Switching Frequency Dependent Averaged Models for PWM DC-DC Converters

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***Abstract*—This paper introduces a new averaging method for PWM dc-dc converters which yields averaged models that are switching frequency dependent. The new models are obtained by using periodic ripple functions to improve the averaging approximation. Two important benefits are the correction of dc offset error in steady-state and the modeling of switching frequency effects on closed-loop performance and stability.**

I. Introduction

Averaging techniques provide the analytical foundation for most power electronic design procedures at the system level. For pulse-width modulated (PWM) dc-dc converters, the three most popular approaches are known as 1) the state space averaging method, 2) the circuit averaging method, and 3) the injected-absorbed-current method.

The state space averaging method formulates the dynamic equations in state space form for each of the topological modes. The averaged model is then obtained by taking a weighted average of the system matrices, where the weighting factor for each topological mode is its duty ratio [1], [2]. The circuit averaging approach replaces each circuit element, whose terminal characteristics are given as instantaneous current and voltage relations, with a corresponding averaged circuit element. The averaged circuit element captures the relationship between the one-cycle average value of terminal quantities. For nonlinear elements, such as power electronic switches, the averaged i-v relation is only approximate [3]. The injected-absorbed-current method is also based on the one-cycle average of instantaneous quantities. The relation between averaged terminal quantities of a switching cell is assumed to exist and is linearized. The exact form of the nonlinear averaged relation is not always known, and a difference approximation to the time derivative is utilized [4].

A theoretical foundation which provides a rigorous mathematical justification for widely used averaging methods in PWM dc-dc converters [5], [6] is the basis for the new averaged models proposed in this paper. This new averaging procedure is based on formal mathematical methods for periodic differential equations. Unlike conventional approaches, the resulting averaged models incorporate switching frequency effects.

Section II highlights two fundamental deficiencies in traditional averaged models. A derivation of new averaged models is outlined in Section III, with a more detailed theoretical interpretation included as the Appendix. Transient simulations for the new and traditional models are compared in Section IV. Section V discusses the practical implications, and the unique features of the averaging method are summarized in Section VI.

II. Limitations of Existing Averaged Models

Averaging is the underlying approximation upon which a large body of work in power electronics is premised. While averaging has been the topic of numerous investigations, fundamental deficiencies and unresolved questions remain. In this section, two limitations of practical consequence are discussed.

A. DC Offset

The dc-dc boost converter with PWM feedback control shown in Fig. 1 will be used to illustrate the steady-state dc offset error exhibited by the conventional averaged model [1]. The equations for this circuit are given in Section IV. For this example, the component values are $E = 5$ V, $L = 50\ \mu$H, $C = 4.4\ \mu$F, and $R = 28\ \Omega$. The controller parameters are $V_{\rm ref} = 0.13$ V, $k_1 = 0.174$, and $k_2 = -0.0435$. A simulated start-up transient is given in Fig. 2. The output voltage transient $v_c(t)$ for the averaged model is shown together with a family of switching transients for several switching frequencies: $f_s = 1/T = 50$ kHz, 100 kHz, and 1 MHz. It can be seen that the approximation improves as switching frequency increases. Conversely, at the slower switching frequencies, the averaged model fails to accurately capture the average value in steady-state. This is a practical concern in applications where semiconductor device capabilities constrain the controller to operate at slower switching frequencies. The dc offset discrepancy has been previously reported for open-loop converters [12]. This affects the linearization. However, it can often be compensated for by including an integrator in the control loop for variables of interest.

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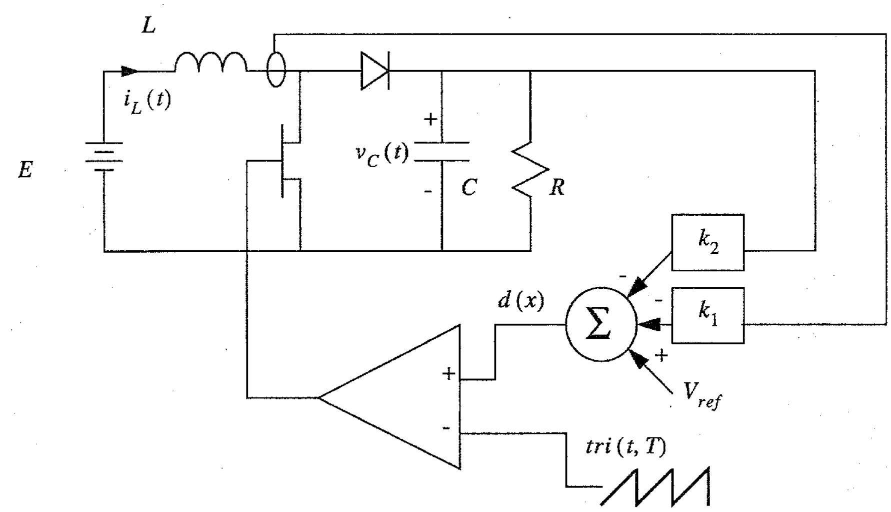

Fig. 1. Boost converter with PWM feedback control.

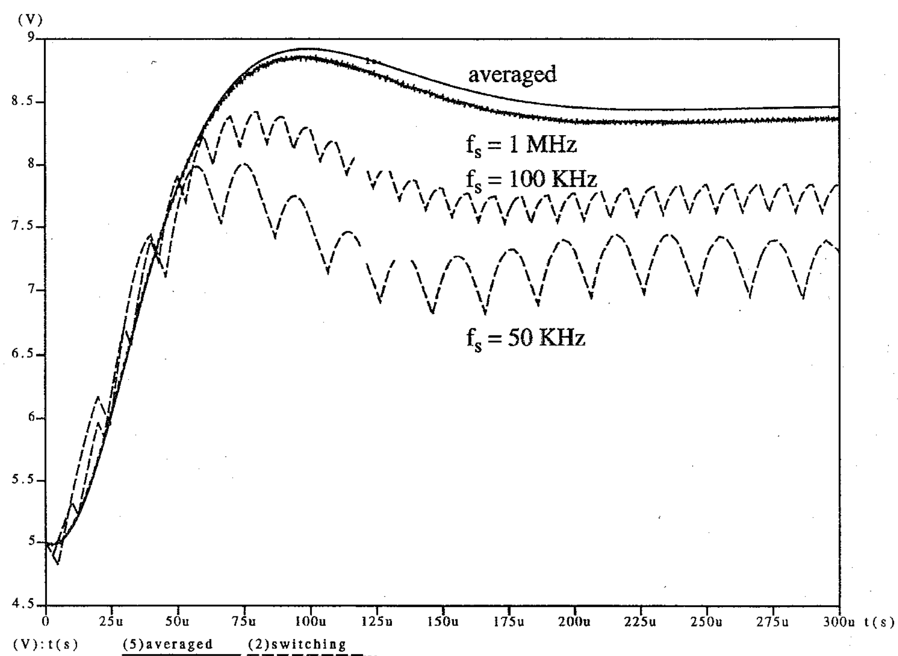

Fig. 2. Output voltage: simulated start-up transient for averaged and switching models.

B. Closed-Loop Stability

The same PWM controlled converter of Fig. 1 will be used to illustrate the effect of switching frequency on closed-loop stability. For this example, the component values are $E = 4$ V, $L = 5.24$ μH, $C = 0.2$ μF, and $R = 16$ Ω. The controller parameters are $V_{\text{ref}} = 0.48$ V, $k_1 = -0.1$, and $k_2 = 0.01$. A simulated transient is given in Fig. 3. The output voltage transient $v_c(t)$ for the averaged model is shown together with the switching transients for the switching frequencies $f_s = 500$ kHz and $f_s = 1$ MHz. Note that the closed-loop system is stable for $f_s = 1$ MHz, but unstable for $f_s = 500$ kHz. Additional simulations confirm the closed-loop system is stable for frequencies greater than $f_s = 500$ kHz and unstable for lower frequencies. As expected, Fig. 4 shows that the averaging approximation improves as switching frequency increases ($f_s = 10$ MHz in this case).

The conventional averaged model [1], which is independent of switching frequency, predicts that the closed-loop system is stable. The averaging approximation is based on the assumption that the switching frequency is "fast enough." While it is not known precisely how fast is "fast enough," designers typically take a rule-of-thumb linearized loop-gain crossover frequency on the order of 1/4–1/10 of the switching frequency. The practical concern here is that the conventional averaged model gives no insight or guidance as to "how fast" the switching frequency needs to be for acceptable closed-loop

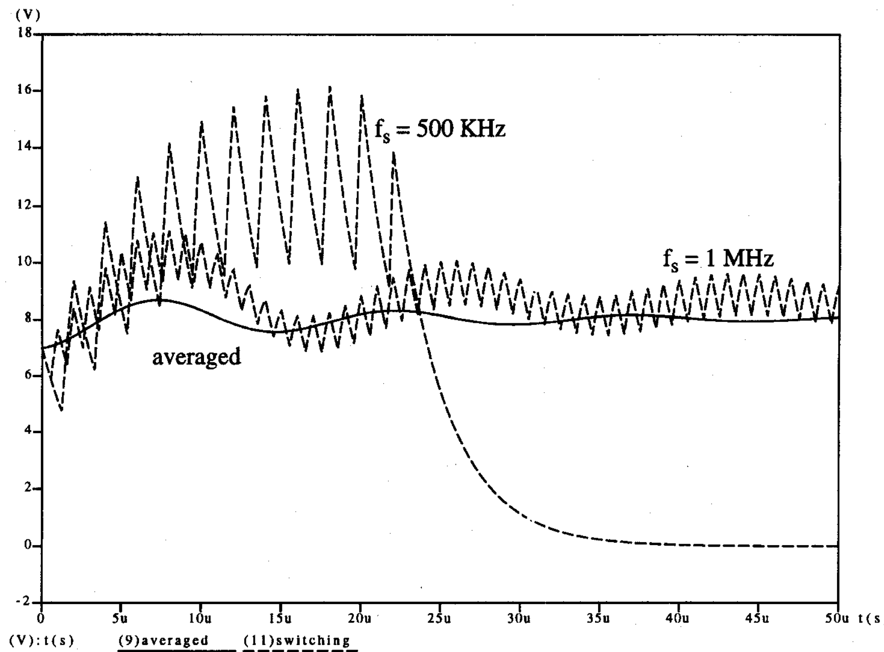

Fig. 3. Output voltage: simulated transient for averaged and switching models.

performance. As a result, transient performance is sacrificed. To our knowledge, the stability discrepancy has not been previously cited, perhaps because designers avoid slow switching frequencies in order to have confidence in the averaged model. However, we have observed this instability phenomenon at slow switching frequencies in laboratory experiments.

III. Derivation of New Averaged Models

In general form, feedback controlled pulse width modulated dc-dc converters in continuous conduction mode can be written as

$$\dot{x} = A_0x + b_0 + [A_1x + b_1]u(d(x) - \text{tri}(t, T)) \tag{1}$$

where $x \in R^n, d(x) = V_{\text{ref}} - [k_1, k_2, \cdots, k_n]x$, $0 \leq d(x) \leq 1$ and $u(\cdot)$ is the Heaviside step function, i.e., $u(s) = 1$ for $s \geq 0$ and $u(s) = 0$ for $s < 0$. The function $\text{tri}(t, T) = \frac{t}{T} - \text{floor}(\frac{t}{T}) = \frac{t \bmod T}{T}$ is shown in Fig. 5.

It has been rigorously shown (see [6]) that when T is sufficiently small, solutions to (1) can be approximated by the autonomous averaged system

$$\dot{y} = A_0y + b_0 + [A_1y + b_1]d(y) \tag{2}$$

which is the conventional averaged model [1]. This section will show how improvements can be made on (2). Note that (1) is a state discontinuous differential equation. Under the proper conditions, given in [6], this causes no additional difficulties.

A. Modified Averaging Technique for PWM DC-DC Converters

The techniques detailed in the Appendix will now be used to derive a more accurate averaged model of (1). In this section, reference will be made to the original equation in "standard form" (19)

$$\dot{x} = \varepsilon f(t, x)$$

the "near identity" periodic transformation (30)

$$x = z + \varepsilon\Psi(t, z, \varepsilon)$$

and the new averaged model (32)

$$\dot{y} = \varepsilon g(y, \varepsilon)$$

which is obtained by simultaneously solving (27) and (29)

$$\frac{1}{T}\int_0^T f(t, z + \varepsilon\Psi)dt \equiv g(z, \varepsilon)$$

$$\Psi(t, z, \varepsilon) = \int [f(t, z + \varepsilon\Psi) - g(z, \varepsilon)]dt + h(z, \varepsilon)$$

where h is a function chosen so that Ψ has zero average. This new model will change as the switching period changes and therefore takes into consideration the effects of error terms that occur because T is not vanishingly small. It is very important to note, however, that the model derived (and, for that matter, all general averaged models) is valid only when T is sufficiently small.

In order to place (1) in "standard form," convert to slow time by letting in (1) $\tau = t/T$ to obtain

$$x'(\tau) = T[A_0x + b_0 + [A_1x + b_1]u(d(x) - \text{tri}(\tau, 1))] \tag{3}$$

where we switch notation so that x' represents the derivative of x with respect to τ and $\dot{x}$ denotes the derivative of x with respect to t. Now (3) is in the "standard form" of (19), with

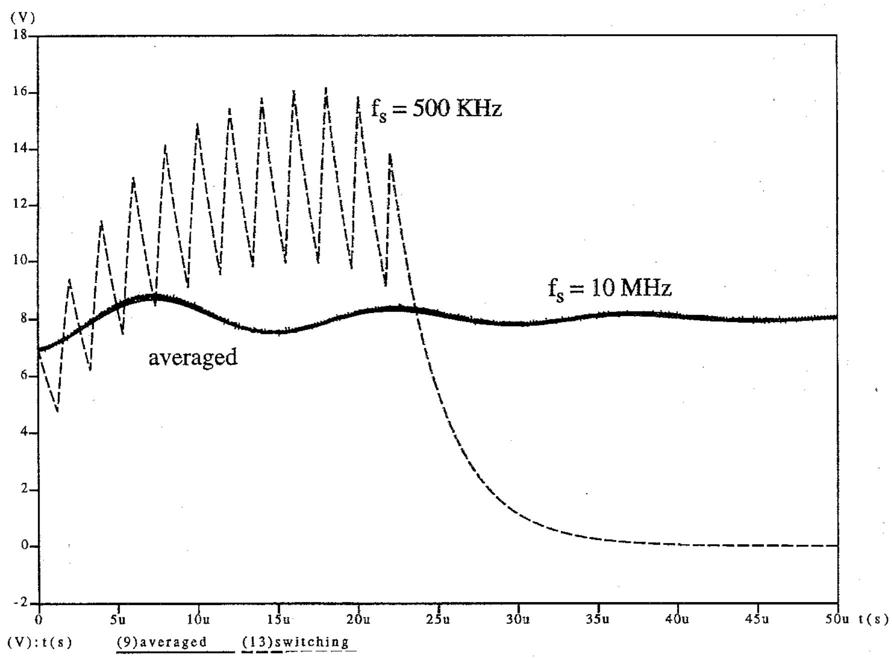

Fig. 4. Output voltage: simulated transient for averaged and switching models.

the perturbation parameter $\varepsilon = T$. Derivation of the averaged model will now be performed in slow time.

Using the "near identity" periodic transformation (30) in (3), (23) becomes (neglecting an $O(T^2)$ term)

$$z'(\tau) = T\Big[A_0(z+T\Psi)+b_0+[A_1(z+T\Psi)+b_1] \times u(d(z+T\Psi)-\mathrm{tri}(\tau,1)) - \frac{\partial\Psi}{\partial\tau}\Big] \qquad (4)$$

where $\Psi(\tau, z, T)$ is a periodic function that gives the ripple in the state variables [6].

The averaged equation in slow time is given by (32) with g given as in (27) as

$$g(y,T) = 1\int_0^1 [A_0(y+T\Psi)+b_0+[A_1(y+T\Psi)+b_1] \times u(d(y+T\Psi)-\mathrm{tri}(\tau,1))]d\tau \qquad (5)$$

and, by (29), Ψ is given in slow time as

$$\Psi(\tau,y,T) = \int [A_0(y+T\Psi)+b_0+([A_1(y+T\Psi)+b_1] \times u(d(y+T\Psi)-\mathrm{tri}(\tau,1))) - g(y,T)]d\tau - h(y,T). \qquad (6)$$

The function h is chosen so that Ψ has zero average. The averaged model requires simultaneous solution of (5) and (6) for g and Ψ.

B. Solving for the New Model

In order to obtain new models for PWM feedback controlled dc-dc converters, it will be necessary to estimate solutions to

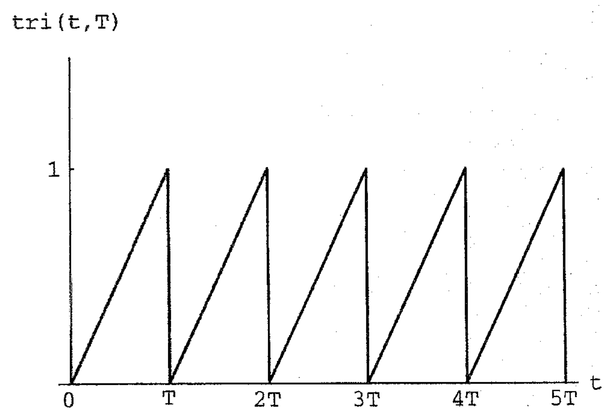

Fig. 5. PWM carrier waveform, $\mathrm{tri}(t, T)$.

(5) and (6). Due to the special properties of the Heaviside step function, it is possible to obtain very close solution estimates.

Let τ_s denote the instant of slow time when $d(z+T\Psi) = \mathrm{tri}(\tau, 1)$. This represents the instant of time when the Heaviside step function in (4) switches from one to zero.

Assume the following, which guarantee that $\Psi = [\Psi_1, \Psi_2]^T$ has components Ψ_i in the form of Fig. 6:

i) Ψ is zero average, and in addition, Ψ has a symmetry with respect to τ, such that

$$\int_0^{\tau_s} \Psi(\tau, y, T)d\tau = 0.$$

ii) Ψ_i is a triangle wave that is strictly increasing or decreasing on the intervals $\tau \in [m, m+\tau_s)$ and $\tau \in [m+\tau_s, m+1), m = 0, 1, \cdots$

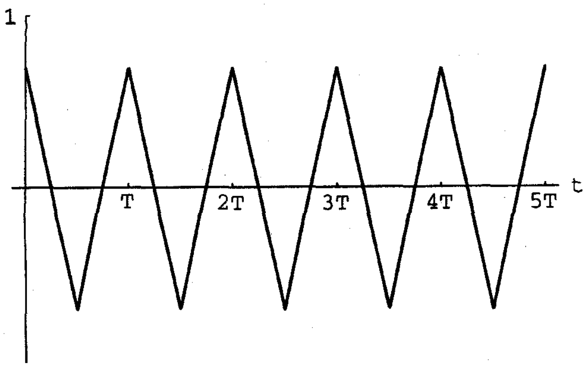

Fig. 6. Periodic ripple function, $\Psi_{\cdot 1}(t, \cdot)$.

Assumption ii) appears to introduce some error. In essence, we are estimating exponential curves by straight lines. The accuracy of this approximation is dependent on the decay rate of solutions to (3).

Under these assumptions, we can explicitly evaluate g in (5) by noting the following facts:

1) $\int_0^1 [A_0(y + T\Psi) + b_0] d\tau = A_0 y + b_0$

2) $\int_0^1 [A_1 \Psi u(d(y + T\Psi) - \mathrm{tri}(\tau, 1)) d\tau = 0$

Proof of Fact 2: Note that

$$\begin{aligned}&\int_0^1 A_1 \Psi u(d(y + T\Psi - \mathrm{tri}(\tau, 1)) d\tau \\ &= \left[\int_0^{\tau_s} A_1 \Psi u(d(y + T\Psi) - \mathrm{tri}(\tau, 1)) d\tau \right. \\ &\quad \left. + \int_{\tau_s}^1 A_1 \Psi u(d(y + T\Psi) - \mathrm{tri}(\tau, 1)) d\tau \right].\end{aligned}$$

It has been assumed that u switches at τ_s. Since $\mathrm{tri}(0, 1) = 0$, and $0 \leq d \leq 1$ (with $d(x) = V_{\mathrm{ref}} - Kx$), this implies that $u(d(y + T\Psi) - \mathrm{tri}(\tau, 1)) = 1$ on $\tau \in [0, \tau_s)$ and equals zero on $[\tau_s, 1)$. Hence the above equation becomes

$$= \left[\int_0^{\tau_s} A_1 \Psi d\tau + 0 \right]$$

which by assumption i) is zero. □

3)

$$\int_0^1 [A_1 y + b_1] u(d(y + T\Psi) - \mathrm{tri}(\tau, 1))\, d\tau = [A_1 y + b_1] \tau_s$$

where τ_s is the instant of time that $d(y + T\Psi) = \mathrm{tri}(\tau, 1)$ $(= \tau_s)$. (Therefore, τ_s is a function of d, y, and T.)

Proof of Fact 3: As in Fact 2, we know that $u(d(y + T\Psi) - \mathrm{tri}(\tau, 1))$ switches from one to zero at $\tau = \tau_s$. Therefore

$$\begin{aligned}&\int_0^1 [A_1 y + b_1] u(d(y + T\Psi) - \mathrm{tri}(\tau, 1)) d\tau \\ &= \int_0^{\tau_s} [A_1 y + b_1] d\tau = [A_1 y + b_1] \tau_s. \ \square\end{aligned}$$

Combining Facts 1–3, the new averaged model of (3) becomes (in slow time)

$$y'(\tau) = T[A_0 y + b_0 + \tau_s (A_1 y + b_1)]. \tag{7}$$

The next step is to estimate the value of τ_s. If the components of Ψ are triangle waves in the form of assumption (ii) above, then this implies that Ψ is obtained by neglecting the $A_0 T\Psi$ and the $A_1 T\Psi$ terms in (6), i.e.

$$\begin{aligned}\Psi(\tau, y, T) \approx \int [A_0 y + b_0 + [A_1 y + b_1] u(d(y + T\Psi) \\ - \mathrm{tri}(\tau, 1)) - g(y, T)] d\tau - h(y, T)\end{aligned} \tag{8}$$

where $g(y, T) = A_0 y + b_0 + \tau_s (A_1 y + b_1)$, as derived above.

Since $u(d(y + T\Psi)) - \mathrm{tri}(\tau, 1))$ switches from 1 to 0 at τ_s, (8) is equivalent to

$$\begin{aligned}\Psi(\tau, y, T) \approx \int ([A_1 y + b_1] u(\tau_s - \mathrm{tri}(\tau, 1)) \\ - [A_1 y + b_1] \tau_s) d\tau - h(y, T)\end{aligned} \tag{9}$$

where h is chosen so that Ψ has zero average. However, (9) is in a form that has explicit solution (see [6]) as

$$\begin{aligned}\Psi \approx \Bigg([A_1 y + b_1] \Bigg\{ [u(\tau_s - \mathrm{tri}(\tau, 1)) - \tau_s] \mathrm{tri}(\tau, 1) \\ + [1 - u(\tau_s - \mathrm{tri}(\tau, 1))] \tau_s + \frac{1}{2} \tau_s (\tau_s - 1) \Bigg\} \Bigg).\end{aligned} \tag{10}$$

It is now possible to explicitly solve for τ_s, using (10) and the fact that

$$d(y + T\Psi(\tau_s, y, T)) = \tau_s. \tag{11}$$

By (10), it is known that

$$\Psi(\tau_s, y, T) = \frac{1}{2} [A_1 y + b_1] [\tau_s - \tau_s^2]. \tag{12}$$

Therefore, when $d(x) = V_{\mathrm{ref}} - Kx$, where $K = [k_1, k_2, \cdots, k_n]$, (11) simply yields a second-order polynomial in τ_s given by

$$V_{\mathrm{ref}} - K\left(y + \frac{T}{2} [A_1 y + b_1] [\tau_s - \tau_s^2] \right) = \tau_s$$

or equivalently

$$d(y) - \frac{T}{2} (\tau_s - \tau_s^2) K [A_1 y + b_1] = \tau_s. \tag{13}$$

As the examples in the paper show, it is fairly simple to solve for the root of the second-order polynomial (in τ_s) given in (13). From (13), it is apparent that τ_s is a function of T, y, and d as expected. It is interesting to note that, by (13), when

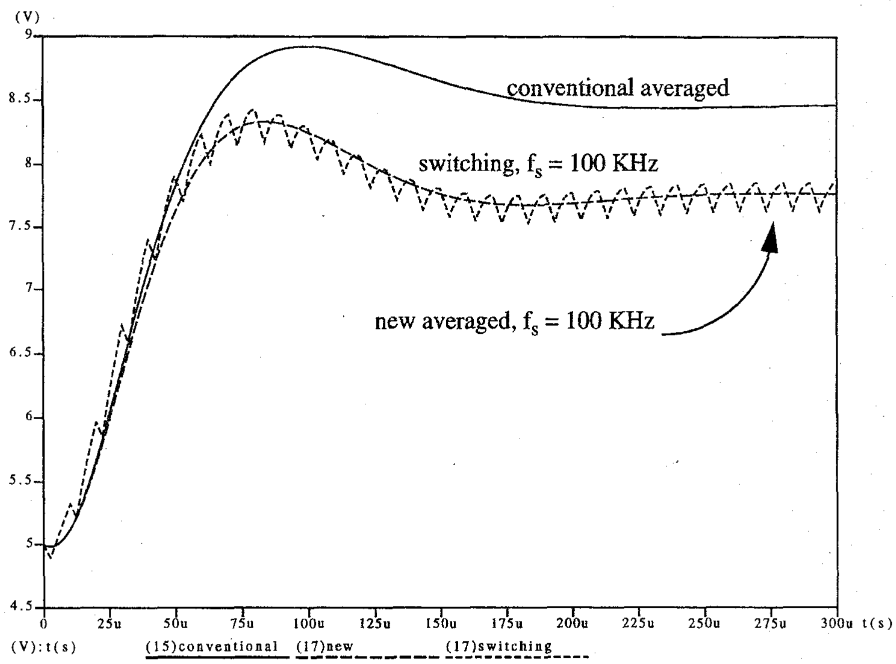

Fig. 7. Output voltage: simulated start-up transient for conventional, new, and switching models.

$T \to 0, \tau_s = d(y)$, which gives the conventional averaged models presently being used.

Using the value for τ_s, given in (13), all parameters in (7) are known, and hence, (7) and (13) completely specify the new averaged model. As a final step, we convert back to fast time and obtain the new averaged model to be

$$\dot{y}(t) = A_0 y + b_0 + \tau_s[A_1 y + b_1] \tag{14}$$

where τ_s is given by (13).

IV. Simulation Results

The dc-dc converter with PWM feedback control considered earlier (Fig. 1) will be used now to illustrate the improvements obtained using the new averaged model. The equation for this converter, formulated in [6], is in the form of (1) with

$$x(t) = \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} \quad A_0 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix}$$

$$b_0 = \begin{bmatrix} \frac{E}{L} \\ 0 \end{bmatrix} \quad b_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Simulations were performed using the SABER simulator [13], which permits the entry of the switching circuit net list and the averaged model equations in the same template.

A. DC Offset

The example considered now is the same as the one considered in Section II-A. Fig. 7 shows a simulated start-up transient for the original switching system (1), together with the start-up transient for the conventional averaged model (2) and the new averaged model (13, 14) for $f_s = 100$ kHz. The new averaged model accurately captures the one-cycle average, while the conventional model exhibits a significant offset error.

B. Closed-Loop Stability

The example considered now is the same as the one considered in Section II-B. Fig. 8 shows a simulated transient for the original system (1), together with the conventional (2) and new (13), (14) averaged models for $f_s = 1$ MHz Note that the new model provides a much more accurate one-cycle average than does the conventional model.

Fig. 9 shows the transients for the new and switching models for two different switching frequencies. Additional simulations confirm that the averaged system accurately predicts the critical switching frequency for closed-loop stability. The reason the "unstable" switching and averaged waveforms approach different steady-state values is because PWM saturation is not included in the averaged model (13), (14). This means $d(y)$ is permitted to take on nonphysical values less than zero or greater than 1. If duty ratio saturation were modeled in (13), then the two waveforms would coincide in steady-state. This simulation result demonstrates that the new model captures the control-loop instability at low switching frequencies. The conventional model transient fails to exhibit this phenomenon, since it only models the behavior in the limit as $f_s \to \infty$.

V. Practical Implications of New Models

It is interesting to note that in *open-loop operation*, the models in this paper are identical to conventional averaged

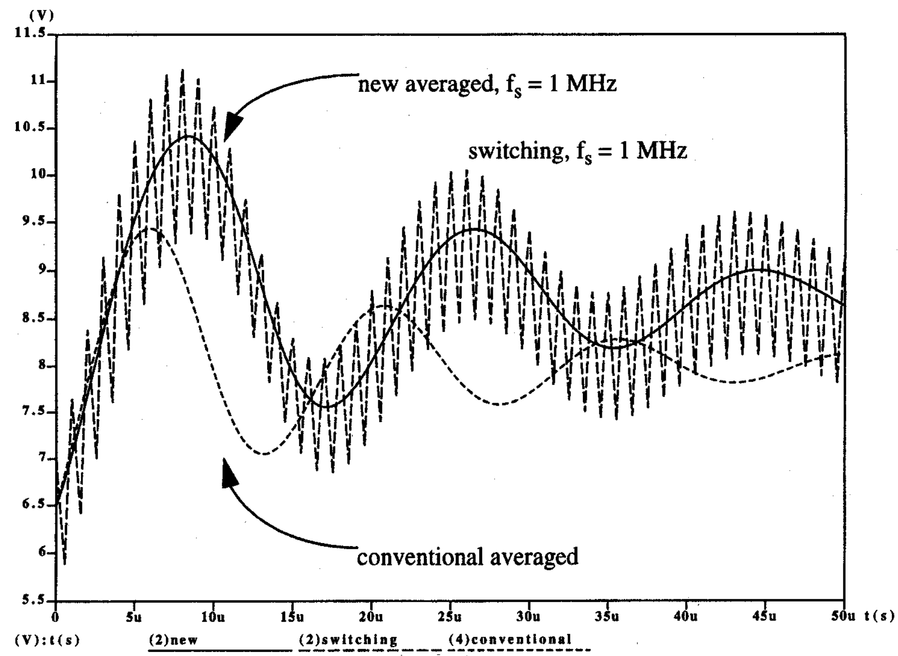

Fig. 8. Output voltage: simulated transient for conventional, new, and switching models.

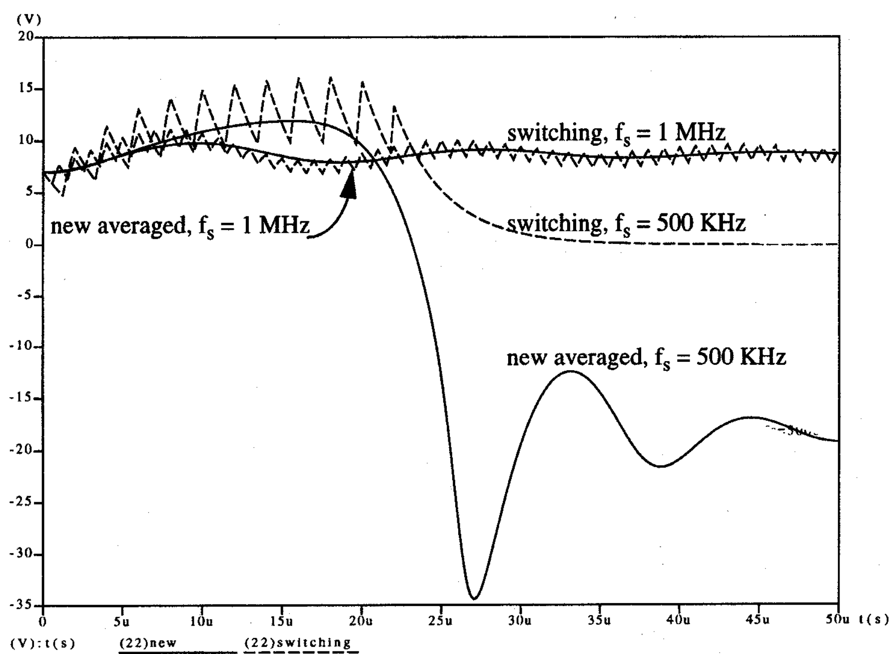

Fig. 9. Output voltage: simulated transient for new averaged and switching models.

models [1]. This can be seen by setting $d(y) = D$ and $K = 0$. Then (13) yields $\tau_s = D$ which is the conventional answer. Therefore, the improvements in accuracy of the proposed averaged models are due to better approximations of the nonlinear feedback dynamics. These nonlinear effects, however, can have influence on stability properties, overshoot, rise time, etc., and in many cases, it is important that they be modeled (see Section II-B).

It is beyond the scope of this paper, due to length constraints, to derive detailed design algorithms which utilize the newly proposed models. The purpose of this paper is to introduce new modeling techniques and we defer in-depth control design for future research. However, to introduce the reader to a few possible advantages of (13) and (14) over conventional averaged models, consider, once again, the specific example of the state feedback controlled boost converter as previously discussed.

TABLE I
EFFECT OF SWITCHING PERIOD ON SYSTEM DYNAMICS

	Ideal (T→0)	T=1μs	T=2.5μs
I_L (Amps)	1	1.1743	complex
V_c (Volts)	8	8.6692	complex
D	0.5	0.53860	complex
$\frac{\hat{v}_c(s)}{\hat{v}_{in}(s)}$	$\frac{3.818\times10^{11}}{s^2+109{,}773s+1.8136\times10^{11}}$	$\frac{3.2292\times10^{11}}{s^2+93{,}819s+1.2317\times10^{11}}$	N.A.

In order to linearize the boost converter, the nominal steady-states must first be obtained by letting the derivatives in (14) equal to zero. In this case, a set of three algebraic relationships are obtained as

$$\begin{aligned} I_L &= \frac{E}{R(1-D)^2} \\ V_C &= \frac{E}{(1-D)} \\ V_{\text{ref}} &- k_1 I_L - k_2 V_C - \frac{T}{2}(D-D^2) \\ &\times \left[\frac{-k_2 I_L}{C} + \frac{k_1 V_C}{L}\right] - D = 0 \end{aligned} \tag{15}$$

where I_L and V_C are the constant steady-state values of current and voltage, respectively, and D is the steady-state constant value of τ_s. If $T \to 0$, then $D \to V_{\text{ref}} - k_1 I_L - k_2 V_C$, which yields the standard steady-state values. However, finite T can have influence on the value of steady-state current and voltage, as seen in Section II-A. This is one advantage of the new models.

Using these steady-state values, the linearization about the nominal steady-states $y = [I_L, V_C]^T$ and $E = V_{\text{IN}}$ is given as

$$\dot{\hat{y}} = [A_0 + DA_1 + A_1 Q]\hat{y} + B_0 \hat{v}_{\text{in}} \tag{16}$$

where A_0, A_1, and D, are as previously defined, $B_0 = [\frac{1}{L}, 0]^T$ and

$$Q = \left.\begin{bmatrix} \left(\frac{\partial \tau_s}{\partial y_1}\right) y_1 & \left(\frac{\partial \tau_s}{\partial y_2}\right) y_1 \\ \left(\frac{\partial \tau_s}{\partial y_1}\right) y_2 & \left(\frac{\partial \tau_s}{\partial y_2}\right) y_2 \end{bmatrix}\right|_{\substack{y1=I_L \\ y2=V_C}}. \tag{17}$$

Such a linearization permits a designer to solve for various closed-loop transfer functions such as line rejection, output impedance, current gains, etc. For example, the line rejection for the above boost converter is given as

$$\frac{\hat{v}_c(s)}{\hat{v}_{\text{in}}(s)} = [0 \quad 1][sI - A_0 - DA_1 - A_1 Q]^{-1} B_0 \tag{18}$$

which yields a scalar transfer function.

For the purpose of illustration, let the component values of the boost converter be as in Section II-B. Table I summarizes some of the switching period effects on system response. For this example, as T increases, the steady-state values I_L, V_C, and τ_s increase also. Notice that when $T = 2.5\ \mu$s, there exist no physically meaningful solutions to (15), i.e., the solutions to (15) are complex. This would explain the reasons for instability at low switching frequencies seen in simulation. Future research will further examine these and other design issues.

VI. CONCLUSION

A new averaged model for PWM dc-dc converters is derived using formal mathematical methods. The new averaged model is switching frequency dependent. Simulated transients demonstrate the improvement over the conventional approach. Two benefits of the new averaged model are the correction of the dc offset error and the modeling of switching frequency effects on closed-loop stability and performance.

APPENDIX
AVERAGING THEORY

A. Introduction

One of the most commonly used methods in the analysis of periodic and almost periodic differential equations is the so-called method of averaging. The classical method of averaging [7] was originally developed for periodic systems in the form

$$\dot{x} = \varepsilon f(t, x) \tag{19}$$

where $f(t+T, \bullet) = f(t, \bullet)$ and $0 < \varepsilon \ll 1$.

Averaging theory addresses the relationship between the original, time varying system (19) and the autonomous averaged system

$$\dot{y} = \varepsilon g_0(y) \tag{20}$$

where function g_0 is obtained by using the averaging operator (*always* treating y as a constant)

$$g_0(y) = \frac{1}{T}\int_0^T f(s, y)ds.$$

Roughly speaking, for sufficiently small ϵ, solutions of (20) provide "good" approximations to solutions of (19). Since there are many mathematical tools that can be used to analyze and control autonomous system (20), the problem of determining the behavior of nonautonomous periodic system (19) has been greatly simplified.

B. Basic Theory Behind Averaging

There are several methods that can be used to explain averaging, including infinite series asymptotic expansions [5], [8], the Fundamental Theorem of Calculus [6], [9], moving averages [10], etc. This paper uses a technique similar to that of [11] because it clearly shows the reasons why the newly proposed averaged models for PWM dc-dc converters are more accurate.

According to [11], the basic idea of averaging is to determine a periodic function Ψ_0 such that the "near identity" periodic transformation

$$x = z + \varepsilon\Psi_0(t, z) \tag{21}$$

takes system (19) into the form

$$\dot{z} = \varepsilon g_0(z) + O(\varepsilon^2) \tag{22}$$

where $\frac{O(\varepsilon^2)}{\varepsilon^2} \to$ constant as $\varepsilon \to 0$ and g_0 is as given in (20). Then by using basic Lyapunov theory, it is known that solutions to (20) approximate solutions to (22) when ε is small. In order to determine Ψ_0, simply substitute (21) into (19) to obtain

$$\left[I + \varepsilon\frac{\partial\Psi_0}{\partial z}\right]\dot{z} = \varepsilon f(t, z + \varepsilon\Psi_0) - \varepsilon\frac{\partial\Psi_0}{\partial t}.$$

Using the fact that $[I + \varepsilon\frac{\partial\Psi_0}{\partial z}]^{-1} = I + O(\varepsilon)$ gives

$$\dot{z} = \varepsilon f(t, z + \varepsilon\Psi_0) - \varepsilon\frac{\partial\Psi_0}{\partial t} + O(\varepsilon^2). \tag{23}$$

The next step (in [11]) to prove averaging is, in fact, the step which our newly proposed averaging algorithms modify in order to improve the accuracy of the PWM averaged models. It is to rewrite (23) as

$$\dot{z} = \varepsilon f(t, z) - \varepsilon\frac{\partial\Psi_0}{\partial t} + \varepsilon[f(t, z+\varepsilon\Psi_0) - f(t, z)] + O(\varepsilon^2). \tag{24}$$

Now, by using continuity and Lipschitz arguments (see [11]), it is known that $\|f(t, z+\varepsilon\Psi_0) - f(t, z)\| = O(\varepsilon)$, and therefore (24) is of the form

$$\dot{z} = \varepsilon f(t, z) - \varepsilon\frac{\partial\Psi_0}{\partial t} + O(\varepsilon^2). \tag{25}$$

The final step is to choose Ψ_0 so that

1) Ψ_0 has zero average, and
2) the following relation is satisfied:

$$f(t, z) - \frac{\partial\Psi_0}{\partial t} = g_0(z)$$

where g_0 is given in (20).

In this case, (25) becomes identical to (22), and the proof is complete.

Hence, if Ψ_0 is defined as (treating z as a constant again)

$$\Psi_0(t, z) \equiv \int [f(t, z) - g_0(z)]dt + h_0(z) \tag{26}$$

where function h is chosen so that Ψ_0 has zero average, then substitution (21) transforms (19) into (22).

C. Modified Averaging Technique

The basic idea of subsection B is to introduce transformation (21) into (19) and then "manipulate" the transformation so that the new differential equation can be written in the form of (25). After this is successfully accomplished, Ψ_0 can be chosen as in (26) in order to complete the proof of averaging.

From B, it is seen that errors came from approximating solutions of (19) by solutions of (20) via the $O(\varepsilon^2)$ terms in (25). In particular, error is introduced in the following two places:

1) by approximating $\left[I + \varepsilon\frac{\partial\Psi_0}{\partial z}\right]^{-1}$ by I, and
2) by approximating $f(t, z + \varepsilon\Psi_0)$ by $f(t, z)$.

As $\varepsilon \to 0$, these errors become extremely small and can be neglected. However, since ε is finite in practice, these errors can sometimes affect the accuracy of the approximation.

This paper proposes a new technique, which eliminates error associated with 2) above. Suppose that (19) has special structure (as it does for PWM dc-dc converters) so that it is possible to determine or closely estimate a new function $g (\neq g_0)$ defined as

$$\frac{1}{T}\int_0^T f(t, z + \varepsilon\Psi)dt \equiv g(z, \varepsilon) \tag{27}$$

where Ψ is a different zero average periodic function given by the relation

$$f(t, z + \varepsilon\Psi) - \frac{\partial\Psi}{\partial t} = g(z, \varepsilon) \tag{28}$$

or equivalently

$$\Psi(t, z, \varepsilon) = \int [f(t, z + \varepsilon\Psi) - g(z, \varepsilon)]dt + h(z, \varepsilon) \tag{29}$$

where h is chosen so that Ψ is zero average. As before, the periodic function Ψ provides a "near identity" periodic transformation

$$x = z + \varepsilon\Psi(t, z, \varepsilon). \tag{30}$$

Substituting Ψ for Ψ_0, system (23) becomes [where $\Psi \equiv \Psi(t, z, \varepsilon)$)]

$$\begin{aligned}\dot{z} &= \varepsilon f(t, z + \varepsilon\Psi) - \varepsilon\frac{\partial\Psi}{\partial t} + O(\varepsilon^2)\\ &= \varepsilon g(z, \varepsilon) + O(\varepsilon^2).\end{aligned} \tag{31}$$

Therefore, the new proposed average model of (19) is

$$\dot{y} = \varepsilon g(y, \varepsilon). \tag{32}$$

One would expect that (32) will give a better approximation of (19) than (20) does, since the error associated with approximating $f(t, z + \varepsilon\Psi)$ by $f(t, z)$ has been eliminated.

For general periodic systems in the form of (19), this technique is *extremely* difficult to apply (almost impossible).

Notice that in (29), Ψ is defined by an integral equation depending on Ψ itself, as well as g. However, to determine g, Ψ must be known, and therefore, it is only rarely possible to explicitly determine g or Ψ.

This paper shows the following: It is possible to explicitly find Ψ and g which *approximately* satisfy (27), (29) in PWM dc-dc converters. The more accurate the approximation of (27), (29), the more accurate the averaged approximation (32) becomes.

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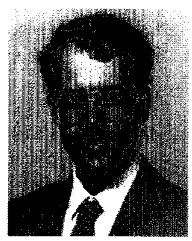

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