sum_1$ and $\sum_2$ are globally δ-equivalent.

Proof: Suppose that $\varepsilon \in [\eta_1, \eta_2]$ such that $\det[sI - \hat{A} + \hat{B}\hat{K}e^{-\tau s}] = 0$ has all solutions with negative real parts. Then by Corollary 2.1, the arguments of Theorems 4.1 and 4.2 become valid for all $t \geq 0$ if $\varepsilon \in [\lambda_1, \lambda_2]$ and n is sufficiently large. By definition, this gives global δ-equivalence. Q.E.D.

Remark 4.2: The above theorem allows for the control of both (average) transient response and stability of $\sum_1$ by examining $\sum_2$ and either (14) or (18) (depending on the relative degree). For example, it is possible to design a δ-equivalent rise time by controlling the rise time of $\sum_2$. Asymptotic stability of $\sum_1$ is equivalent to designing a controller so that the conditions in Theorem 4.4 are satisfied. In fact, the conditions of Theorem 4.4, Theorem 2.2, and Corollary 2.1 guarantee that $y(t)$ of $\sum_1$ will approach a periodic orbit oscillating in the vicinity of the steady state value of $\hat{y}(t)$. In terms of $\overline{y}(t)$, this implies that $\lim_{t\to\infty} \overline{y}(t) \approx \hat{C}(-\hat{A} + \hat{B}\hat{K})^{-1}\hat{B}\hat{l}(t)$, where $\hat{l}(t) = $ constant. □

Remark 4.3: The problem of vibrational feedback control has been reduced to the design of a controller for a time-invariant system. Controller parameters can be selected using any number of methods. For example, the closed-loop transfer function of $\sum_2$ can be designed to satisfy specified requirements for step response, which guarantee stability and other design criteria. □

Alternatively, the technique [25] can be modified to design a stabilizing controller.

Definition 4.1: Suppose a SISO closed-loop feedback system has characteristic equation $\Delta(s) = 1 + G(s)$, where $G(s)$ is a rational proper transfer function. Suppose further that $\Delta(s) = 0$ has all solutions with negative real parts. Let the solutions, (ω, ϕ), of the equation

$$1 + G(j\omega)e^{-j\phi} = 0, \qquad \text{for all } 0 \le \phi \le 2\pi \text{ and } \omega > 0$$

be denoted by the set of pairs $\{(\omega_i, \phi_i)\}$. Then the *phase-frequency ratio*, β, of the system is defined as

$$\beta = \begin{cases} \min_i \left(\dfrac{\phi_i}{\omega_i}\right), & \text{if } \{(\omega_i, \phi_i)\} \ne \{\emptyset\} \\ \infty, & \text{if } \{(\omega_i, \phi_i)\} = \{\emptyset\}. \end{cases}$$

□

In many cases, ϕ_i will equal the positive phase margin of the system and ω_i will equal its corresponding phase crossover frequency. However, for the case when $|G(j\omega)| = 1$ has multiple frequency solutions, then one has to be extremely careful in defining phase margin. The phase-frequency ratio is a helpful stability analysis tool for time lag systems and will be used as part of the design algorithm presented below.

Theorem 4.5: Assume $l(t)$ is a step input, and define $\hat{G}_{\hat{K}}(s) = \hat{K}(sI - \hat{A})^{-1}\hat{B}$, where $\hat{A}$, $\hat{B}$, and $\hat{K}$ are as given in (14) or (18) (depending on relative degree). Suppose that there exist positive constants η_1 and η_2, $0 < \eta_1 \le \eta_2$, such that for any $\varepsilon \in [\eta_1, \eta_2]$, a closed-loop unity feedback system with loop gain $\hat{G}_{\hat{K}}(s)$ has positive phase-frequency ratio β. Define constants $\lambda_i = \tau\eta_i/(\tau + nT\eta_i)$, $i = 1, 2$, where n is an integer and T is the period of $F_0(t)$ and $K_r(t)$ in (8). Finally suppose that $0 \le \tau < \beta$, where $\tau \ge 0$ is the delay of the system.

Then for any $\delta > 0$ there exists an integer $n_0 = n_0(\delta) > 0$ such that, when $n \ge n_0$ and $\varepsilon \in [\lambda_1, \lambda_2]$, $\sum_1$ and $\sum_2$ are globally δ-equivalent.

Proof: It is known that the unity feedback system

$$\begin{aligned} \dot{x}(t) &= \hat{A}x(t) + \hat{B}u(t) \\ w(t) &= \hat{K}x(t - \tau) \\ u(t) &= l(t) - w(t) \end{aligned}$$

has positive phase margin under the conditions of the theorem. This occurs if and only if $\det[sI - \hat{A} + \hat{B}\hat{K}e^{-s\tau}] = 0$ has all solutions with negative real part. Therefore, Theorem 4.4 guarantees that, for $\varepsilon \in [\lambda_1, \lambda_2]$ and ε sufficiently small, $\sum_1$ is globally δ-equivalent to $\sum_2$. Q.E.D.

Remark 4.4: An algorithm for controlling (7) can now be derived. Parameters for the periodic compensator can be selected so that the conditions of Theorem 4.6 are true. It should be noted that when $\tau = 0$, it is shown in [25] that the zeros of $\hat{G}_{\hat{K}}(s)$ can be arbitrarily placed provided that (A, B, C) in (7) is controllable and observable. This in no way implies that δ-equivalent zeros of $\sum_1$ can be arbitrarily placed. As we previously suggested, the output w is not a direct measure of the true output y. However, zero placement of $\hat{G}_{\hat{K}}(s)$ does allow for the phase-frequency ratio of $\hat{G}_{\hat{K}}(s)$ to increase, making global δ-equivalence possible when $\tau = 0$. When $\tau \ne 0$, this statement is not necessarily true unless the controller is "tuned" in a special manner, as below. In order to apply these methods, it is beneficial to fix ε to be a specific value, related to the period and delay. This is demonstrated in the following theorem. □

Theorem 4.6: Let $\hat{G}_{\hat{K}}(s)$ be as in Theorem 4.6. Assume that $F_0(t)$ and $K_r(t)$ are T-periodic, and suppose that (A, B, C) in (7) is controllable and observable when $\tau = 0$.

Then the zeros of $\hat{G}_{\hat{K}}(s)$ can be arbitrarily placed, provided that $\tau/\varepsilon = nT$, where n is any nonnegative integer and τ is the fixed nonnegative constant delay.

Proof: The transfer function $\hat{G}_{\hat{K}}(s)$ when $r = 0$ is given as

$$\begin{aligned} \hat{G}_{\hat{K}}(s) &= T_1(s) \\ &= [\overline{-p^{-1}(t/\varepsilon)}C(sI - A)^{-1}BE \\ &\quad \times (sI - \overline{\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)})^{-1}G \\ &\quad + H(sI - \overline{\Phi^{-1}(t/\varepsilon)F(t/\varepsilon)})^{-1}G]e^{-\tau s} \end{aligned} \tag{25}$$

where

$$\begin{aligned} H &= d\left[\overline{Kp^{-1}(t/\varepsilon)\Phi\left(\frac{t-\tau}{\varepsilon}\right)} \right. \\ &\quad \left. + \overline{K_0\left(\frac{t-\tau}{\varepsilon}\right)p^{-1}(t/\varepsilon)\Phi\left(\frac{t-\tau}{\varepsilon}\right)}\right] \\ E &= [K\overline{\Phi(t/\varepsilon)} + \overline{K_0(t/\varepsilon)\Phi(t/\varepsilon)}]. \end{aligned}$$

When $r \ne 0$, $\hat{G}_{\hat{K}}(s)$ is given by

$$\begin{aligned} \hat{G}_{\hat{K}}(s) &= T_2(s) \\ &= [\overline{p^{-1}(t/\varepsilon)}C(sI - A)^{-1} \\ &\quad \times BM_1(sI - \overline{\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)})^{-1}G \\ &\quad + N(sI - \overline{\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)})^{-1}G]e^{-\tau s} \end{aligned} \tag{26}$$

where

$$\begin{aligned} M_1 &= K\overline{\Phi(t/\varepsilon)} \\ &\quad + (-1)^r\overline{L_r(t/\varepsilon)F^{r-1}\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)} \\ N &= \overline{CA^{r-1}Bp^{-1}(t/\varepsilon)L_r\left(\frac{t-\tau}{\varepsilon}\right)}. \end{aligned}$$

Letting $\tau/\varepsilon = nT$, the transfer functions (25) and (26) simplify to the two transfer functions

$$\begin{aligned} T_1(s) &= [-\overline{p^{-1}(t/\varepsilon)}C(sI - A)^{-1} \\ &\quad \times BE(sI - \overline{\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)})^{-1}G \\ &\quad + Q(sI - \overline{\Phi^{-1}(t/\varepsilon)F(t/\varepsilon)})^{-1}G]e^{-\tau s} \end{aligned} \tag{27}$$

where

$$\begin{aligned} Q &= d[K\overline{p^{-1}(t/\varepsilon)\Phi(t/\varepsilon)} + \overline{K_0(t/\varepsilon)p^{-1}(t/\varepsilon)\Phi(t/\varepsilon)}] \\ E &= [K\overline{\Phi(t/\varepsilon)} + \overline{K_0(t/\varepsilon)\Phi(t/\varepsilon)}] \end{aligned}$$

and

$$T_2(s) = T_3(s) + T_4(s) \tag{28}$$

where

$$
\begin{aligned}
T_3(s) &= \overline{p^{-1}(t/\varepsilon)}C(sI-A)^{-1} \\
&\quad \times BR(sI-\overline{\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)})^{-1}G \\
T_4(s) &= CA^{r-1}B\overline{p^{-1}L_r(t/\varepsilon)}(sI-\overline{\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)})^{-1} \\
&\quad \times Ge^{-\tau s} \\
R &= K\overline{\Phi(t/\varepsilon)}+(-1)^r\overline{L_r(t/\varepsilon)F^{r-1}\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)} \\
&= K\overline{\Phi(t/\varepsilon)}+[0 \quad 0 \quad \cdots \quad 0 \quad \overline{(-1)^r\gamma_{n-r}(t/\varepsilon)p(t/\varepsilon)}].
\end{aligned}
$$

Transfer functions (27) and (28) are exactly the same as the transfer functions defined in [25, Th. 4.1], only now multiplied by $e^{-\tau s}$. In [25], it is shown that the zeros of $T_1(s)$ and $T_2(s)$ can be arbitrarily placed. Therefore, the proof now follows from the arguments made in [25]. Q.E.D.

Remark 4.5: When $\tau/\varepsilon \neq nT$, it is not known whether zero placement capabilities of $\hat{G}_{\hat{K}}(s)$ are always possible. However, by tuning ε in a sufficient manner, the zeros of $\hat{G}_{\hat{K}}(s)$ can be moved arbitrarily. Often this will permit an increase in the phase-frequency ratio in Theorem 4.5. In particular, when $|\hat{G}_{\hat{K}}(j\omega)| = 1$ has only one phase crossover frequency, then $\beta = \phi_m/\omega_\phi$, where ϕ_m denotes the positive phase margin and ω_ϕ is equal to the corresponding *phase crossover frequency*, i.e., $1 + G(j\omega_\phi)e^{-j\phi_m} = 0$. In this case, the goal to design large β is equivalent to maximizing ϕ_m and minimizing ω_ϕ. While there are no guarantees (necessary and sufficient algorithms) that this procedure always works for general systems, in many applications it is successful. □

To summarize, one possible design procedure is now given.

Algorithm for Controller Design:

Step 1: Assume $\tau = 0$, and select controller gains so that $\hat{G}_{\hat{K}}(s)$ has large phase frequency ratio. This is often equivalent to designing a system to have a large phase margin ϕ_m and small (unique) phase crossover frequency ω_ϕ. A large phase frequency ratio can usually be accomplished by using the techniques of zero placement proposed in [25].

Step 2: Verify that $\beta > \tau$, where τ is the fixed positive delay. If not, change the controller parameters and repeat Step 1.

Step 3: Select $\varepsilon = \varepsilon_1 = \tau/nT$, where n is a sufficiently large positive integer. Completion of Steps 1–3 guarantees stabilization of $\sum_1$ and global dynamic equivalence to $\sum_2$.

Remark 4.6: It is now possible to understand why (7) is not robust with respect to small time delays. Theorem 4.6 is valid for $\tau/\varepsilon_1 = nT$. If ε_1 is small, then small perturbations in the delay move the zeros of $\hat{G}_{\hat{K}}(s)$ to undesirable locations, often causing a decrease in ϕ_m. This is because the elements of $\hat{K}$ explicitly depend on the ratio of τ/ε_1. Alternatively, for the same reasons it is seen that (7) is not robust to ε or the period. This is a surprising contrast to the results of [34] which suggest that periodic controllers can be robust to perturbations in the period. However, for the controllers presented in this paper, there is a clear lack of robustness with respect to both unmodeled delays and the period of the controller. This fact is demonstrated in the following section, and related results are given in [4]. □

Remark 4.7: For simplicity, this research has assumed that the reference input $l(t)$ was a step input. Similar results can be obtained when $l(t)$ is a continuous periodic or almost periodic function. □

E. Estimation of Output Ripple

Under the conditions of global δ-equivalence in Theorems 4.4 and 4.5, the output of $\sum_1$ will asymptotically tend to a periodic orbit, i.e., $y(t) \longrightarrow y^*(t, \varepsilon)$ as $t \longrightarrow \infty$, where $y^*(t+\varepsilon T, \varepsilon) = y^*(t, \varepsilon)$. Furthermore, by the definition of δ-equivalence, it is known that $|\overline{y^*(t, \varepsilon)} - \hat{y}_{ss}| < \delta(\varepsilon)$, where $\hat{y}_{ss}$ denotes the steady-state value of the output $\sum_2$. Theorems 4.4 and 4.5 provide important information on the qualitative changes in dynamic behavior caused by periodic control. However, the notion of δ-equivalence studies only the moving average of the system output, which does not completely describe the overall transient behavior.

It is possible to view $y(t)$, the output of $\sum_1$, as being composed of a slow trajectory superimposed with fast zero average harmonics. Furthermore, $\overline{y}(t)$ in Definition 3.3 represents a measure of this slow trajectory. As a result, it is possible to write $y(t) = \overline{y}(t) + \Psi(t, \varepsilon, \cdot)$, where Ψ represents the higher harmonics of the system. Under the conditions of global δ-equivalence, it is known that $|\overline{y}(t) - \hat{y}(t)| < \delta$. Therefore, when δ is small, an updated estimate on $y(t)$ is given by $y(t) \approx \hat{y}(t) + \Psi(t, \varepsilon, \cdot)$.

To estimate Ψ, it is possible to explicitly estimate the difference $y(t) - \hat{y}(t)$ by analyzing the proofs Theorems 4.1 and 4.2. Let $\Gamma(t, \varepsilon)$ be the $1 \times 2n$ vector function given by

$$
\Gamma(t, \varepsilon) \equiv \left[C \quad d\left(K + K_0\left(\frac{t-\tau}{\varepsilon}\right)\right)\Phi\left(\frac{t-\tau}{\varepsilon}\right)\right] \quad \text{for } r = 0 \tag{29a}
$$

$$
\Gamma(t, \varepsilon) \equiv \left[C \quad CA^{r-1}BL_r\left(\frac{t-\tau}{\varepsilon}\right)\right] \quad \text{for } r \neq 0 \tag{29b}
$$

where $\Phi(t)$ and $L_r(t)$ are defined in the statements of Theorems 4.1 and 4.2, respectively. Notice in Theorems 4.1 and 4.2 that $\hat{C} = \overline{\Gamma(t, \varepsilon)}$ in $\sum_2$. From (16) and (21) we have

$$
y(t) - \hat{y}(t) = \Gamma(t, \varepsilon)\chi(t-\tau) - \hat{C}\hat{x}(t-\tau)
$$

where $\chi(t)$ is given in the proofs of Theorems 4.1 and 4.2. If $\sum_1$ and $\sum_2$ are δ-equivalent for small fixed positive δ, then $\chi(t-\tau) \approx \hat{x}(t-\tau)$. Therefore, an estimate on the zero average output ripple can be given by

$$
\Psi(t, \varepsilon, \cdot) \approx \Theta(t, \varepsilon) \equiv [\Gamma(t, \varepsilon) - \hat{C}]\hat{x}(t-\tau).
$$

This gives (30a) and (30b), as shown at the bottom of the next page, where each matrix in (29) is known and given in Theorems 4.1 and 4.2.

An estimate of the output of $\sum_1$ can now be written as $y(t) \approx \hat{y}(t) + \Theta(t, \varepsilon)$. Calculation of Θ from (30) provides information on peak-to-peak deviations of $y(t)$ from $\hat{y}(t)$, as well as information on the shape of $y^*(t, \varepsilon)$.

Theorem 4.7: Define constants η_1, η_2, λ_1, λ_2, T, and τ as in Theorem 4.4. Let n denote an integer, and let $\Theta(t, \varepsilon)$ be as given in (30). Suppose that the assumptions of Theorem 4.4 are true.

Then for any $\beta > 0$ there exists an integer $n_0 = n_0(\beta)$ such that for $n \geq n_0$ and $\varepsilon \in [\lambda_1, \lambda_2]$

$$|y(t) - \hat{y}(t) - \Theta(t, \varepsilon)| < \beta, \qquad \text{for all } t \geq 0.$$

Proof: Let $\chi(t)$ be as defined in the proofs of Theorem 4.1 (for $r = 0$) and Theorem 4.2 (for $r \neq 0$). Then

$$\begin{aligned}|y(t) - \hat{y}(t) - \Theta(t, \varepsilon)| &= |\Gamma(t, \varepsilon)[\chi(t-\tau) - \hat{x}(t-\tau)]| \\ &\leq |\Gamma(t, \varepsilon)|\,|\chi(t-\tau) - \hat{x}(t-\tau)|\end{aligned}$$

where $\Gamma(t, \varepsilon)$ is defined in (29). Under the assumptions of Theorem 4.4, it is known that $\hat{x}(t)$ approaches a uniformly asymptotically stable equilibrium point. Therefore, by Corollary 2.1 and the proofs of Theorems 4.1 and 4.2, there will always exist an integer $n \geq n_0$ such that $|\chi(t-\tau) - \hat{x}(t-\tau)| < \sigma$ for all $t \geq 0$. Let $m = \sup_{s \in \Re} |\Gamma(s, \varepsilon)|$ (due to periodicity m is also given as $m = \sup_{s \in [0, T]} |\Gamma(s, 1)|$). Setting $\beta = \sigma/m$ completes the proof. Q.E.D.

Remark 4.8: Theorem 4.7 provides insight into the relationship between y and $\hat{y}$, whereas the previous theorems relate $\overline{y}$ and $\hat{y}$. A similar theorem can be written under the conditions of Theorem 4.5. □

V. Example and Robustness

A. Example

Consider the system

$$\begin{aligned}\dot{x}(t) &= x(t) + u(t) \\ y(t) &= -2x(t-\tau) + u(t-\tau)\end{aligned}$$

which has a transfer function given by $G(s) = G_p(s)e^{-\tau s} = (s-3)e^{-\tau s}/(s-1)$.

For $\tau = 0$, this plant was examined in [25]. Since the plant has a delay, a zero in the right half-plane, and a pole in the right half-plane, time-invariant controllers have limited capabilities [14], [21]. In fact, the delay in the system makes stabilization of this plant by previously known finite-dimensional techniques extremely difficult [14] (we could find no previously published techniques in the literature that could adequately stabilize and control this plant with finite-dimensional unity output feedback). We will now show that the proposed vibrational feedback controllers can robustly stabilize this system. Employ periodic controller (8) with $r = 0$ in the form of

$$\begin{aligned}\dot{x}_c(t) &= \left[f + \frac{a}{\varepsilon}\cos(t/\varepsilon)\right]x_c(t) + e(t) \\ u(t) &= [k + k^{(0)}\sin(t/\varepsilon)]x_c(t) \\ e(t) &= l(t) - y(t)\end{aligned}$$

where it is assumed that $l(t)$ is a step input. The following closed-loop averaged equation is obtained as

$$\begin{aligned}\dot{\hat{x}}(t) &= \begin{bmatrix}1 & k\overline{p(t/\varepsilon)} + k^{(0)}\overline{q(t/\varepsilon)} \\ 0 & f\end{bmatrix}\hat{x}(t) + \begin{bmatrix}0 \\ 1\end{bmatrix}\hat{u}(t) \\ \hat{y}(t) &= [-2 \quad [k\overline{p(t/\varepsilon)} + k^{(0)}\overline{q(t/\varepsilon)}]]\hat{x}(t-\tau) \\ \hat{u}(t) &= \hat{l}(t) - \left[-2\overline{p(t/\varepsilon)}\left[\overline{kp^{-1}(t/\varepsilon)p\left(\frac{t-\tau}{\varepsilon}\right)}\right.\right. \\ &\qquad \left.\left. + \overline{k^{(0)}\sin\left(\frac{t-\tau}{\varepsilon}\right)p^{-1}(t/\varepsilon)p\left(\frac{t-\tau}{\varepsilon}\right)}\right]\right]\hat{x}(t-\tau)\end{aligned}$$

where

$$\begin{aligned}p(t/\varepsilon) &= e^{\alpha \sin(t/\varepsilon)} \\ q(t/\varepsilon) &= \sin(t/\varepsilon)e^{\alpha \sin(t/\varepsilon)} \\ \hat{l}(t) &= \overline{p^{-1}(t/\varepsilon)}l(t).\end{aligned}$$

The closed-loop transfer function of $\sum_2$ is given by

$$\hat{H}(s,\tau) = \frac{[k\overline{p(t/\varepsilon)} + k^{(0)}\overline{q(t/\varepsilon)}](s-3)e^{-\tau s}}{(s-1)(s-f+M_0e^{-\tau s}) - 2\overline{p(t/\varepsilon)}[k\overline{p(t/\varepsilon)} + k^{(0)}\overline{q(t/\varepsilon)}]e^{-\tau s}}$$

where

$$M_0 = \left[\overline{kp^{-1}(t/\varepsilon)p\left(\frac{t-\tau}{\varepsilon}\right)} + \overline{k^{(0)}\sin\left(\frac{t-\tau}{\varepsilon}\right)p^{-1}(t/\varepsilon)p\left(\frac{t-\tau}{\varepsilon}\right)}\right].$$

Notice that the zero at $s = 3$ has not been moved. This is consistent with Theorem 4.3 which states that the δ-equivalent zeros contain the open-loop plant zeros.

In order to select proper parameters for a stabilizing controller, the control algorithm following Remark 4.5 can be followed directly.

$$\Theta(t, \varepsilon) \equiv \begin{bmatrix}0 & d\left(K + K_0\left(\frac{t-\tau}{\varepsilon}\right)\right)\Phi\left(\frac{t-\tau}{\varepsilon}\right) \\ & -d\left(\overline{K\Phi\left(\frac{t}{\varepsilon}\right)} + \overline{K_0\left(\frac{t}{\varepsilon}\right)\Phi\left(\frac{t}{\varepsilon}\right)}\right)\end{bmatrix}\hat{x}(t-\tau) \quad \text{for } r = 0 \tag{30a}$$

$$\Theta(t, \varepsilon) \equiv \left[0 \quad CA^{r-1}BL_r\left(\frac{t-\tau}{\varepsilon}\right)\right]\hat{x}(t-\tau) \quad \text{for } r \neq 0 \tag{30b}$$

Step 1: Assume $\tau = 0$. Then

$$\begin{aligned}\hat{G}_{\hat{K}}(s) &= [-2\overline{p(t/\varepsilon)} \quad k] \\ &\times \begin{bmatrix} s-1 & -[k\overline{p(t/\varepsilon)} + k^{(0)}\overline{q(t/\varepsilon)}] \\ 0 & s-f \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{-[2\overline{p(t/\varepsilon)}(k\overline{p(t/\varepsilon)} + k^{(0)}\overline{q(t/\varepsilon)}) - k(s-1)]}{(s-1)(s-f)}. \end{aligned} \tag{31}$$

The goal, now, is to select the pole and zero of $\hat{G}_{\hat{K}}(s)$ in such a manner as to maximize the phase frequency ratio β. Only the pole at $s = 1$ in (31) is fixed. After a few trial and error iterations on MATLAB, it is found that choosing a pole at $s = -3$ and a zero at $s = -4$ has high ratio β $(= \phi_m/\omega_\phi$ for this example). Then straightforward calculations, utilizing (31), yield the parameters $k = 1$, $k^{(0)} = -5.743\,271$, $\alpha = 1$, $f = -3$, which, in turn, yields $\overline{p} = 1.266\,07$ and $\overline{q} = 0.565\,15$. This gives

$$\hat{G}_{\hat{K}}(s) = \frac{s+4}{(s-1)(s+3)}$$

as desired.

Step 2: Verify that $\beta > \tau$. In this case, the system phase margin is $\phi_m = 0.637$ (rad) with a unique phase crossover frequency of $\omega_\phi = 0.8486$ and $\beta = \phi_m/\omega_\phi$. Therefore, τ can be any fixed constant satisfying $0 \le \tau < 0.75$.

Step 3: Choose $\varepsilon = \varepsilon_1$ fixed and sufficiently small to satisfy $\tau/\varepsilon_1 = 2\pi n$, where n is a sufficiently large positive integer. Then, $y(t)$ is globally δ-equivalent to $\hat{y}(t)$ for fixed nonnegative $\tau < 0.75$, i.e., the system output is bounded, and $\overline{y}(t)$ can be approximated by $\hat{y}(t)$. Hence, choosing the controller parameters as above will stabilize the system.

B. Simulation and Transient Response

Consider the above example with $k = 1, k^{(0)} = -5.734\,271$, $f = -3$, $\varepsilon = \frac{1}{50}$, and $l(t) = 10$. Assume that all states and inputs have zero initial functions. Fig. 1(a) plots $y(t)$, the output of the time-varying system, with $\hat{y}(t)$, the output of $\sum_2$, when $\tau = 0.376\,987\,1$. As the figure shows, the two systems are globally δ-equivalent and $\hat{y}(t)$ predicts the moving average of $y(t)$. Fig. 1(a) also demonstrates that the averaged equation approaches a steady value $\hat{y}_{ss} \approx 75$, while $y(t)$ approaches a periodic orbit $y^*(t, \varepsilon)$ with average value approximately equal to $\hat{y}_{ss}$. Notice that $\hat{y}_{ss}$ does not approximate half the peak-to-peak value of $y^*(t, \varepsilon)$. This is because $y^*(t, \varepsilon)$ is not symmetric about $\hat{y}_{ss}$, as Fig. 1(b) shows on a blown up time scale. As can be seen from Fig. 1(a), the output is stable. It should be mentioned that it is extremely difficult to stabilize this system with time-invariant unity output feedback controllers when $\tau = 0.376\,987\,1$. The theory predicts that for $\tau > 0.75$ the system is unstable. This has been verified by numerical simulations.

Notice that in Fig. 1(a) the outputs are initially in the opposite direction as the step input, i.e., the outputs first become negative. This is indicative of there being a δ-equivalent zero with positive real part at $s = 3$, which is consistent with Theorem 4.3.

The power of the results in this paper is that it is possible to control the transient behavior of $y(t)$ based on the characteristics of $\hat{y}(t)$. For example, in Fig 1(a) it is seen that the rise time of $\sum_2$ is given as $t_r = 2.15$ s, and hence, the δ-equivalent rise time of $y(t)$ (for a step input) is given as $t_{\delta r} = 2.15$ s. (Rise time represents the amount of time it takes for the moving average of the output to rise from 10 to 90% of its averaged steady-state value.) Other δ-equivalent transient response characteristics can also be read from Fig. 1(a) simply by measuring the characteristics of $\hat{y}(t)$, e.g., the δ-equivalent percent overshoot is 1.9% and the δ-equivalent 2% settling time is $t_{\delta s} = 4.05$ s. Therefore, the averaged transient design specifications for $\sum_1$ can be controlled by controlling the transient response of $\sum_2$.

Additionally, Theorem 4.7 suggests that an improved estimate on $y(t)$ is given by $y(t) \approx \hat{y}(t) + \Theta(t, \varepsilon)$, where Θ is given in (30). For the parameters in this example, $\Theta(t, \varepsilon) = ([1 - 5.743\,271 \sin(t/\varepsilon - \tau/\varepsilon)] \exp\{\sin(t/\varepsilon - \tau/\varepsilon)\} - 2)\hat{x}_2(t - \tau)$. Fig. 1(c) plots both $y(t)$ and $\hat{y}(t) + \Theta(t, \varepsilon)$ for $\varepsilon = 0.02$ and $\tau = 0.376\,987$. As Fig. 1(c) illustrates, including a ripple estimate in the approximation gives a better estimate on $y(t)$, although there still exist approximation errors. Fig. 1(d) shows, on a blown-up time scale, the improvement of including a ripple estimate. By Theorem 4.7, the approximation can be improved letting $\varepsilon = \tau/2\pi n$ and taking n larger. Numerical simulations verify that for n large enough, the approximation becomes almost exact.

C. Lack of Robustness with Respect to Delay

When the delay in the above system is assumed to be zero and the reference input is zero, then the trivial solution in $\sum_1$ is asymptotically stable for $\varepsilon = \frac{1}{50}$. This has been shown in [25] for similar parameters. However, for the case when $\tau \ne 0$, the transfer function for $\hat{G}_{\hat{K}}(s)$ is given by

$$\hat{G}_{\hat{K}}(s) = \frac{-[2\overline{p(t/\varepsilon)}(k\overline{p(t/\varepsilon)} + k^{(0)}\overline{q(t/\varepsilon)}) - M_0(s-1)]}{(s-1)(s-f)}$$

where M_0 is defined previously in this section. Consider the case when $\tau = 0.02$. Then, for the controller parameters above, i.e., $k = 1$, $k^{(0)} = -5.734\,271$, $f = -3$, and $\varepsilon = \frac{1}{50}$, the transfer function is

$$\hat{G}_{\hat{K}}(s) = \frac{[5 - 0.232(s-1)]}{(s-1)(s+3)}.$$

There is both an open-loop zero and open-loop pole in the right half-plane, obviously implying that the output of the system is unstable, as shown in Fig. 2. However, for $\tau = 2\pi n\varepsilon \le 0.75$, $n = 0, 1, \cdots$, the output, $y(t)$, approaches an asymptotically stable periodic orbit for fixed $\varepsilon > 0$ sufficiently small. That is, the system can be stable for fixed delays larger than 0.02, even though it is unstable for $\tau = 0.02$, e.g., the system is stable for $\varepsilon = 0.02$ and $\tau = 0.125\,66$. (As previously mentioned, the output is stable for $\tau = 0$.) This type of stability–instability sensitivity to the delay appears to be unique to fast time-varying systems with delay and illustrates the importance of exact modeling of the delay when employing vibrational feedback control. In order to guarantee that the output remains

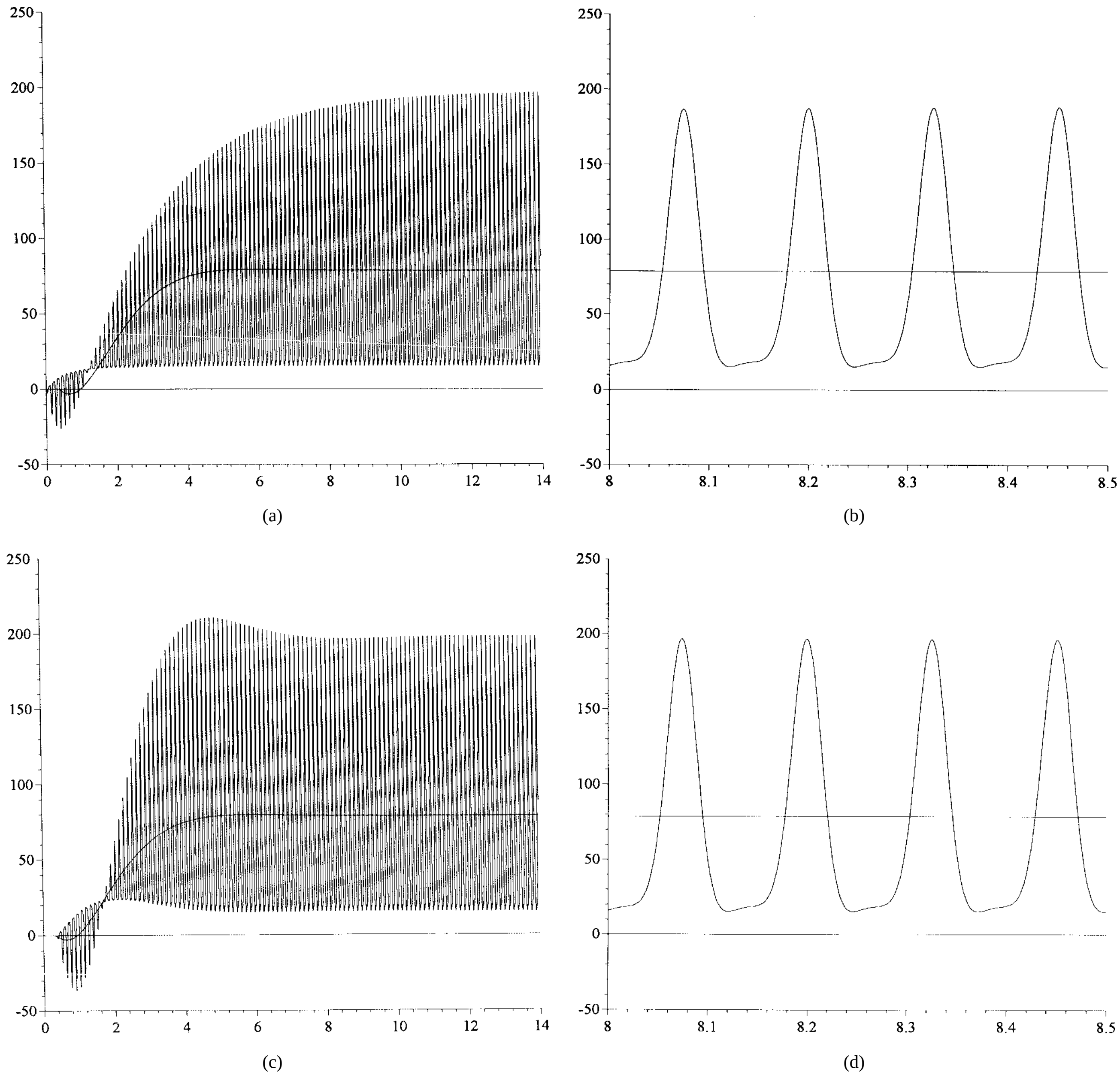

Fig. 1. Plots of the outputs versus t for the example with $\varepsilon = 0.02$, $\tau = 0.376\,987\,1$, $k = 1$, $k^{(0)} = -5.734\,271$, $f = -3$, and $\alpha = 1$. In (a) and (b), $y(t)$ and $\hat{y}(t)$ are plotted. In (c) and (d), $\hat{y}(t) + \Theta(t, \varepsilon)$ is plotted.

bounded for $\tau = 0.02$ and a step input, the parameter ε should be tuned so that $\varepsilon = \tau/2\pi n$. For example, when $\tau = 0.02$, one value of ε which guarantees stability is $\varepsilon = 0.003\,183$. In this case, $\hat{G}_{\hat{K}}(s)$ is as given in (31) and $y(t)$ becomes bounded.

The smaller the value of ε, the more sensitive the stability properties are to the delay. In fact, delays of the order $O(\varepsilon)$ must be modeled. This leads to a design tradeoff. Smaller ε improves the averaging approximation of $\overline{y}(t) \approx \hat{y}(t)$. On the other hand, smaller ε increases the system sensitivity to unmodeled delays.

D. Gain Margin

It is the interest of this section to show the effect of vibrational feedback controllers on the gain margin of the nonminimum phase system $G_\mu = \mu G_p(s) e^{-\tau s}$.

Definition 5.1: Let a and b be constant, $0 < a < b < \infty$. Assume that $1 + G_\mu(s) = 0$ has all roots with negative real parts for $\mu \in [a, b]$. Then the maximum gain margin $\equiv \max_{0<a<b<\infty} (b/a)$. □

It is known that if $G_p(s)$ has at least one zero in the right half-plane, any linear time-invariant controller will result in a finite-gain margin for $\tau = 0$. Consider $G_p(s) = (s - 3)e^{-\tau s}/(s - 1)$. When there is no delay, $\tau = 0$, the $\max(b/a) = 9$ (see [21] and [23]). In [25], the maximum gain margin improved to 32 by using a vibrational feedback controller when $\tau = 0$ and $\varepsilon = \frac{1}{50}$.

It will now be demonstrated that the proposed vibrational feedback controller has superior gain margin performance compared to any time-invariant controller when applied to time lag systems, provided the previously described methods

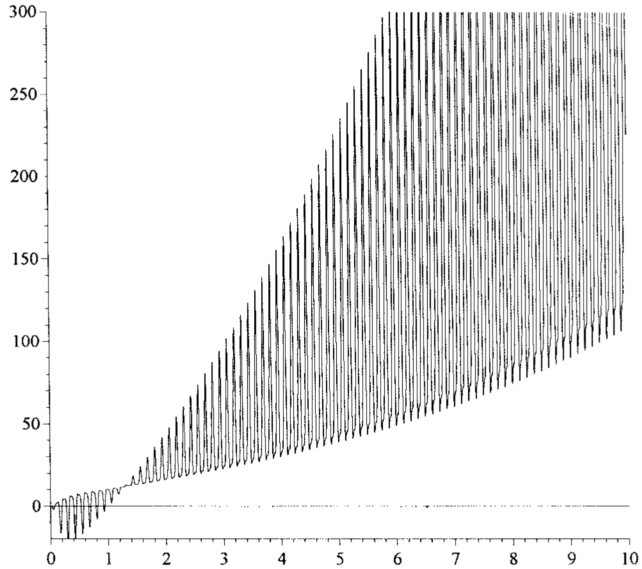

Fig. 2. Output $y(t)$ versus t for the example with $\varepsilon = 0.02$, $k = 1$, $k^{(0)} = -5.734\,271$, $f = -3$, $\alpha = 1$, and $\tau = 0.02$.

are employed. Select k, $k^{(0)}$, and f as before. Let $\tau = 0$ and $\varepsilon = 0.02$. Then, numerical simulations indicate that the system is stable for $\mu \in [0.75, 17.39]$, which yields a gain margin of 23.19. Keeping $\varepsilon = 0.02$ and increasing the value of the delay to $\tau = 0.125\,663\,7$ $(= 2\pi\varepsilon)$, numerical simulations give a stability range of $\mu \in [0.76, 10.76]$, yielding a gain margin of 14.16. Hence, even though there is a delay present, vibrational feedback controllers still demonstrate superior gain margins over LTI compensators applied to the system when there is no delay. Theoretical predictions of the gain margin, based on Nyquist plots of $\hat{G}_{\hat{K}}(s)$, predict stability for $\mu \in [0.76, 11.01]$ when $\tau = 0.125\,663\,7$.

Increasing the delay will, as expected, decrease the gain margin. For $\tau = 0.376\,987\,1$ and $\varepsilon = 0.02$, numerical simulations show that stability is maintained for $\mu \in [0.76, 2.49]$. For $\tau = 0.628\,318\,5$ and $\varepsilon = 0.02$, stability is maintained for $\mu \in [0.76, 1.22]$. The theoretical predictions for $\tau = 0.376\,987\,1$ and $\tau = 0.628\,318\,5$ are $\mu \in [0.76, 2.61]$ and $\mu \in [0.76, 1.28]$, respectively. As expected, theoretical predictions of stability ranges become more accurate when $\varepsilon = \tau/2\pi n$ and integer n is taken to be larger.

VI. Conclusions

This paper extends the technique of vibrational feedback control to systems with delay. Averaging theory for differential delay equations is presented and then applied to aid in the controller design. A method of control design is introduced and examples are presented to illustrate issues such as stability, gain margin, and robustness with respect to delay. The results of this research indicate that vibrational feedback control for delay systems can substantially improve performance when applied correctly.

Appendix

In order to prove Theorem 2.1 and 2.2, we must use the following lemmas.

Lemma A.1: Let $F_i(t)$ be continuous T-periodic $n \times n$ matrices, $i = 1, 2$. Let μ and ε denote arbitrary constants. Then, for all (t, μ, ε) in compact subsets of $\Re \times \Re \times \Re$, there exists a continuous function $\gamma(s)$: $(0, \infty) \rightarrow [0, \infty)$, monotonically decreasing, such that $\gamma(s) \rightarrow 0$ as $s \rightarrow \infty$ and

$$\left\| \frac{1}{s} \int_t^{t+s} F_1(\lambda) F_2\Big(\lambda - \frac{\mu}{\varepsilon}\Big) d\lambda - \frac{1}{T} \int_0^T F_1(\lambda) F_2\Big(\lambda - \frac{\mu}{\varepsilon}\Big) d\lambda \right\| \le \gamma(s) \tag{32}$$

for all $s > 0$. Furthermore, γ is completely independent of μ and ε.

Proof: Since $F_i(\sigma + T) = F_i(\sigma)$, it is also true that $F_1(\sigma)F_2(\sigma - \mu/\varepsilon)$ is T-periodic. The proof is now divided into the following two intervals.

Case 1) $0 \le s < 2T$: Because $F_1(\sigma)F_2(\sigma - \mu/\varepsilon)$ is continuous and periodic, there exists an $M > 0$ such that $\|F_1(\sigma)F_2(\sigma - \mu/\varepsilon)\| \le M$ for all $(\sigma, \mu, \varepsilon)$ in compact subsets of $\Re \times \Re \times \Re$. Therefore, on $0 \le s < 2T$

$$\left\| \frac{1}{s} \int_t^{t+s} F_1(\lambda) F_2\Big(\lambda - \frac{\mu}{\varepsilon}\Big) d\lambda - \frac{1}{T} \int_0^T F_1(\lambda) F_2\Big(\lambda - \frac{\mu}{\varepsilon}\Big) d\lambda \right\| \le 2M.$$

Case 2) $s \ge 2T$: For $s \ge 2T$, there always exists an integer $n \ge 2$ and a constant $R < T$ such that $s = nT + R$. Therefore, $1/s = 1/(nT + R) = 1/nT - R/(nT(nT + R))$. This implies that

$$\begin{aligned} &\left\| \frac{1}{s} \int_t^{t+s} F_1(\lambda) F_2\Big(\lambda - \frac{\mu}{\varepsilon}\Big) d\lambda - \frac{1}{T} \int_0^T F_1(\lambda) F_2\Big(\lambda - \frac{\mu}{\varepsilon}\Big) d\lambda \right\| \\ &= \left\| \frac{1}{nT} \int_t^{t+nT} F_1(\lambda) F_2\Big(\lambda - \frac{\mu}{\varepsilon}\Big) d\lambda - \frac{1}{T} \int_0^T F_1(\lambda) F_2\Big(\lambda - \frac{\mu}{\varepsilon}\Big) d\lambda \right. \\ &\quad + \frac{1}{s} \int_{t+nT}^{t+nT+R} F_1(\lambda) F_2\Big(\lambda - \frac{\mu}{\varepsilon}\Big) d\lambda \\ &\quad \left. - \frac{R}{s(s-R)} \int_t^{t+nT} F_1(\lambda) F_2\Big(\lambda - \frac{\mu}{\varepsilon}\Big) d\lambda \right\| \\ &\le 0 + \frac{1}{s} \left[\int_{t+nT}^{t+nT+R} \Big\| F_1(\lambda) F_2\Big(\lambda - \frac{\mu}{\varepsilon}\Big) \Big\| d\lambda \right. \\ &\quad \left. + \frac{R}{(s-R)} \int_t^{t+nT} \Big\| F_1(\lambda) F_2\Big(\lambda - \frac{\mu}{\varepsilon}\Big) \Big\| d\lambda \right] \end{aligned}$$

where we have used the fact that for a T-periodic function, $f(t)$, the integral $(1/nT) \int_t^{t+nT} f(\lambda) d\lambda = (1/T) \int_0^T f(\lambda)\, d\lambda$.

Therefore, for $s \geq 2T$ and $n \geq 2$

$$\left\| \frac{1}{s}\int_t^{t+s} F_1(\lambda)F_2\Big(\lambda - \frac{\mu}{\varepsilon}\Big)d\lambda - \frac{1}{T}\int_0^T F_1(\lambda)F_2\Big(\lambda - \frac{\mu}{\varepsilon}\Big)d\lambda \right\| \leq \frac{1}{s}\left[\int_{t+nT}^{t+nT+R} M d\lambda + \frac{T}{(n-1)T}\int_t^{t+nT} M d\lambda\right] \leq \frac{3MT}{s}.$$

The proof is now complete by letting $\gamma(s) = 4MT/s$ in (32). Q.E.D.

Lemma A.2: Let $F_i(t) \in \Re^{n\times n}$ and $b(t) \in \Re^{n\times 1}$ be continuous and T-periodic. Let G_1, $G_2(\mu/\varepsilon)$ and v be as defined in (2). Then for any piecewise constant function $\tilde{x}(t)$ and any constant $L > 0$, there exists a continuous function $\gamma(s)$: $(0, \infty) \to [0, \infty)$, monotonically decreasing such that $\gamma(s) \to 0$ as $s \to \infty$ and

$$\left|\int_0^t \left[\Big(F_1\Big(\frac{\lambda}{\varepsilon}\Big) - G_1\Big)\tilde{x}(\lambda) + \Big(F_2\Big(\frac{\lambda}{\varepsilon}\Big)F_3\Big(\frac{\lambda-\mu}{\varepsilon}\Big) - G_2(\mu/\varepsilon)\Big)\tilde{x}(\lambda-\tau) + \Big(b\Big(\frac{\lambda}{\varepsilon}\Big) - v\Big)\right]d\lambda\right| \leq \gamma\Big(\frac{1}{\varepsilon}\Big)$$

for any $t \in [0, L]$, any finite $\mu > 0$, and any $\varepsilon > 0$.

Proof: Since $\tilde{x}(t)$ is piecewise constant, $\tilde{x}(t-\tau)$ is also piecewise constant. Therefore, there exists an increasing sequence $\{\alpha_i\}$, $i = 0, \cdots, N$ with $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_N = t$, $t \leq L$, and sets of constants $\{m_j\}$, $\{n_j\}$, $j = 0, 1, \cdots, N$, such that

$$\int_0^t \left[\Big(F_1\Big(\frac{\lambda}{\varepsilon}\Big) - G_1\Big)\tilde{x}(\lambda) + \Big(F_2\Big(\frac{\lambda}{\varepsilon}\Big)F_3\Big(\frac{\lambda-\mu}{\varepsilon}\Big) - G_2(\mu/\varepsilon)\Big)\tilde{x}(\lambda-\tau) + \Big(b\Big(\frac{\lambda}{\varepsilon}\Big) - v\Big)\right]d\lambda$$
$$= \sum_{i=1}^N \int_{\alpha_{i-1}}^{\alpha_i}\left[\Big(F_1\Big(\frac{\lambda}{\varepsilon}\Big) - G_1\Big)m_i + \Big(F_2\Big(\frac{\lambda}{\varepsilon}\Big)F_3\Big(\frac{\lambda-\mu}{\varepsilon}\Big) - G_2(\mu/\varepsilon)\Big)n_i + \Big(b\Big(\frac{\lambda}{\varepsilon}\Big) - v\Big)\right]d\lambda.$$

Notice that by Lemma A.1

$$\left|\sum_{i=1}^N \int_{\alpha_{i-1}}^{\alpha_i}\left[\Big(F_1\Big(\frac{\lambda}{\varepsilon}\Big) - G_1\Big)m_i + \Big(F_2\Big(\frac{\lambda}{\varepsilon}\Big)F_3\Big(\frac{\lambda-\mu}{\varepsilon}\Big) - G_2(\mu/\varepsilon)\Big)n_i + \Big(b\Big(\frac{\lambda}{\varepsilon}\Big) - v\Big)\right]d\lambda\right|$$
$$\leq \sum_{i=1}^N (\alpha_i - \alpha_{i-1})\left|\frac{\varepsilon}{(\alpha_i - \alpha_{i-1})} \times \int_{\alpha_{i-1}/\varepsilon}^{\alpha_{i-1}/\varepsilon + (\alpha_i - \alpha_{i-1})/\varepsilon} \times \Big[(F_1(s) - G_1)m_i + \Big(F_2(s)F_3\Big(s - \frac{\mu}{\varepsilon}\Big) - G_2(\mu/\varepsilon)\Big)n_i + (b(s) - v)\Big]ds\right|$$
$$\leq \sum_{i=1}^N (\alpha_i - \alpha_{i-1})\gamma_i\Big(\frac{\alpha_i - \alpha_{i-1}}{\varepsilon}\Big)$$

where $\gamma_i(s)$: $(0, \infty) \to [0, \infty)$ are monotonically decreasing continuous functions with $\gamma_i(s) \to 0$ as $s \to \infty$. Since the sum of monotonically decreasing functions is another monotonically decreasing function, the proof is complete. Q.E.D.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1: Integrating (3) and (4) one can obtain

$$\begin{aligned}|x(t;\varphi) - y(t;\varphi)| &= \left|\int_0^t \left[F_1\Big(\frac{\lambda}{\varepsilon}\Big)x(\lambda;\varphi) + F_2\Big(\frac{\lambda}{\varepsilon}\Big)F_3\Big(\frac{\lambda-\mu}{\varepsilon}\Big)x(\lambda-\tau;\varphi) + b\Big(\frac{\lambda}{\varepsilon}\Big)\right]d\lambda \right.\\ &\quad \left. - \int_0^t [G_1 y(\lambda;\varphi) + G_2(\mu/\varepsilon)y(\lambda-\tau;\varphi) + v]d\lambda\right|. \end{aligned} \tag{33}$$

Using the techniques of [28, Th. 2.2] (applying the Fundamental Theorem of Calculus), a piecewise constant function, $\tilde{x}(t)$ can always be constructed such that for any $\sigma > 0$ and any $k > 0$

$$\begin{aligned}&\left|\int_0^t \left[\Big(F_1\Big(\frac{\lambda}{\varepsilon}\Big) - G_1\Big)x(\lambda;\varphi) + \Big(F_2\Big(\frac{\lambda}{\varepsilon}\Big)F_3\Big(\frac{\lambda-\mu}{\varepsilon}\Big) - G_2(\mu/\varepsilon)\Big)x(\lambda-\tau;\varphi)\right]d\lambda\right| \\ &\leq \frac{2\sigma}{3}e^{-kL} + \left|\int_0^t \left[\Big(F_1\Big(\frac{\lambda}{\varepsilon}\Big) - G_1\Big)\tilde{x}(\lambda;\varphi) + \Big(F_2\Big(\frac{\lambda}{\varepsilon}\Big)F_3\Big(\frac{\lambda-\tau}{\varepsilon}\Big) - G_2(\mu/\varepsilon)\Big) \times \tilde{x}(\lambda-\tau;\varphi)\right]d\lambda\right| \end{aligned} \tag{34}$$

for any $t \in [0, L]$. The constant k will be selected later in the proof.

Therefore (33) becomes

$$|x(t;\varphi)-y(t;\varphi)| \le \left|\int_0^t \left[\left(F_1\left(\frac{\lambda}{\varepsilon}\right)-G_1\right)x(\lambda;\varphi) + \left(F_2\left(\frac{\lambda}{\varepsilon}\right)F_3\left(\frac{\lambda-\tau}{\varepsilon}\right)-G_2(\mu/\varepsilon)\right) \times x(\lambda-\tau;\varphi)\right]d\lambda\right| + \left|\int_0^t [G_1(x(\lambda;\varphi)-y(\lambda;\varphi)) + G_2(\mu/\varepsilon)(x(\lambda-\tau;\varphi)-y(\lambda-\tau;\varphi))]\,d\lambda\right| + \left|\int_0^t \left[b\left(\frac{\lambda}{\varepsilon}\right)-v\right]d\lambda\right|$$

which, due to (34) and Lemma A.2, yields

$$|x(t;\varphi)-y(t;\varphi)| \le \frac{2\sigma}{3}e^{-kL}+\gamma\left(\frac{1}{\varepsilon}\right)+\left|\int_0^t [G_1(x(\lambda;\varphi)-y(\lambda;\varphi)) + G_2(\mu/\varepsilon)(x(\lambda-\tau;\varphi)-y(\lambda-\tau;\varphi))]\,d\lambda\right|.$$

By [10, Remark 54, p. 46], $G_2(\mu/\varepsilon)$ is continuous and periodic in ε, and therefore, there exists a constant $M>0$ such that

$$|x(s;\varphi)-y(s;\varphi)| \le \frac{2\sigma}{3}e^{-kL}+\gamma\left(\frac{1}{\varepsilon}\right)+M\int_0^t |x(\lambda-\tau;\varphi) - y(\lambda-\tau;\varphi)|\,d\lambda \tag{35}$$

for all $t\in[0,L]$. The right-hand side of (35) is an increasing function, and therefore, this implies that

$$\sup_{s\in[-\tau,t]}|x(s;\varphi)-y(s;\varphi)| \le \frac{2\sigma}{3}e^{-kL}+\gamma\left(\frac{1}{\varepsilon}\right)+M\int_0^t \sup_{s\in[-\tau,\lambda]}|x(s;\varphi) - y(s;\varphi)|\,d\lambda$$

for all $t\in[0,L]$. Choosing ε sufficiently small so that $\gamma(1/\varepsilon)=\sigma/3\,e^{-kL}$, letting $k=M$, and applying a modified Gronwall's inequality for delay systems (see [11, p. 190] for a detailed discussion of application of modified Gronwall's inequality to functional differential equations, or see [30] for related results), (35) yields

$$\sup_{s\in[-\tau,L]}|x(s;\varphi)-y(t;\varphi)|\le\sigma\exp\{M(t-L)\}$$

for all $t\in[0,L]$. Therefore, for all $t\in[0,L]$, this implies that $|x(t;\varphi)-y(t;\varphi)|\le\sigma$.

Proof of Theorem 2.2: Under the conditions of the theorem, y_s is uniformly asymptotically stable (see [15, ch. 3.2]). Therefore, due to Theorem 2.1, Part 1) of Theorem 2.2 follows using similar arguments as those used in [28, Th. 2.3].

Since (1) is linear, it has a unique periodic solution $x^*(t,\varepsilon)$ (see [18, ch. 8]). By Part 1) of the theorem, it is seen that the ball $B_r=\{x:|x-y_s|<\sigma\}$ contains a global attractor of (1). By uniqueness, this implies that $x(t;\varphi)\to x^*(t,\varepsilon)$ as $t\to\infty$. Furthermore, the fact that $y(t)$ exponentially decays to y_s, combined with 1), immediately implies uniform asymptotic stability of $x^*(t,\varepsilon)$.

Now we will show that $|x^*(t,\varepsilon)-y_s|\le\rho$. By the properties of uniform asymptotic stability, for any $\rho>0$, there exists an $L>0$ such that $|x(L;\varphi)-x^*(t;\varepsilon)|\le\rho/3$ and $|y(L;\varphi)-y_s|\le\rho/3$ for all $t>L$. Let $\sigma=\rho/3$, where σ is given in Part 1) of the theorem. Then for any $\varepsilon\in[\eta_1,\eta_2]$ and $\varepsilon\le\varepsilon_0(\rho/3)$, $|x(L;\varphi)-y(L;\varphi)|<\rho/3$. Therefore

$$|x^*(t,\varepsilon)-y_s| \le |x(L;\varphi)-x^*(t;\varepsilon)| + |x(L;\varphi)-y(L;\varphi)|+|y(L;\varphi)-y_s| < \frac{\rho}{3}+\frac{\rho}{3}+\frac{\rho}{3}=\rho$$

which completes the proof. Q.E.D.

Proof of Corollary 2.1: By [10, paragraph 27], it is known that $G_2(\mu/\varepsilon+T)=G_2(\mu/\varepsilon)$. From the assumptions of Corollary 2.1, $\det[sI-G_1-G_2(\mu/\varepsilon)e^{-\tau s}]=0$ has all solutions with negative real part when $\varepsilon\in[\eta_1,\eta_2]$. Due to periodicity of $G_2(\cdot)$, this implies that all solutions to this characteristic equation will also have negative real parts for $\varepsilon\in[\mu\eta_1/(\mu+nT\eta_1),\ \mu\eta_2/(\mu+nT\eta_2)]$ for all integers n (provided that the denominator is not equal to zero). Taking n sufficiently large guarantees that $[\mu\eta_1/(\mu+nT\eta_1),\mu\eta_2/(\mu+nT\eta_2)]\cap(0,\varepsilon_0]\ne\{\emptyset\}$. Defining $\lambda_i=\mu\eta_i/(\mu+nT\eta_i)$, $i=1,2$, completes the proof. Q.E.D.

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