

Extensions of Averaging Theory for Power Electronic Systems

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Abstract— This paper extends averaging theory for power electronic systems to include feedback controlled converters. New averaging techniques based on the integral equation description provide theoretical justification for commonly used averaging methods. The new theory provides a basis for answering fundamental questions about the averaging approximation. A ripple estimate expression is presented, along with the simulation results for a feedback controlled boost converter.

I. INTRODUCTION

STATE space averaging techniques are commonly used in the analysis and control design of pulse width modulated (PWM) power electronic systems [1]–[3]. However, it was not until recently that rigorous mathematical justification [3], [4] was given that theoretically explained the applications of these averaging techniques. As [3] and [5] have pointed out, the theoretical development of PWM systems lags far behind the many practical control applications.

In [3], classical Russian averaging techniques [6], [7] are shown to be applicable to several types of PWM power electronic systems, such as open loop dc–dc converters. Besides using these classical averaging techniques to prove stability, [3] also gives a ripple estimate for improving the accuracy of the averaging technique, even for systems with large ripple. However, the application of the results of [3] is limited to systems with time discontinuities.¹

In fact, the classical averaging theory used in [3] is not applicable when there are state discontinuities. This is significant because all feedback controlled converters are state discontinuous. In [3], the argument is made that smooth commutation models can be used in place of the discontinuous Heaviside unit step function to avoid any state discontinuity in the mathematical system model. In essence, this idea was introduced by Filippov [8] to justify what is meant by solutions to state discontinuous differential equations. The work of [9] continues this line of thinking by presenting

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¹In this paper, a system with “time discontinuity” is described by a differential equation whose right-hand side is discontinuous with respect to time. A system with “state discontinuity” is described by a differential equation whose right-hand side is discontinuous with respect to a state variable.

stability results which rely on abstract averaging theory (see references in [9]) that partially combine the results of [6] and [7] with the theory of Filippov [8].

It is the purpose of this paper to introduce averaging techniques that are general enough to encompass both time discontinuity and large classes of state discontinuity, without utilizing the (difficult) theory of Filippov. Because the proofs are straightforward (essentially relying on the Fundamental Theorem of Calculus and Gronwall’s inequality), insight on both transient and asymptotic behavior of PWM feedback controlled dc–dc converters is obtained. The results of this paper begin to provide theoretical justification for commonly used averaging techniques. In addition, this work points out some shortcomings in the averaging technique (which to our knowledge have not been documented before). Some readers may question whether there is a significant contribution in writing a paper that theoretically justifies models that have been in use for so many years. However, we believe that it is vital to bridge theory with practice in order for future fundamental contributions to be made. In fact, the theoretical results of this paper have led to the discoveries of new, more accurate switching-frequency-dependent-averaged models [10], published in a separate paper.

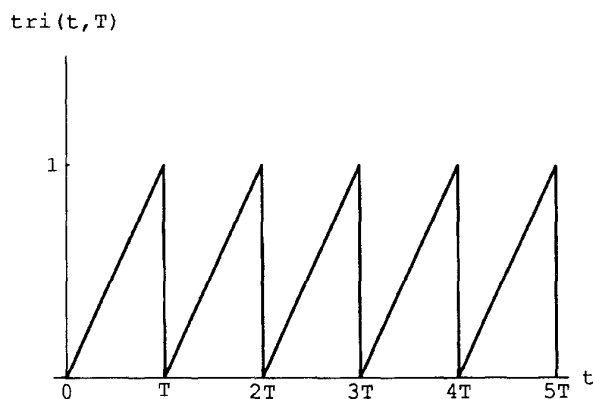
Section II reviews some of the mathematical issues associated with state discontinuous systems. The primary theoretical contribution of this paper is contained in two theorems presented in Section III. Section IV discusses the practical implications of the results of Section III and gives numerical examples and computer simulations. Section V draws conclusions.

II. THEORETICAL PRELIMINARIES

The difficulty in mathematically justifying averaging approximation techniques of state discontinuous differential equations can be best explained through an example. Consider the state discontinuous differential equation

$$\dot{x}(t) = f(x) + bu(d(x) - \text{tri}(t, T)) \quad (2.1)$$

where $x \in \mathcal{R}^n$, $b \in \mathcal{R}^n$, $f: \mathcal{R}^n \rightarrow \mathcal{R}^n$ and $d: \mathcal{R}^n \rightarrow \mathcal{R}$ are both continuous functions with $0 \leq d(x) \leq 1$, and $u(\cdot)$ is the Heaviside step function, i.e., $u(s) = 1$ for $s \geq 0$ and $u(s) = 0$ for $s < 0$. The function $\text{tri}(t, T) = (t/T) - \text{floor}(t/T) = (t \bmod T)/T$ is shown in Fig. 1. Equation (2.1) is a typical representation of a feedback controlled PWM Buck converter [3]. The theory presented in [3], however, only applies to

Fig. 1. $\text{Tri}(t, T)$.

open-loop control and does not extend to feedback controlled converters.

The usual condition for a unique solution of (2.1) to exist is that the right-hand side satisfy a Lipschitz condition. (A function, $f(x)$ is said to be Lipschitz with constant $k > 0$ if $\|f(x) - f(y)\| \leq k\|x - y\|$ for any $x \in \mathcal{R}^n, y \in \mathcal{R}^n$.) However (2.1) is not Lipschitz since it is discontinuous with respect to x . Hence, standard approaches fail when trying to prove the existence of a unique solution—which implies that formal averaging approximations of (2.1) cannot, in general, be directly derived. There is an extensive amount of literature on differential inclusions that shows how one can redefine what is meant by a unique solution to (2.1) (see Filippov [8]). However, this paper shows that, under the standard operating conditions of power electronic systems (no chattering), the theory of differential inclusions is not needed to theoretically justify averaging approximations.

While in general “standard” solutions to (2.1) are not known to exist, under the proper conditions (see Section II-A), there are a finite number of jumps in the right-hand side of (2.1) on any finite time interval, and each jump (switch) is norm bounded due to the fact that $0 \leq u(\cdot) \leq 1$. This implies that (under these conditions) the right-hand side of (2.1) is Lebesgue integrable for all $t \geq t_0$ and that the solution of the integral equation

$$\begin{aligned} x(t; t_0, x(t_0)) &= x(t) \\ &\equiv x(t_0) + \int_{t_0}^t [f(x(s)) \\ &\quad + bu(d(x(s)) - \text{tri}(s, T))] ds \end{aligned} \quad (2.2)$$

is unique and *satisfies state differential equation (2.1) almost everywhere*. Hence, when no chattering occurs in the system, the “standard” solution to (2.1) can be derived and will be equal to the solution of integral equation (2.2) almost everywhere.

Furthermore, when there is no chattering, $x(t; t_0, x(t_0)) = x(t)$, as given by (2.2), is a continuous function that depends continuously on its switching period, T . Using this fact, [4] develops approximation techniques by examining (2.2) instead of (2.1). This work by Sira-Ramirez shows that the solution

of (2.2) can be accurately approximated by an autonomous averaged system by letting $T \rightarrow 0$. In [4], it is shown that there always exists a sufficiently small sampling period T , for which the deviations between the actual PWM controlled responses (of an integral equation) and those of an averaged model, under identical initial conditions, remain arbitrarily close to each other. This, of course, is an immediate consequence of continuity on T .

Therefore, it seems reasonable to approach the problem of approximating the dynamics of (2.1) by using classical averaging techniques on integral equation (2.2). Classical averaging techniques have the advantage over the techniques of [4] because they provide answers to fundamental questions about the validity of the approximation. By performing averaging on an integral equation instead of a differential equation, this paper will show that the difficulties due to many types of state discontinuities are eliminated. This approach allows a rigorous explanation, which was not provided in [3] and [9].

Most classical averaging techniques [6], [7], though, are not directly applicable to integral equations. However, recently, new state space averaging theory has been developed that relies entirely on the representation of solutions of differential equations by their corresponding integral equation [11], [12]. The results of [11] and [12] are written for infinite dimensional dynamical systems, but the techniques, as this paper shows, can also be applied to ordinary differential equations.

III. AVERAGING OF STATE DISCONTINUOUS POWER ELECTRONIC SYSTEMS

In general form, feedback pulse width modulated systems considered in this paper will be modeled by the integral equation

$$\begin{aligned} x(t; t_0, x(t_0)) &= x(t) \\ &\equiv x(t_0) + \int_{t_0}^t \left[f_0(x(s)) ds + \sum_{i=1}^N f_i(x(s)) \right. \\ &\quad \left. \cdot u(d_i(x(s)) - \text{tri}(s, T)) \right] ds \end{aligned} \quad (3.1)$$

where it will always be assumed that $x \in \mathcal{R}^n$, t_0 denotes initial time, and $f_i: \mathcal{R}^n \rightarrow \mathcal{R}^n$ are locally Lipschitz functions, i.e., there exists an open neighborhood $\Omega \subset \mathcal{R}^n$ such that for every $x_1 \in \Omega, x_2 \in \Omega$, there are constant positive k_i satisfying $\|f_i(x_1) - f_i(x_2)\| \leq k_i\|x_1 - x_2\|$. The functions $d_i: \mathcal{R}^n \rightarrow \mathcal{R}$ are the duty ratios and will also be assumed locally Lipschitz in Ω with Lipschitz constant m_i . Furthermore, they will always satisfy $0 \leq d_i(x) \leq 1$.

Along with (3.1), consider the corresponding “averaged” integral equation

$$\begin{aligned} y(t; t_0, y(t_0)) &= y(t) \\ &\equiv y(t_0) + \int_{t_0}^t f_0(y(s)) ds \\ &\quad + \sum_{i=1}^N \int_{t_0}^t f_i(y(s)) d_i(y(s)) ds \end{aligned} \quad (3.2)$$

where f_i , and d_i are as previously defined and $y \in \mathcal{R}^n$. This section will discuss the conditions under which solutions to (3.2) can approximate solutions to (3.1). Since (3.2) is both continuous and autonomous, its analysis is much simpler than that of discontinuous and nonautonomous (3.1). For example, if f_i and d_i have continuous partial derivatives with respect to x , then the stability properties of (3.2) may be determined by examining the eigenvalues of the linearization of (3.2) about each steady state. No such simple statement can be said about determining the stability of (3.1). The two theorems presented in this section extend the results of [3] to the state discontinuous case, i.e., to the feedback control case.

A. Chattering

By representing state discontinuous differential equations by a corresponding integral equation, it is possible to rigorously explain averaging approximations in power electronic systems. However, it will always be necessary to assume that the models under consideration have a finite number of right-hand side state discontinuities on any bounded time interval and that each discontinuity is Lebesgue integrable. This, however, is not always true for mathematical models of power electronic systems. For example, when systems are switching infinitely often (chattering), there exists no compact time interval in which the right-hand side of the state discontinuous differential equation is continuous. Hence, a unique solution to a corresponding integral equation will not exist in the usual sense unless the theory of differential inclusions [8] is used.

In this paper, we will always assume that the system is not chattering. The physical implication of this assumption is that power electronic switches turn on and off only once each PWM switching period. Conditions for guaranteeing this are presented in [13] and will not be discussed here. However, it is important to note that the averaging results presented below, are only valid when chattering does not occur.

B. Theoretical Results

We begin this section by outlining the general averaging procedure that will be taken in this paper to justify the approximation of (3.1) by (3.2).

Given a nonautonomous, integral equation [such as (2.1) or (3.1)] $x(t) = x(t_0) + \int_{t_0}^t g(s, x(s), T) ds$, consider the corresponding autonomous "averaged" integral equation $y(t) = y(t_0) + \int_{t_0}^t \bar{g}(y(s)) ds$, where $\bar{g}(\cdot)$ is an "average value" of $g(t, \cdot, \cdot)$ and $\bar{g}(\cdot)$ does not depend on time, t , or on the switching period, T .

Step 1: Take the difference between the two integral equations to obtain

$$\|x(t) - y(t)\| \leq \|x(t_0) - y(t_0)\| + \left\| \int_{t_0}^t [g(s(x(s), T) - \bar{g}(y(s)))] ds \right\|.$$

Step 2: Show that for any $\delta > 0$, however small, and any $L > t_0$, however large, there will always exist a $T_0 = T_0(\delta, L)$

and a constant $K > 0$ such that for $0 < T \leq T_0$

$$\left\| \int_{t_0}^t [g(s, x(s), T) - \bar{g}(y(s))] ds \right\| \leq \delta + K \int_{t_0}^t \|x(s) - y(s)\| ds$$

for any $t \in [t_0, L]$.

Step 3: Immediately from Step 1, Step 2, and Gronwall's inequality, this implies that for $t \in [t_0, L]$, $L > t_0$, and $0 < T \leq T_0$

$$\|x(t) - y(t)\| \leq (\|x(t_0) - y(t_0)\| + \delta)e^{K(L-t_0)}$$

where $\delta \rightarrow 0$ as $T \rightarrow 0$. This implies that on any arbitrarily large but bounded time interval, if $x(t_0) = y(t_0)$, then $x(t)$ and $y(t)$ can remain arbitrarily close to each other for a sufficiently small switching period.

Step 4: Assume that $x(t_0) = y(t_0)$ and that $y(t)$ approaches a uniformly asymptotically stable equilibrium point, y_s . Then, there will always exist a sufficiently small $T_0 = T_0(\delta)$ such that, for $0 < T \leq T_0$

$$\|x(t) - y(t)\| < \delta, \quad t \geq t_0.$$

Furthermore, this result will remain valid for initial conditions that satisfy $\|x(t_0) - y(t_0)\| \leq \beta$, where $\beta > 0$ is sufficiently small.

Step 4 basically states that if averaging can be proven on a finite time interval, then it can always be extended to an infinite time interval in the special case when the averaged solution approaches a uniformly asymptotically stable equilibrium point. This statement has been proven by many authors [3], [6], [7], [12] and is standard to averaging theory.

Once Step 2 is completed, Steps 3 and 4 will immediately follow. However, it turns out that, for PWM systems, completing Step 2 is extremely difficult and relies on some very recently developed mathematical tools [11], [12]. Keeping the above algorithm in mind, it is now possible to prove the main results of this paper. The proof of Theorem 3.1 relies on several Lemmata, which are presented in the Appendix.

Theorem 3.1: Let $x(t)$ and $y(t)$ denote the solutions to (3.1) and (3.2), respectively. Then, for any constant $L > t_0$ and for any constant $\eta > 0$, there exists a $T_0 = T_0(\eta, L) > 0$ and a constant $K > 0$ such that, for $0 < T \leq T_0$,

$$\|x(t) - y(t)\| \leq (\|x(t_0) - y(t_0)\| + \eta) \exp\{K(t - t_0)\} \quad (3.3)$$

for all $t \in [t_0, L]$.

Proof of Theorem 3.1: For simplicity, define operators $\mathcal{J}: \mathcal{R}^n \rightarrow \mathcal{R}^n$ and $\mathcal{W}: \mathcal{R}^n \rightarrow \mathcal{R}^n$ as

$$\begin{aligned} (\mathcal{J}x)(t) &\equiv x(t_0) + \int_{t_0}^t f_0(x(s)) ds \\ &+ \sum_{i=1}^N \int_{t_0}^t f_i(x(s)) \\ &\cdot u(d_i(x(s)) - \text{tri}(s, T)) ds \end{aligned} \quad (3.4)$$

$$\begin{aligned}
 (\mathcal{W}y)(t) &\equiv y(t_0) + \int_{t_0}^t f_0(y(s)) ds \\
 &+ \sum_{i=1}^N \int_{t_0}^t f_i(y(s)) d_i(y(s)) ds. \quad (3.5)
 \end{aligned}$$

Under the assumption of no chattering, $x(t)$, the solution to (3.1) will be continuous. Therefore, it is well known (Theorems 24.4 and 24.5, [14]) that $x(t)$ can be approximated by piecewise constant functions. Construct $N+1$ such piecewise constant functions $\tilde{x}_i(t) \in \mathcal{R}^n$, $i = 0, 1, \dots, N$, such that for any $t \in [t_0, L]$, $0 \leq [d_i(x(t)) - d_i(\tilde{x}_i(t))] \leq \delta_i$ for $i = 1, 2, \dots, N$, where $\delta_i > 0$ are a set of positive constants.

Furthermore, choose $\tilde{x}_i(t)$ such that for any $t \in [t_0, L]$, $\|f_i(x(t)) - f_i(\tilde{x}_i(t))\| \leq \delta_i$ also, for $i = 0, 1, 2, \dots, N$. Since $f_i(\cdot)$ and $d_i(\cdot)$ are Lipschitz functions, such $\tilde{x}_i(t)$ can always be constructed for arbitrary $\delta_i > 0$. Define $(\mathcal{J}\tilde{x})(t)$ as

$$\begin{aligned}
 (\mathcal{J}\tilde{x})(t) &\equiv x(t_0) + \int_{t_0}^t f_0(\tilde{x}_0(s)) ds \\
 &+ \sum_{i=1}^N \int_{t_0}^t f_i(\tilde{x}_i(s)) \\
 &\cdot u(d_i(\tilde{x}_i(s)) - \text{tri}(s, T)) ds. \quad (3.6)
 \end{aligned}$$

Consider

$$\begin{aligned}
 \|(\mathcal{J}x)(t) - (\mathcal{J}\tilde{x})(t)\| &\leq \\
 &\int_{t_0}^t \|f_0(x(s)) - f_0(\tilde{x}_0(s))\| ds \\
 &+ \sum_{i=1}^N \int_{t_0}^t \|f_i(x(s)) - f_i(\tilde{x}_i(s))\| \\
 &\cdot \|u(d_i(x(s)) - \text{tri}(s, T))\| ds \\
 &+ \sum_{i=1}^N \int_{t_0}^t \|f_i(\tilde{x}_i(s))\| \cdot \|u(d_i(x(s)) - \text{tri}(s, T)) \\
 &- u(d_i(\tilde{x}_i(s)) - \text{tri}(s, T))\| ds. \quad (3.7)
 \end{aligned}$$

By Lemma A.2, for any $t \in [t_0, L]$, $\|f_i(\tilde{x}_i(t))\| \leq M_i$; $i = 1, 2, \dots, N$. Let $M = \max\{M_i\}$; $i = 1, 2, \dots, N$. Then, using the fact that $\|u(\cdot)\| \leq 1$ and using the fact that \tilde{x}_i have been constructed so that $\|f_i(x(t)) - f_i(\tilde{x}_i(t))\| \leq \delta_i$ for any $t \in [t_0, L]$, $i = 0, 1, 2, \dots, N$, (3.7) becomes

$$\begin{aligned}
 \|(\mathcal{J}x)(t) - (\mathcal{J}\tilde{x})(t)\| &\leq \\
 &\delta_0(N+1)(t-t_0) \\
 &+ M \sum_{i=1}^N \int_{t_0}^t \|u(d_i(x(s)) - \text{tri}(s, T)) \\
 &- u(d_i(\tilde{x}_i(s)) - \text{tri}(s, T))\| ds \quad (3.8)
 \end{aligned}$$

for any $t \in [t_0, L]$. However, $\tilde{x}_i(\cdot)$ and δ_i have been chosen so that $d_i(\tilde{x}_i(t)) \leq d_i(x(t)) \leq d_i(\tilde{x}_i(t)) + \delta_i$, $i = 1, 2, \dots, N$ for any t . Define N new piecewise constant functions, $h_i(\tilde{x} - i(t))$, where $h_i(\tilde{x}_i(t)) = \min\{1, d_i(\tilde{x}_i(t)) + \delta_i\}$; $i = 1, 2, \dots, N$. Note that $d_i(\tilde{x}_i(t)) \leq d_i(x(t)) \leq$

$h_i(\tilde{x}_i(t))$ for all $t \in (t_0, L)$; $i = 1, 2, \dots, N$. Then, by Lemma A.1, this implies

$$\begin{aligned}
 \|(\mathcal{J}x)(t) - (\mathcal{J}\tilde{x})(t)\| &\leq \\
 &\delta_0(N+1)(t-t_0) \\
 &+ M \sum_{i=1}^N \int_{t_0}^t \|u(h_i(\tilde{x}_i(s)) - \text{tri}(s, T)) \\
 &- u(d_i(\tilde{x}_i(s)) - \text{tri}(s, T))\| ds. \quad (3.9)
 \end{aligned}$$

Using Lemma A.5, there will always exist a $T_0 = T_0(\sigma_i, L)$ such that, for $0 < T \leq T_0$

$$\begin{aligned}
 \|(\mathcal{J}x)(t) - (\mathcal{J}\tilde{x})(t)\| &\leq \delta_0(N+1)(t-t_0) \\
 &+ M \sum_{i=1}^N [\sigma_i + \delta_i(t-t_0)] \\
 &= \sigma + \gamma_1(\delta) \quad (3.10)
 \end{aligned}$$

where $\sigma = M \sum_{i=1}^N \sigma_i$ goes to zero as $T \rightarrow 0$, $\delta = [\delta_0, \dots, \delta_N]$ and $\gamma_1(\delta)$ is a positive constant that approaches zero as $\delta_i \rightarrow 0$.

Similarly, for any $t \in [t_0, L]$

$$\begin{aligned}
 \|(\mathcal{W}\tilde{x})(t) - (\mathcal{W}x)(t)\| &\leq \\
 &\int_{t_0}^t \|f_0(\tilde{x}_0(s)) - f_0(x(s))\| ds \\
 &+ \sum_{i=1}^N \int_{t_0}^t \|f_i(\tilde{x}_i(s))\| \cdot \|d_i(\tilde{x}_i(s)) - d_i(x(s))\| ds \\
 &+ \sum_{i=1}^N \int_{t_0}^t \|d_i(x(s))\| \cdot \|f_i(\tilde{x}_i(s)) - f_i(x(s))\| ds \quad (3.11)
 \end{aligned}$$

where

$$\begin{aligned}
 (\mathcal{W}\tilde{x})(t) &\equiv x(t_0) + \int_{t_0}^t f_0(\tilde{x}_0(s)) ds \\
 &+ \sum_{i=1}^N \int_{t_0}^t f_i(\tilde{x}_i(s)) d_i(\tilde{x}_i(s)) ds.
 \end{aligned}$$

Noting that $\|f_i(\tilde{x}_i(t))\| \leq M$, for any $t \in [t_0, L]$ and that $\|d_i(\cdot)\| \leq 1$, (3.11) becomes

$$\begin{aligned}
 \|(\mathcal{W}\tilde{x})(t) - (\mathcal{W}x)(t)\| &\leq \delta_0(t-t_0) + M \sum_{i=1}^N \delta_i(t-t_0) \\
 &+ \sum_{i=1}^N \delta_i(t-t_0) \\
 &\leq \gamma_2(\delta),
 \end{aligned}$$

for any $t \in [t_0, L]$. Clearly $\gamma_2(\delta) \rightarrow 0$ as $\delta_i \rightarrow 0$.

Consider now the inequality

$$\begin{aligned}
 \|(\mathcal{J}x)(t) - (\mathcal{W}x)(t)\| &\leq \|(\mathcal{J}x)(t) - (\mathcal{J}\tilde{x})(t)\| \\
 &+ \|(\mathcal{J}\tilde{x})(t) - (\mathcal{W}\tilde{x})(t)\| \\
 &+ \|(\mathcal{W}\tilde{x})(t) - (\mathcal{W}x)(t)\| \quad (3.12)
 \end{aligned}$$

which is true for all t . Using the above discussion and Lemma A.4, there exists a $T_0 = T_0(\sigma, \beta, L)$ such that, for

$0 < T \leq T_0$,

$$\|(\mathcal{J}x)(t) - (\mathcal{W}x)(t)\| \leq \sigma + \beta + \gamma_1(\delta) + \gamma_2(\delta);$$

$$t \in [t_0, L] \quad (3.13)$$

where σ and β are positive constants that approach zero as $T \rightarrow 0$, as defined in (3.10) and Lemma A.4, respectively. Constants $\gamma_1(\delta)$ and $\gamma_2(\delta)$ can be made arbitrary small by making $\tilde{x}_i(\cdot)$ approximate $x(\cdot)$ with arbitrary accuracy. Therefore, without loss of generality, it can be assumed that $\delta_i \rightarrow 0$, which implies that for a sufficiently small switching period

$$\|(\mathcal{J}x)(t) - (\mathcal{W}x)(t)\| \leq \eta; \quad t \in [t_0, L] \quad (3.14)$$

where $\eta = \sigma + \beta$, and $\eta \rightarrow 0$ as $T \rightarrow 0$.

Finally, consider the inequality

$$\|x(t) - y(t)\| \equiv \|(\mathcal{J}x)(t) - (\mathcal{W}y)(t)\|$$

$$\leq \|(\mathcal{J}x)(t) - (\mathcal{W}x)(t)\|$$

$$+ \|(\mathcal{W}x)(t) - (\mathcal{W}y)(t)\|. \quad (3.15)$$

The following is always true:

$$\|(\mathcal{W}x)(t) - (\mathcal{W}y)(t)\| \leq \|x(t_0) - y(t_0)\|$$

$$+ \int_{t_0}^t \|f_0(x(s)) - f_0(y(s))\| ds$$

$$+ \sum_{i=1}^N \int_{t_0}^t \|f_i(x(s))\| \cdot \|d_i(x(s)) - d_i(y(s))\| ds$$

$$+ \sum_{i=1}^N \int_{t_0}^t \|d_i(y(s))\| \cdot \|f_i(x(s)) - f_i(y(s))\| ds.$$

Noting that $f_i(\cdot)$ and $d_i(\cdot)$ are Lipschitz and that $0 \leq d_i(\cdot) \leq 1$, one obtains

$$\|(\mathcal{W}x)(t) - (\mathcal{W}y)(t)\|$$

$$\leq \|x(t_0) - y(t_0)\| + \left(M \sum_{i=1}^N m_i + \sum_{i=0}^N k_i \right)$$

$$\cdot \int_{t_0}^t \|x(s) - y(s)\| ds \quad (3.16)$$

where m_i are the Lipschitz constants of $d_i(\cdot)$ and k_i are the Lipschitz constants for $f_i(\cdot)$. Let $K = M \sum_{i=1}^N m_i + \sum_{i=0}^N k_i$. Then (3.15) becomes

$$\|x(t) - y(t)\| \leq p(\|x(t_0) - y(t_0)\| + \eta)$$

$$+ K \int_{t_0}^t \|x(s) - y(s)\| ds$$

for any $t \in [t_0, L]$. Applying Gronwall's inequality completes the proof of the theorem. Q.E.D.

Remark 3.1: The main trick of the proof of Theorem 3.1 is to construct $N + 1$ piecewise constant functions $\tilde{x}_i(t)$, $i = 0, 1, \dots, N$, which accurately approximate $x(t)$ on $t \in [t_0, L]$. Such functions can always be constructed since $x(t)$ is continuous.

Then, using the notation defined in (3.4)–(3.6) and (3.11)

$$\|x(t) - y(t)\| = \|(\mathcal{J}x)(t) - (\mathcal{W}y)(t)\|$$

$$= \|(\mathcal{J}x)(t) - (\mathcal{J}\tilde{x})(t) + (\mathcal{J}\tilde{x})(t)$$

$$- (\mathcal{W}\tilde{x})(t) + (\mathcal{W}\tilde{x})(t)$$

$$- (\mathcal{W}x)(t) + (\mathcal{W}x)(t) - (\mathcal{W}y)(t)\|$$

$$\leq \|(\mathcal{J}x)(t) - (\mathcal{J}\tilde{x})(t)\|$$

$$+ \|(\mathcal{J}\tilde{x})(t) - (\mathcal{W}\tilde{x})(t)\|$$

$$+ \|(\mathcal{W}\tilde{x})(t) - (\mathcal{W}x)(t)\|$$

$$+ \|(\mathcal{W}x)(t) - (\mathcal{W}y)(t)\|. \quad (3.17)$$

Now, Step 2 of the averaging algorithm must be performed. Each term on the right-hand side of (3.17) is considered separately. By constructing $\tilde{x}_i(t)$ to approximate $x(t)$ with arbitrary accuracy, the quantities $\|(\mathcal{J}x)(t) - (\mathcal{J}\tilde{x})(t)\|$ and $\|(\mathcal{W}\tilde{x})(t) - (\mathcal{W}x)(t)\|$ can be made arbitrarily small. In essence, this is due to the Fundamental Theorem of Calculus, which states that any integral can be estimated by the sums of the areas of rectangles. Since $\tilde{x}_i(t)$ is piecewise constant, $(\mathcal{J}\tilde{x})(t)$ and $(\mathcal{W}\tilde{x})(t)$ represent nothing more than areas under the curve of a piecewise constant function which is equivalent to summing the areas of rectangles. Of course, due to the discontinuities that appear in $\mathcal{J}(\cdot)$, more advanced theoretical arguments must be made in order to justify these approximations.

Likewise, because $f_i(\cdot)$ and $d_i(\cdot)$ have been assumed Lipschitz, it is not too difficult to show that for any $t \in [t_0, L]$

$$\|(\mathcal{W}x)(t) - (\mathcal{W}y)(t)\| \leq \|x(t_0) - y(t_0)\|$$

$$+ K \int_{t_0}^t \|x(s) - y(s)\| ds. \quad (3.18)$$

Now, the only term left to consider in (3.17) is $\|(\mathcal{J}\tilde{x})(t) - (\mathcal{W}\tilde{x})(t)\|$. However, this term only considers the difference between the integrals of piecewise constant functions, which, as the theorem shows, is a much simpler problem to handle (based on the lemma in the Appendix).

Remark 3.2: When $x(t_0) = y(t_0)$, Theorem 3.1 guarantees that there will always exist a sufficiently small switching period such that for any $\eta > 0$, however small, $\|x(t) - y(t)\| < \eta$ on any finite time interval. This bound is true, even when (3.1) or (3.2) are unstable. For the case when solutions are bounded, however, more powerful theorems can be stated.

Remark 3.3: The choice of T_0 is best found through numerical simulation, since theoretical estimates are often extremely conservative. One reason for poor theoretical estimates of T_0 is that Theorem 3.1 does not distinguish between stable and unstable systems. For unstable systems, it is possible that solutions to (3.1) and (3.2) grow exponentially, making it difficult to estimate the difference, $\|x(t) - y(t)\|$. With this in mind, we make these general statements:

For general systems, from the proof of Theorem 3.1 and from basic averaging theory, it can be derived that T_0 is

sufficiently small if all three of the following conditions are satisfied:

- 1) there exists no chattering in the system;
- 2) $T_0 \ll e^{-k_i(L-t_0)}$, where k_i are the Lipschitz constants for $f_i(\cdot)$;
- 3) $T_0 \ll e^{m_i(L-t_0)}$, where m_i are the Lipschitz constants for $d_i(\cdot)$.

This is not to say that for every system in question, the switching period must be chosen so that 1)–3) are satisfied. For example, if solutions to (3.2) decay exponentially to an equilibrium point, then condition 2) can often be relaxed. It is important to remark that condition 1) must always be fulfilled or else the solutions of (3.1) will not be defined in the usual sense.

Remark 3.4: Based on the Theorem 3.1 and the above discussion, it is possible to determine general conditions that suggest the improvement of the accuracy of approximation between the original (3.1) and the approximate (3.2) system. Clearly, the approximation becomes better as the switching period becomes smaller, but also, as Remark 3.3 notes, the approximations will tend to improve for systems with smaller Lipschitz constants, i.e., the smaller k_i and m_i are, the more accurate the averaging technique will tend to be (for general systems) and the better for linear systems than for nonlinear systems. Additionally, as Theorem 3.2 suggests below, if the averaged system is stable, then the averaging approximations will also improve. Conversely, if the averaged system is unstable, the averaging approximation tends to worsen. Finally, as is clear from (3.3), a necessary condition for the solutions of (3.2) to approximate the solutions of (3.3) is that the initial conditions of the two systems must be chosen in appropriate neighborhoods.

Remark 3.5: One of the main advantages of the averaging technique is that nonlinearities are maintained in the averaged system. Hence, the approximation of (3.1) by (3.2) is valid even when the states, x , become large, which would not be true if a linearization technique were to be used. The averaging approximation is, therefore, valid for large signals.

As stated earlier, when the solution to the averaged equation approaches a uniformly asymptotically stable equilibrium point, the solutions of (3.1) and of (3.2) will remain close to each other on an infinite time interval for a sufficiently small switching period. The following theorem is an immediate consequence of this fact. The proof is almost identical to Proposition 4 of [3] or Theorem 2.2 of [12], and therefore, is omitted.

Theorem 3.2: Let $x(t)$ and $y(t)$ denote the solutions to (3.1) and (3.2), respectively, and let $y_s \in \Omega$ ($y_s \neq y(t_0)$) denote a uniformly asymptotically stable equilibrium point. Suppose that $y(t) \rightarrow y_s$ as $t \rightarrow \infty$.

Then there are constants $\beta_0(\eta)$ and $T_0(\eta)$ such that, for any $\eta > 0$, any $\|x(t_0) - y(t_0)\| < \beta$, $0 \leq \beta < \beta_0 < \eta$, and $0 < T \leq T_0$

$$\|x(t) - y(t)\| < \eta \quad (3.19)$$

for all $t \geq t_0$.

Remark 3.6: The above theorem gives conditions in which the interval in Theorem 3.1 can be made infinite. For the case when $y(t)$ approaches a uniformly asymptotically stable equilibrium point, y_s , the difference, $\|x(t) - y(t)\|$, can be made arbitrarily small for all $t \geq t_0$ assuming $\|x(t_0) - y(t_0)\|$ and the switching period are sufficiently small.

Remark 3.7: Suppose $f_i(\cdot)$ and $d_i(\cdot)$ have continuous partial derivatives. Then, for an equilibrium point, y_s , of (3.2) to be uniformly asymptotically stable, it is possible to check that

$$\text{Det} \left\{ sI - \frac{\partial f_0(y_s)}{\partial y} - \sum_{i=1}^N \left[\frac{\partial f_i(y_s)}{\partial y} d_i(y_s) - f_i(y_s) \frac{\partial d_i(y_s)}{\partial y} \right] \right\} = 0$$

have all solutions with $\text{Re}(s) < 0$.

Remark 3.8: Theorem 3.2 guarantees that under the proper conditions, when (3.2) is stable, then so is (3.1). Unlike (3.2), however, the solution to (3.1) will not in general approach an equilibrium point as $t \rightarrow \infty$, since (3.1) is a time varying integral equation. In general, the solution to (3.1) will (assuming it is stable) approach a periodic orbit. However, this periodic orbit will not necessarily be in the vicinity of the equilibrium point of the averaged equation, unless T is sufficiently small. In fact, (the theory clearly shows that) it is possible to construct examples in which (3.1) has an asymptotically stable periodic orbit for all T , but is only in the vicinity of the equilibrium point of (3.2) when $T \rightarrow 0$ (see Section IV). This behavior becomes more pronounced in feedback controlled (as opposed to open loop) PWM dc–dc converters due to the nonlinearities, and is not noted in [3] and [9]. We further explain this phenomenon in [10].

Remark 3.9: In Theorems 3.1 and 3.2, the feedback signals are compared with $\text{tri}(t, T)$, shown in Fig. 1. However, all the above theorems remain valid for triangle waves as shown in Fig. 2 also, provided that they are rescaled to vary between zero and one (see Section IV). Furthermore, it is not necessary to compare each $d_i(\cdot)$ with the same function with the same period. For instance, in (3.1) we might have $u(d_i(\cdot) - \text{tri}(\cdot, T_i))$ instead of $u(d_i(\cdot) - \text{tri}(\cdot, T))$, where T_i might not equal T_j , for $i \neq j$. As long as each T_i is sufficiently small, all previous results will remain valid.

C. Ripple Estimate

It is often desirable to obtain an estimate on the ripple of the system, which will be denoted in this paper as $\Psi(t, T, \cdot)$. Then, practical applications of averaging tell us that a better approximation of the solution to (3.1) will be given by

$$x(t) \approx y(t) + \Psi(t, T, y(t)) \quad (3.20)$$

where $x(t)$ and $y(t)$ are the solutions of (3.1) and (3.2), respectively, T is the switching period, and $\Psi(t, T, \cdot)$ is the ripple estimate obtained by the following algorithm.

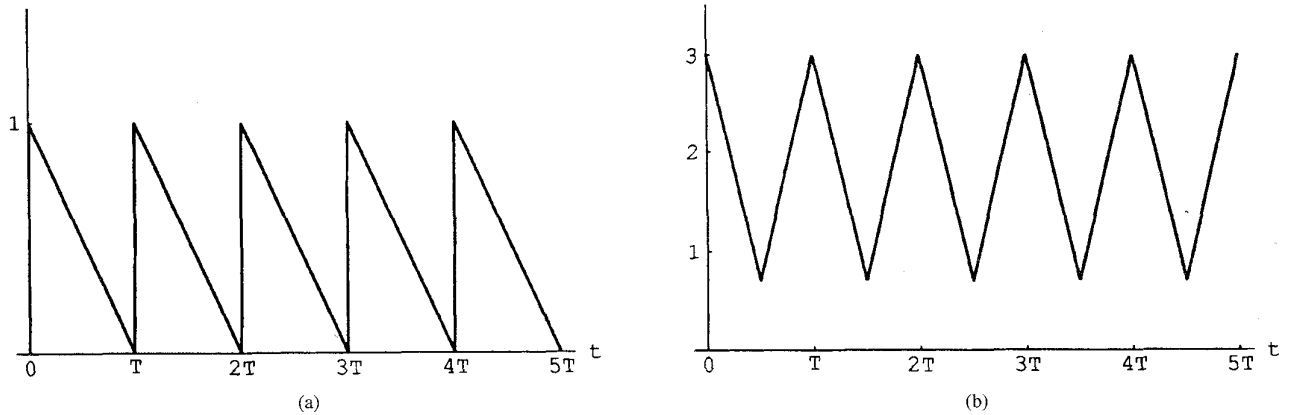


Fig. 2. Other possible triangle waves.

Consider only the right-hand sides of (3.1) and (3.2). Let $x(t_0) = y(t_0)$, and replace every $x(s)$ and $y(s)$ in (3.1) and (3.2) by the constant $c \in \mathcal{R}^n$. Now take the difference between (3.1) and (3.2) to obtain

$$\Gamma(t, T, c) = \sum_{i=1}^N f_i(c) \int_t^T [u(d_i(c) - \text{tri}(t, T)) - d_i(c)] dt \quad (3.21)$$

where $\int_t^T h(t) dt$ denotes the indefinite integral of $h(t)$ (mathematically referred to as the primitive). The ripple estimate is given as

$$\Psi(t, c) = \Gamma(t, T, c) - \frac{1}{T} \int_0^T \Gamma(t, T, c) dt. \quad (3.22)$$

Replacing c by $y(t)$ yields $\Psi(t, T, \cdot)$. Performing integrations (3.21) and (3.22), using (3.1) and (3.2), an estimate on the ripple is computed to be

$$\begin{aligned} \Psi(t, T, y(t)) = T \sum_{i=1}^N f_i(y(t)) \{ & [u(d_i(y(t)) - \text{tri}(t, T)) \\ & - d_i(y(t))] \text{tri}(t, T) \\ & + [1 - u(d_i(y(t)) - \text{tri}(t, T))] d_i(y(t)) \\ & + \frac{1}{2} d_i(y(t)) [d_i(y(t)) - 1] \}. \end{aligned} \quad (3.23)$$

As the switching period becomes smaller, the amplitude of $\Psi(t, T, \cdot)$ will also become smaller and ripple of the system will become negligible. Additionally, an adjustment on the initial condition can be made by solving the equation $x(t_0) = y(t_0) + \Psi(t, T, y(t_0))$, for $y(t_0)$ in terms of $x(t_0)$. The general expression for the ripple estimate (3.23) is an important contribution of this work and has been used in [10] to help model the effects of switching at lower frequencies.

IV. APPLICATION EXAMPLE

Consider the PWM boost converter with feedback control structure as shown in Fig. 3. Open-loop operation of this con-

verter was considered in [3]; however, the theory developed in [3] does not extend to closed-loop operation (as do the theorems in this paper). Assuming the converter is operating in the continuous conduction mode, the closed loop (rescaled) system description is given by

$$\begin{aligned} x(t) = x(t_0) + \int_{t_0}^t [A_0 x(s) + b] ds \\ + \int_{t_0}^t A_1 x(s) u(d(x(s)) - \text{tri}(s, T)) ds \\ d(x) = V_{REF} - k_1 x_1(t) - k_2 x_2(t), \end{aligned} \quad (4.1)$$

$$A_0 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} \frac{E}{L} \\ 0 \end{bmatrix}$$

where the components of $x(t) = [i_L(t), v_C(t)]^T$ are the inductor current and capacitor voltage. Note, that since the triangle wave in Fig. 3 varies from 0.7–3 V, it is necessary to rescale the system into (4.1) so that Theorems 3.1 and 3.2 can be applied. This is easily done scaling the duty ratio function using the minimum ($\text{trimin} = 0.7$ V) and maximum ($\text{trimax} = 3.0$ V) values of the triangle wave:

$$d(x) = \frac{g(x) - \text{trimin}}{\text{trimax} - \text{trimin}}$$

where $g(x)$ is defined in Fig. 3. For this specific $g(x)$, we have $V_{REF} = 0.3/2.3$, $k_1 = 0.4/2.3$, and $k_2 = -0.1/2.3$.

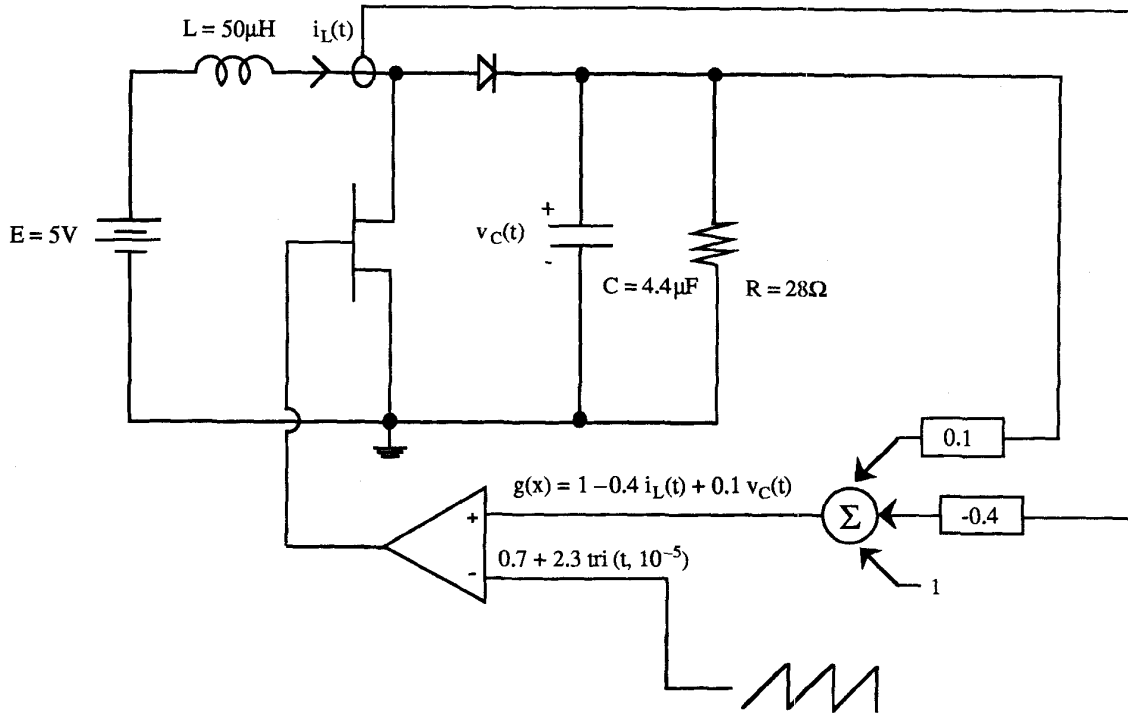


Fig. 3. Feedback control boost converter

The corresponding average of (4.1) is then given by

$$y(t) = y(t_0) + \int_{t_0}^t [A_0 y(s) + b] ds + \int_{t_0}^t A_1 y(s) d(y(s)) ds. \quad (4.2)$$

Application of Theorems 3.1 and 3.2 is now immediate upon noting that, using the previous notation, $f_0(x) = A_0 x + b$, $f_1(x) = A_1 x$, and $N = 1$. The closed loop switching and averaged models were simulated using Saber [15]. Fig. 4 illustrates the switching and averaged trajectories of the capacitor voltage for different switching periods. As the frequency of the system, f_s , increases, or equivalently as the switching period decreases (since $f_s = T^{-1}$), the approximation of $x(t)$ by $y(t)$ improves. For example, when $f_s = 50$ KHz, system (4.1) has a capacitor voltage that, in steady state, oscillates about (approximately) 7.3 V. The averaged system, on the other hand, approaches (approximately) 8.5 V. As the frequency of the system increases (the switching period decreases) the capacitor voltage for (4.2) more closely approximates the capacitor voltage of (4.1). For $f_s = 1$ MHz, system (4.1) has steady state capacitor voltage that oscillates about (approximately) 8.4 V, representing a significant improvement. Additionally, for larger frequency, the amplitude of the ripple decreases. This further verifies Theorems 3.1 and 3.2, which state that the approximation between the averaged system and the original system improves as the switching period decreases and is consistent with Remark 3.8. Similar results can be obtained for the inductor current.

Using (3.23), it is possible to directly compute an estimate on the ripple of the system as

$$\begin{aligned} \Psi(t, T, y(t)) &= T A_1 y(t) \{ [u(d(y(t)) - \text{tri}(t, T)) - d(y(t))] \text{tri}(t, T) \\ &\quad + [1 - u(d(y(t)) - \text{tri}(t, T))] d(y(t)) \\ &\quad + \frac{1}{2} d(y(t)) [d(y(t)) - 1] \}. \end{aligned} \quad (4.3)$$

Fig. 5 plots the capacitor voltage and inductor current of the original system (4.1) when $f_s = 100$ KHz. A comparison of these plots can be made with Fig. 6, which shows the improvement of the averaging technique by approximating $x(t)$ by $x(t) \approx y(t) + \Psi(t, T, y(t))$ and updating the initial condition, $y(t_0)$, by solving [given $x(t_0)$] the nonlinear equation

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} 1 & \frac{T}{2L} [d^2(y(t_0)) - d(y(t_0))] \\ -\frac{T}{2C} [d^2(y(t_0)) - d(y(t_0))] & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1(t_0) \\ y_2(t_0) \end{bmatrix}.$$

Fig. 6 indicates that the "shape" of solutions to averaged system (4.2) added to the ripple estimate closely resembles the "shape" of solutions to the original system (plus, perhaps, a dc offset). Therefore, the ripple estimate may provide important system information, even at a low frequency (large switching period).

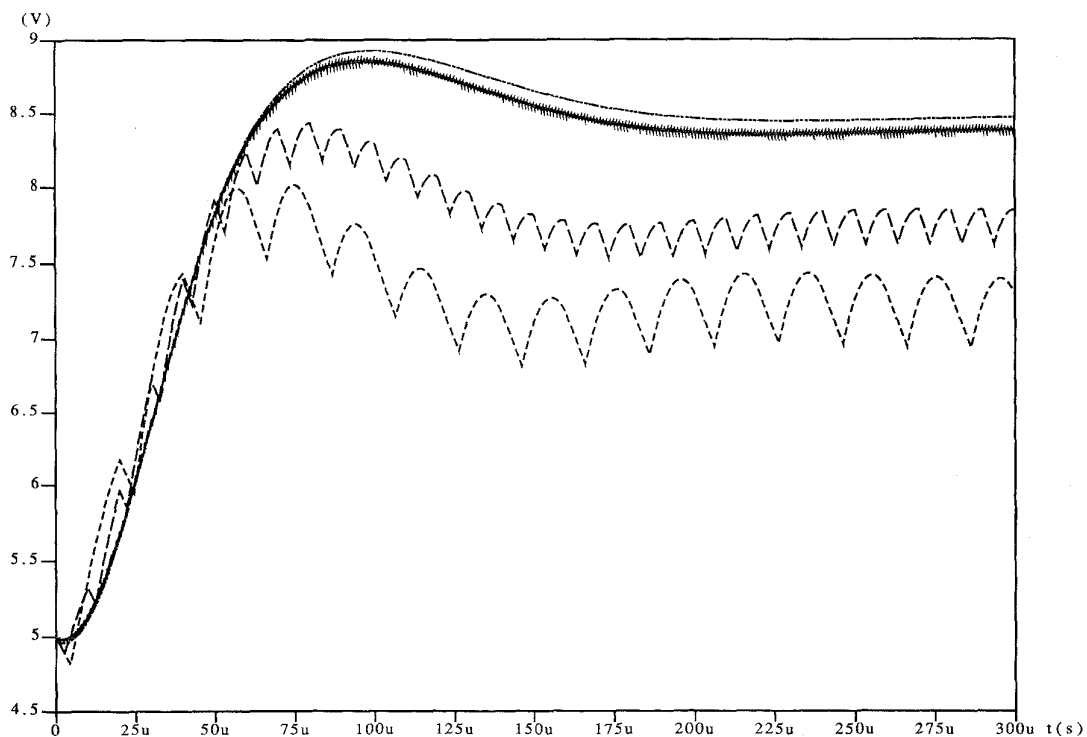


Fig. 4. Simulated start up transient response of capacitor voltage for (4.1) and (4.2) for different values of switching frequency. — $f_s = 1$ MHz; - - - $f_s = 100$ kHz; - · - $f_s = 50$ kHz; · · · average.

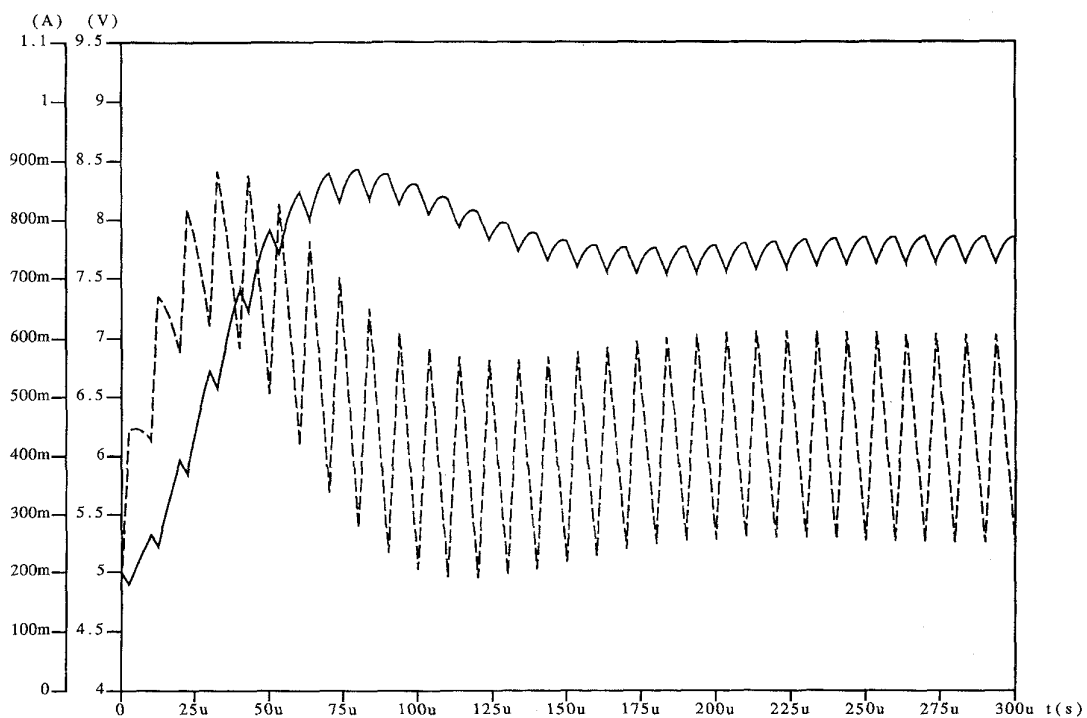


Fig. 5. Simulated start up transient response of both capacitor voltage and inductor current for (4.1) when switching frequency equals 100 kHz. — $v_C(t)$; - - - $i_L(t)$.

V. CONCLUSION

A rigorous averaging theory for power electronic systems has been developed. This new theory extends previous work

to include state discontinuous (feedback controlled) PWM systems. The two theorems presented in this paper provide a basis for answering fundamental questions about the averaged

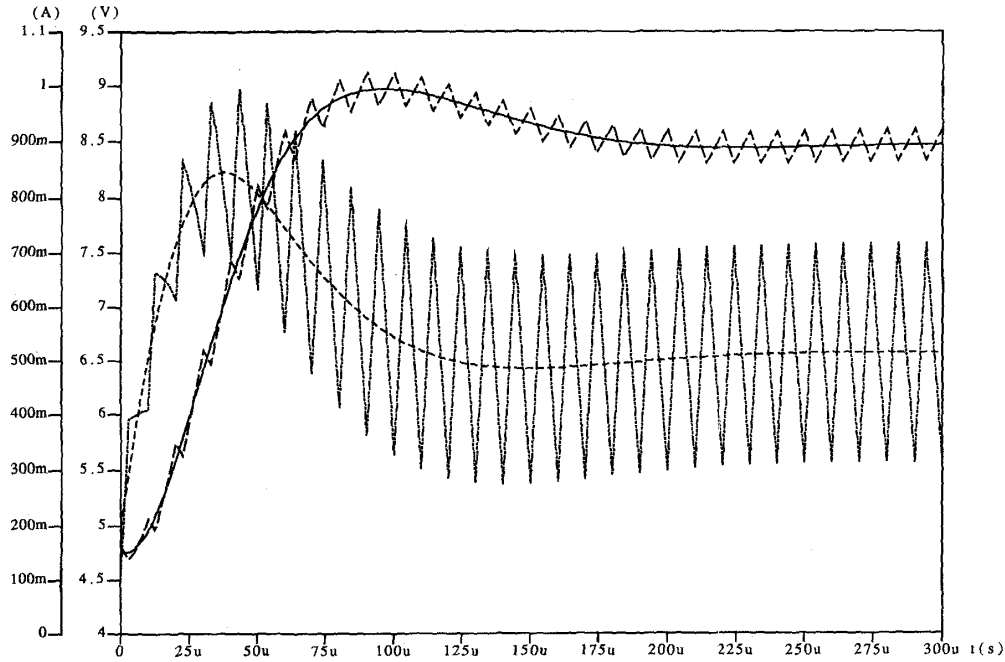


Fig. 6. Simulated start up transient response of both capacitor voltage and inductor current for (4.2) and ripple correction (4.3) when switching frequency equals 100 KHz. — $v_{c\text{avg}}(t)$; - - - $v_{c\text{rip}}(t)$; ···· $i_{l\text{avg}}(t)$; - · - · $i_{l\text{rip}}(t)$.

model and its relation to the original switching model. First-order ripple estimates are derived, and an application of the theory to a feedback controlled boost converter is presented.

APPENDIX

Lemma A.1: Let $g_1(x)$ and $g_2(x)$ be functions mapping $\mathcal{R}^n \rightarrow \mathcal{R}$. Suppose that, for any $x \in \mathcal{R}^n$, $g_1(x) \leq g_2(x)$.

Then, for any $x \in \mathcal{R}^n$, any $T > 0$, and any $t \in \mathcal{R}$ the following inequality is always true:

$$u(g_1(x) - \text{tri}(t, T)) \leq u(g_2(x) - \text{tri}(t, T)). \quad (\text{A.1})$$

Proof of Lemma A.1: If $g_1(x) \leq g_2(x)$, then at no time can $g_1(x) - \text{tri}(t, T) > 0$ while $g_2(x) - \text{tri}(t, T) < 0$. Using this fact and applying the definition of the Heaviside step function, the proof is immediate. Q.E.D.

Lemma A.2: Suppose that $x(t)$ and $y(t)$ are given by (3.1) and (3.2), respectively. Then for any $t \in [t_0, L]$, $L \geq t_0$

$$\|x(t)\| \leq \|x(t_0)\| \exp \left[\sum_{i=0}^N k_i(t - t_0) \right] \quad (\text{A.2})$$

$$\|y(t)\| \leq \|y(t_0)\| \exp \left[\sum_{i=0}^N k_i(t - t_0) \right] \quad (\text{A.3})$$

where k_i are the Lipschitz constants of f_i , previously defined.

Proof of Lemma A.2: By (3.1)

$$\|x(t)\| \leq \|x(t_0)\| + \int_{t_0}^t \|f_0(x(s))\| ds$$

$$+ \sum_{i=1}^N \int_{t_0}^t \|f_i(x(s))\| \cdot \|u(d_i(x(s)) - \text{tri}(s, T))\| ds. \quad (\text{A.4})$$

Since f_i are Lipschitz functions with Lipschitz constants k_i and since $\|u(\cdot)\| \leq 1$, we have

$$\|x(t)\| \leq \|x(t_0)\| + \sum_{i=0}^N \int_{t_0}^t k_i \|x(s)\| ds \quad (\text{A.5})$$

which, by Gronwall's inequality, implies (A.2). Upon noting that $\|d_i(\cdot)\| \leq 1$, (A.3) can be obtained using almost the same arguments. Q.E.D.

Lemma A.3: Let D be a constant satisfying $0 \leq D \leq 1$. Then, for any $t \geq t_0$

$$\left\| \int_{t_0}^t [u(D - \text{tri}(s, T)) - D] ds \right\| \leq \|DT(1 - D)\|. \quad (\text{A.6})$$

Proof of Lemma A.3: Without loss of generality, assume that $t_0 = 0$ (initial time can always be redefined so that this is the case.) By definition

$$u(D - \text{tri}(t, T)) = \begin{cases} 1 & t \in [nT, nT + DT] \\ 0 & t \in [nT + DT, (n+1)T] \end{cases} \quad n = 0, 1, 2, \dots \quad (\text{A.7})$$

Assume that $D \neq 0$. (The case when $D = 0$ is trivially proved since both the left and right-hand side of (A.6) are identically equal to zero). Suppose $0 \leq t \leq DT$. Then

$$\left\| \int_0^t [u(D - \text{tri}(s, T)) - D] ds \right\| = \left\| \int_0^t (1 - D) ds \right\|$$

$$\begin{aligned}
&= \|(1-D)t\| \\
&\leq \|DT(1-D)\|. \quad (\text{A.8})
\end{aligned}$$

Now consider $DT < t < T$. Then

$$\begin{aligned}
&\left\| \int_0^t [u(D - \text{tri}(s, T)) - D] ds \right\| \\
&= \left\| \int_0^{DT} (1-D) ds + \int_{DT}^t (0-D) ds \right\| \\
&= \|DT - Dt\| \\
&\leq \|DT(1-DT)\|. \quad (\text{A.9})
\end{aligned}$$

Finally, suppose that $t \geq T$. Then, there always exists an integer $M = M(t, T)$, depending on t and T , such that $M \geq 1$ and $MT \leq t \leq (M+1)T$. Therefore

$$\begin{aligned}
&\left\| \int_0^t [u(D - \text{tri}(s, T)) - D] ds \right\| \\
&= \left\| \int_0^{MT} [u(D - \text{tri}(s, T)) - D] ds \right. \\
&\quad \left. + \int_{MT}^t [u(D - \text{tri}(s, T)) - D] ds \right\|. \quad (\text{A.10})
\end{aligned}$$

Due to periodicity, $\int_0^{MT} [u(D - \text{tri}(s, T)) - D] ds = M \int_0^T [u(D - \text{tri}(s, T)) - D] ds$ and $\int_{MT}^t [u(D - \text{tri}(s, T)) - D] ds = \int_0^{t-MT} [u(D - \text{tri}(s, T)) - D] ds$. Note that

$$\begin{aligned}
&\int_0^T [u(D - \text{tri}(s, T)) - D] ds \\
&= \int_0^{DT} (1-D) ds + \int_{DT}^T (0-D) ds \\
&= 0. \quad (\text{A.11})
\end{aligned}$$

Furthermore, by (A.8) and (A.9)

$$\left\| \int_0^{t-MT} [u(D - \text{tri}(s, T)) - D] ds \right\| \leq \|DT(1-D)\|. \quad (\text{A.12})$$

Therefore, by (A.10)–(A.12), for $t \geq T$

$$\left\| \int_0^t [u(D - \text{tri}(s, T)) - D] ds \right\| \leq M(0) + \|DT(1-D)\| \quad (\text{A.13})$$

which completes the proof. Q.E.D.

Lemma A.4: Let $\tilde{x}(t)$ be a piecewise constant function. Then for any constant $L > t_0$ and any constant $\beta > 0$, there exists a $T_0 = T_0(\beta, L)$ such that, for $0 < T \leq T_0$

$$\begin{aligned}
&\left\| \sum_{i=1}^N \int_{t_0}^t f_i(\tilde{x}(s)) u(d_i(\tilde{x}(s)) - \text{tri}(s, T)) ds \right. \\
&\quad \left. - \sum_{i=1}^N \int_{t_0}^t f_i(\tilde{x}(s)) d_i(\tilde{x}(s)) ds \right\| \leq \beta; \quad t \in [t_0, L]. \quad (\text{A.14})
\end{aligned}$$

Proof of Lemma A.4: If c is a constant vector, then $d_i(c)$ and $f_i(c)$ are constants also. Therefore

$$\begin{aligned}
&\sum_{i=1}^N \int_{t_0}^t f_i(c) u(d_i(c) - \text{tri}(s, T)) ds - \sum_{i=1}^N \int_{t_0}^t f_i(c) d_i(c) ds \\
&= \sum_{i=1}^N f_i(c) \int_{t_0}^t [u(d_i(c) - \text{tri}(s, T)) - d_i(c)] ds \\
&\equiv M(c, f, d). \quad (\text{A.15})
\end{aligned}$$

By Lemma A.3, there exists a $T_0 = T_0(\gamma_i, L)$ such that, for $0 < T \leq T_0$

$$M(c, f, d) \leq \sum_{i=1}^N f_i(c) \gamma_i; \quad t \in [t_0, L] \quad (\text{A.16})$$

where $M(\cdot)$ is defined in (A.15), and γ_i are arbitrary small positive constants. From here, it follows that

$$\begin{aligned}
&\left\| \sum_{i=1}^N f_i(c) \int_{t_1}^{t_2} [u(d_i(c) - \text{tri}(s, T)) - d_i(c)] ds \right\| \\
&\leq \sum_{i=1}^N \|f_i(c)\| \gamma_i \quad (\text{A.17})
\end{aligned}$$

for any $t_0 \leq t_1 \leq t_2 \leq L$.

Since $\tilde{x}(t)$ is a piecewise constant function, there will always exist a sequence $t_0 = a_0 < a_1 < a_2 < \dots < a_p = t$, $t \leq L$, and a set of constants $\{c_j\}$; $j = 1, 2, \dots, p$, with $c_j = \tilde{x}(t)$ on the interval $t \in [a_{j-1}, a_j]$, such that

$$\begin{aligned}
&\left\| \int_{t_0}^t f_i(\tilde{x}(s)) [u(d_i(\tilde{x}(s)) - \text{tri}(s, T)) - d_i(\tilde{x}(s))] ds \right\| \\
&= \left\| \sum_{j=1}^p \int_{a_{j-1}}^{a_j} f_i(c_j) [u(d_i(c_j) - \text{tri}(s, T)) - d_i(c_j)] ds \right\|. \quad (\text{A.18})
\end{aligned}$$

Assuming no chattering and using the previous discussion this implies that there exists a $T_0 = T_0(\gamma_i, L)$ such that

$$\begin{aligned}
&\left\| \int_{t_0}^t f_i(\tilde{x}(s)) [u(d_i(\tilde{x}(s)) - \text{tri}(s, T)) - d_i(\tilde{x}(s))] ds \right\| \\
&\leq \frac{p}{2} (p+1) \sup_j \|f_i(c_j)\| \gamma_i, \quad j = 1, 2, \dots, p \quad (\text{A.19})
\end{aligned}$$

Noting that $\sup_j \|f_i(c_j)\| \leq k_i \sup_j \|c_j\| < \infty$, it is easy to see that (A.19) can be made arbitrarily small by making γ_i arbitrarily small (by choosing T_0 sufficiently small). Defining

$$\beta \equiv \sum_{i=1}^N \frac{p}{2} (p+1) \sup_j \|f_i(c_j)\| \gamma_i; \quad j = 1, 2, \dots, p \quad (\text{A.20})$$

(A.14) the proof is complete. Q.E.D.

Lemma A.5: Let $g_1(x)$ and $g_2(x)$ be continuous functions mapping $\mathcal{R}^n \rightarrow \mathcal{R}$, with $0 \leq g_1(x) \leq 1$ and $0 \leq g_2(x) \leq 1$. Suppose that $\tilde{x}(t)$ is a piecewise constant function and that $0 \leq g_1(\tilde{x}(t)) - g_2(\tilde{x}(t)) \leq \delta$, for some constant $\delta > 0$ and for all $t \in [t_0, L]$, $L > t_0$.

Then for any constant $L > t_0$ and any constant $\sigma > 0$, there exists a constant $T_0 = T_0(\sigma, L)$ such that, for $0 < T \leq T_0$

$$\int_{t_0}^t \|u(g_1(\tilde{x}(s)) - \text{tri}(s, T)) - u(g_2(\tilde{x}(s)) - \text{tri}(s, T))\| ds \leq \sigma + \delta(t - t_0). \quad (\text{A.21})$$

Proof of Lemma A.5: By Lemma A.1 and simple algebra

$$\begin{aligned} \int_{t_0}^t \|u(g_1(\tilde{x}(s)) - \text{tri}(s, T)) - u(g_2(\tilde{x}(s)) - \text{tri}(s, T))\| ds \\ = \int_{t_0}^t [u(g_1(\tilde{x}(s)) - \text{tri}(s, T)) \\ - u(g_2(\tilde{x}(s)) - \text{tri}(s, T))] ds \\ = \int_{t_0}^t [u(g_1(\tilde{x}(s)) - \text{tri}(s, T)) - g_1(\tilde{x}(s))] ds \\ + \int_{t_0}^t [g_1(\tilde{x}(s)) - g_2(\tilde{x}(s))] ds \\ + \int_{t_0}^t [g_2(\tilde{x}(s)) - u(g_2(\tilde{x}(s)) - \text{tri}(s, T))] ds. \end{aligned} \quad (\text{A.22})$$

By basic norm inequalities, this implies that

$$\begin{aligned} \int_{t_0}^t \|u(g_1(\tilde{x}(s)) - \text{tri}(s, T)) - u(g_2(\tilde{x}(s)) - \text{tri}(s, T))\| ds \\ \leq \left\| \int_{t_0}^t [u(g_1(\tilde{x}(s)) - \text{tri}(s, T)) - g_1(\tilde{x}(s))] ds \right\| \\ + \int_{t_0}^t \|g_1(\tilde{x}(s)) - g_2(\tilde{x}(s))\| ds \\ + \left\| \int_{t_0}^t [g_2(\tilde{x}(s)) - u(g_2(\tilde{x}(s)) - \text{tri}(s, T))] ds \right\|. \end{aligned} \quad (\text{A.23})$$

By Lemma A.4, for any $\gamma > 0$, there exist a $T_0 = T_0(\gamma, L)$ such that, for $0 < T \leq T_0$,

$$\left\| \int_{t_0}^t [u(g_1(\tilde{x}(s)) - \text{tri}(s, T)) - g_i(\tilde{x}(s))] ds \right\| < \gamma \quad (\text{A.24})$$

$$i = 1, 2.$$

Defining $\sigma = 2\gamma$ and noting that $\|g_1(\tilde{x}(t)) - g_2(\tilde{x}(t))\| \leq \delta$ for all $t \in [t_0, L]$, (A.23) immediately gives (A.21). Q.E.D.

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