Corollary 1: Consider the system given by (2), (4), (26), and (27) were $\boldsymbol{Y}_m$ is defined by (25), $\boldsymbol{A} = \lambda \boldsymbol{I}$ where $\lambda > 0$ is a constant, and $\boldsymbol{\Gamma}$ is assumed to be constant, symmetric, and positive definite.

Then the equilibrium $\boldsymbol{x} = \boldsymbol{0}$ is stable in the sense of Lyapunov.

Proof: By combining (4), (25), and (27) the closed-loop dynamic system

$$\boldsymbol{H}\dot{\boldsymbol{s}} + (\lambda \boldsymbol{H} + \boldsymbol{C})\boldsymbol{s} = \boldsymbol{Y}_m \tilde{\boldsymbol{a}} \tag{28}$$

is obtained. The system given by (26), (2), and (28) is identical to the system (1)–(3) with $\boldsymbol{Y}_m$ replacing $\boldsymbol{Y}$ and $\lambda \boldsymbol{H}$ replacing $\boldsymbol{K}_D$. Then since the inertia matrix $\boldsymbol{H}$ is uniformly positive definite and $\lambda > 0$ is a constant, Proposition 1 applies, and the result follows.

Remark 2: Lyapunov stability for the system in Corollary 1 cannot be shown with the proof in [2] since $\lambda^2 \boldsymbol{H}$ is not constant in general (see Remark 1). In Proposition 1 less restrictive assumptions are made, and Lyapunov stability can be established.

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Delay Independent Stability Conditions and Decay Estimates for Time-Varying Functional Differential Equations

Brad Lehman and Khalil Shujaee

Abstract—**This paper presents sufficient delay independent conditions that guarantee stability of nonlinear time varying functional differential equations (FDE's). Estimates on the rate of decay of solutions are also obtained.**

I. Introduction

There has been a great amount of literature over the past 30 years discussing delay independent stability conditions for functional differential equations (FDE's) [1]–[14]. Much of this literature focuses on linear time invariant point delay systems [1]–[8]. There are several different approaches to developing stability conditions of such types of systems. Two of the most common techniques are to either use Lyapunov functions [1]–[4] or to analyze the FDE from a completely algebraic point of view [4]–[5]. While both techniques provide a powerful theoretical framework for stability analysis, there are several associated disadvantages: 1) the results are generally valid for linear time invariant (LTI) point delay systems only and/or 2) the results are often difficult to verify.

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Recently, [6], [7] derived delay independent stability conditions for LTI point delay systems in terms of a matrix measure (logarithmic norm). Furthermore, [8] calculated estimates on the decay rate of these systems, and therefore, information about both stability properties and transient responses of special classes of FDE's could be obtained.

The use of matrix measures in the analysis of delay equations has the benefits of being both an algorithmic procedure and being simple computationally. The work of [9] generalizes [6]–[8] by presenting delay independent stability conditions of point delay systems where the nominal plant is LTI but the perturbations are special classes of nonlinear, time-varying, bounded globally Lipschitz functions with point delay only. Presenting an algorithmic approach, [9] extends the work of [6]–[8] and the older works of [10]–[12] (which provide highly theoretical and computationally difficult to apply delay independent stability conditions for FDE's).

One drawback with [6]–[9] is that the nominal plant is assumed time invariant. Some work has been performed on linear time-varying point delay systems [10]–[17]; however, once again, most of the stability conditions derived are either computationally difficult to verify [10]–[14] or have strong restrictions on the time varying part of the system [15], [16]. None of the conditions provide estimates on the transient decay rate of the system.

The work presented in this paper is an extension of [6]–[14], [17]; in particular we have been greatly influenced by [10]–[12]. The results of this paper provide: 1) sufficient delay independent stability conditions for general classes of nonlinear time varying systems which are computationally simple to verify and 2) estimates on the decay rate of stable solutions of such systems. It is, in fact, these estimates which turn out to be the most difficult to prove.

Section II discusses the mathematical preliminaries necessary to present the main results of this work, which are found in Section III. Section III also provides examples which demonstrate the theory, and Section IV summarizes the results.

II. Preliminaries and Problem Formulation

This paper considers the stability properties of functional differential equations in the form

$$x'(t) = A(t)x(t) + f(t, x(t-g_1(t)), \cdots, x(t-g_m(t))) \tag{2.1}$$

where $\boldsymbol{x} \in \mathcal{R}^n$, $A(t) \in \mathcal{R}^{n\times n}$, $A(\cdot)$ is continuous on $t \geq t_0$, the function f: $(t_0 - r, \infty) \times \Omega \times \Omega \times \cdots \times \Omega \to \mathcal{R}^n$, $\Omega \subset \mathcal{R}^n$, is continuous in time, and $'$ denotes the right-hand derivative. It will always be assumed that $0 \leq g_i(t) \leq r$ for $t \geq t_0$, $i = 1, \cdots, m$ and some $0 \leq r < \infty$. Assume that the continuous initial condition of (2.1) takes the form $x(t) = \psi(t)$ for $t_0 - r \leq t \leq t_0$.

For simplicity, we will define the function $x_t = x_t(\sigma) \equiv x(t+\sigma)$ for $-r \leq \sigma \leq 0$. Then it is known [12] that (2.1) can be rewritten as

$$x'(t) = A(t)x(t) + F(t, x_t); \qquad x_{t_0} = \psi_{t_0}. \tag{2.2}$$

Let $C([-r, 0]; \Omega)$ be the space of all continuous functions mappings $[-r, 0] \to \Omega$, and define $J = [t_0 - r, \infty)$. Then F: $J \times C([-r, 0]; \Omega) \to \mathcal{R}^n$, and F is continuous in time. For a function $\psi \in C([-r, 0]; \Omega)$ define $\|\psi\|_r = \sup_{-r\leq\sigma\leq 0} \|\psi(\sigma)\|$.

The norm of a real vector x will be denoted by $\|x\|_k$, and the corresponding induced matrix norm is given as $\|A\|_k$. The matrix measure, sometimes referred to as the logarithmic norm, is defined by

$$\mu_k(A) = \lim_{\delta \to 0^+} \frac{\|I + \delta A\|_k - 1}{\delta}. \tag{2.3}$$

The matrix measure has the property that it can have both negative and positive values. Further, it is specifically induced by the corresponding norm $\|\cdot\|_k$. We have

$$\mu_k(A) = \begin{cases} \max_i\left\{a_{ii} + \sum_{j\neq i}|a_{ij}|\right\} & k = \infty \\ \max_j\left\{a_{jj} + \sum_{i\neq j}|a_{ij}|\right\} & k = 1 \\ \lambda_{\max}[A^* + A]/2 & k = 2. \end{cases} \tag{2.4}$$

III. Stability and Transient Decay

It is the purpose of this section to determine conditions in which the trivial solution of (2.2) is uniformly asymptotically stable and, more importantly, give estimates on the decay rates of solutions. The results presented are more general than those presented in [6]–[9] because 1) $A(t)$ is time varying, and 2) the restrictions on perturbation $F(t, x_t)$ are less restrictive. In particular, [9] considers (2.1) when $A(t) = A =$ constant and

$$F(t, x_t) = f_0(t, x(t)) + \sum_{i=1}^{m} f_i(t, x(t - r_i)).$$

In addition, the results of this section generalize and improve the works of [10]–[12], [17] by giving solution decay rates and by taking into consideration the special property that a time varying system can decay faster than $ke^{-\lambda(t-t_0)}$ (where constants $k \geq 1$, $\lambda > 0$).

In the course of stability analysis of (2.2), it will be necessary to use the following lemma, which is a generalization of [10, pp. 389], [11].

Lemma: Let $v(t)$ and $f(t)$ be continuous real valued nonnegative functions on $[t_0 - r, \beta)$ and $[t_0, \beta)$, respectively. Assume that $f(t)$ is positive and nondecreasing for all $t \in [t_0, \beta)$. Assume further that for all $t \in [t_0, \beta)$

$$v'(t) \leq -f(t)v(t) + b\|v_t\|_r \tag{3.1}$$

where b is some positive constant satisfying $0 < b < f(t)$ on $t \in [t_0, \beta)$.

Then

$$v(t) \leq \|v_{t_0}\|_r e^{-\int_{t_0}^{t} \gamma(s)\, ds} \quad \text{for } t_0 - r \leq t < \beta \tag{3.2}$$

where $\gamma(t)$ is the nondecreasing unique continuous solution to

$$\gamma(t) = \begin{cases} f(t_0) - be^{\gamma(t)r} & t < t_0 \\ f(t) - be^{\gamma(t)r} & t \in [t_0, \beta). \end{cases} \tag{3.3}$$

Furthermore, $\gamma(t)$ satisfies the inequality $0 < \gamma(t) < f(t)$ on $t \in [t_0 - r, \beta)$.

Proof: Let $\Delta(\gamma, t) = \gamma(t) - f(t) + be^{\gamma(t)r}$. Fix $t = t_1 \geq t_0$. Let $\gamma_1 = \gamma(t_1) =$ constant and let $a_1 = f(t_1) =$ constant > 0. Then

$$\Delta(\gamma_1, t_1) = \Delta(\gamma_1) = \gamma_1 - a_1 + be^{\gamma_1 r}. \tag{3.4}$$

Equation (3.4) is not time varying, and therefore the techniques of [10], [11] can be applied: Since $\Delta(0) = -a_1 + b < 0$, $\Delta(a_1) = be^{a_1 r} > 0$ and $(\partial\Delta/\partial\gamma) = 1 + \gamma be^{\gamma r} > 0$ for all $\gamma \geq 0$, it follows that there exists a unique γ_1, $0 < \gamma_1 < \alpha$, for which $\Delta(\gamma_1) = 0$. Since t_1 is arbitrary, it also follows that $\Delta(\gamma, t) = 0$ uniquely defines a function $\gamma(t)$. Further, $\gamma(t)$ is continuous since $\Delta(\cdot)$ depends continuously on $\gamma(\cdot)$ and $f(\cdot)$, where $f(\cdot)$ has been assumed continuous. This also implies that $0 < \gamma(t) < f(t)$ on $t \in [t_0, \beta)$.

Define $w(t) = \|v_{t_0}\|_r \exp\{-\int_{t_0}^{t} \gamma(s)\, ds\}$ for $t_0 - r \leq t < \beta$. Then on $t_0 - r \leq t \leq t_0$, $v(t) \leq w(t)$ since on this interval $(-\int_{t_0}^{t} \gamma(s)\, ds) > 0$.

Suppose there exists some $t_2 > t_0$ such that $v(t) \leq w(t)$ for $t_0 - r \leq t \leq t_2$, $v(t_2) = w(t_2)$ and $v(t) > w(t)$ on $t \in (t_2, t_2 + \tau)$, for some $\tau > 0$. Then it must be that $v'(t_2) > w'(t_2)$.

By (3.1)$v'(t_2) \leq -f(t_2)v(t_2) + b\|v_{t_2}\|_r$. Furthermore, it has been assumed that $v(t_2) = w(t_2)$, and that $v(t) \leq w(t)$ on $t_0 - r \leq t \leq t_2$. Therefore

$$v'(t_2) \leq -f(t_2)w(t_2) + b\|w_{t_2}\|_r. \tag{3.5}$$

Since $w(t)$ is monotone decreasing, we have

$$v'(t_2) \leq -f(t_2)w(t_2) + bw(t_2 - r). \tag{3.6}$$

Taking the derivative of the definition of $w(t)$, it is easy to show that

$$w'(t_2) = w(t_2)[-f(t_2) + be^{\gamma(t_2)r}]. \tag{3.7}$$

By assumption, $v'(t_2) > w'(t_2)$. Using (3.6) and (3.7), this implies that $w(t_2)e^{\gamma(t_2)r} < w(t_2 - r)$. Using the definition of $w(t)$, this gives

$$e^{\gamma(t_2)r} < e^{\int_{t_2-r}^{t_2} \gamma(s)\, ds}. \tag{3.8}$$

Since $(\partial\gamma/\partial f) > 1$, and $f(t)$ has been assumed nondecreasing, this implies that $\gamma(t)$ is nondecreasing also. Therefore, $\|\gamma_{t_2}\|_r = \gamma(t_2)$, and hence (3.8) yields equivalently

$$e^{\int_{t_2-r}^{t_2} \sup_{-r\leq\sigma\leq 0}\|\gamma(t_2+\sigma)\|_r\, ds} < e^{\int_{t_2-r}^{t_2} \gamma(s)\, ds} \tag{3.9}$$

which is never true. Contradiction. Q.E.D.

Using the lemma, it is possible to determine conditions in which the trivial solution of (2.2) is uniformly asymptotically stable. For a function $\psi \in C([-r, 0]; \Omega)$, let

$$\|\psi\|_{kr} = \sup_{-r\leq\sigma\leq 0} \|\psi(\sigma)\|_k \tag{3.10}$$

where $\|\psi(\sigma)\|_k$ is as previously defined.

Theorem: Assume in system (2.2) that $A(\cdot)$ is continuous and that there exists a constant $M > 0$ with an open neighborhood Ω $(0 \in \Omega)$ such that $\|F(t, \xi)\|_k \leq M\|\xi\|_{kr}$ for all $(t, \xi) \in (t_0, \infty) \times C([-r, 0]; \Omega)$. Suppose, further, that for all $t \geq t_0$

$$\mu_k(A(t)) < -M < 0 \tag{3.11}$$

and $\mu_k(A(t))$ is nonincreasing.

Then: i) the trivial solution of (2.2) is uniformly asymptotically stable independent of r, and ii) if $\psi \in C([-r, 0]; \Omega)$, an estimate on the transient response is given by

$$\|x(t)\|_k \leq \|\psi\|_{kr} \exp\left\{-\int_{t_0}^{t} \gamma_k(s)\, ds\right\} \tag{3.12}$$

where $0 < \gamma_k(t) < -\mu_k(A(t))$ and $\gamma_k(t)$ is the solution to

$$\gamma_k(t) = -\mu_k(A(t)) - Me^{\gamma_k(t)r} \qquad t \geq t_0. \tag{3.13}$$

Proof: Since $x(t)$ is right differentiable for $t \geq t_0$, (2.2) is equivalent to

$$\frac{x(t+\delta) - x(t)}{\delta} = A(t)x(t) + F(t, x_t) + O(\delta),$$

$$x_{t_0} = \psi_{t_0} \tag{3.14}$$

where $\delta > 0$ and $O(\delta) \to 0$ as $\delta \to 0^+$. This implies that for $t \geq t_0$

$$\|x(t+\delta)\|_k \leq \|I + \delta A(t)\|_k \cdot \|x(t)\|_k + \delta\|F(t, x_t)\|_k + \delta O(\delta). \tag{3.15}$$

Therefore,

$$\lim_{\delta \to 0+} \frac{\|x(t+\delta)\|_k - \|x(t)\|_k}{\delta} \le \lim_{\delta \to 0+} \frac{(\|I + \delta A(t)\| - 1)\|x(t)\|_k}{\delta} + M\|x_t\|_r + O(\delta). \quad (3.16)$$

The left-hand side of (3.16) is the right-hand derivative of $\|x(t)\|_k$. So we have

$$\frac{d^+}{dt}\|x(t)\|_k \le \mu_k(A(t))\|x(t)\|_k + M\|x_t\|_{kr} \quad (3.17)$$

where (d^+/dt) denotes the right-hand derivative. Letting $v(t) = \|x(t)\|_k$ for $t > t_0$, $v(t) = \|\psi(t)\|$ for $t \in [t_0 - r, t_0]$, and defining $f(t) = -\mu_k(A(t))$, the lemma can be directly applied (as long as $\psi \in C([-r, 0]; \Omega)$), since it has been assumed that for $t \ge t_0$, $0 < M < -\mu_k(A(t))$. This proves ii) of the theorem which, in turn, immediately implies i). Q.E.D.

Remark 1: The above theorem provides stability conditions which are similar to the results of [17]. The strength of the theorem, however, is that it provides an estimate on the exponential decay of solutions to (2.2) (which cannot be obtained using the techniques of [17]). These estimates of decay are, however, delay dependent. The function $\gamma(t)$ actually decreases as r increases, as is shown by (3.13).

Remark 2: In the proof of the lemma, it is shown that if $\mu_k(A(t))$ decreases, $\gamma_k(t)$ will increase. Therefore, the transient response could decay extremely quick (faster than $\exp[-(t - t_0)\gamma(t_0)]$, where $\gamma(t_0) =$ constant). In the special case when $A(t) = A$ and $F(t, x_t) = \sum_{i=1}^{m} f(t, x(t - r_i))$, and $\Omega = \mathcal{R}^n$, then $\gamma_k(t) = \gamma_k =$ constant, and the results of [6]–[9] are obtained.

Corollary: Assume in system (2.2) that $A(\cdot)$ is continuous and that there exists a neighborhood Ω $(0 \in \Omega)$ such that $\|F(t, \xi)\|_k \le M\|\xi\|_{kr}$ for all $(t, \xi) \in (t_0, \infty) \times C([-r, 0]; \Omega)$. Suppose that $\sup_{t \ge t_0}[\mu_k(A(t))] < -M < 0$, but $\mu_k(A(t))$ is not nonincreasing (as in the theorem).

Then i) the trivial solution of (2.2) is uniformly asymptotically stable independent of r, and ii) if $\psi \in C([-r, 0]; \Omega)$, then an estimate of the transient response is given by

$$\|x(t)\|_k \le \|\psi\|_{kr} \exp\{-(t - t_0)\gamma\} \quad (3.18)$$

where $0 < \gamma < |\mu_k(A(t))|$ for all $t \ge t_0$ and γ is the constant solution to

$$\gamma = \inf_{t \ge t_0}[-\mu_k(A(t))] - Me^{\gamma r}. \quad (3.19)$$

Proof: Almost immediate from the proof of the theorem. Q.E.D.

Remark 3: The above results can be applied to systems not exactly written in the form of (2.1) or (2.2). Consider the delay differential equation

$$x'(t) = f(t, x(t), x(t - g_1(t)), \cdots, x(t - g_m(t))) \quad (3.20)$$

where x, f, and g_i are previously defined in (2.1). Clearly, if f is continuously Frechet differentiable with respect to its second argument, then (3.20) can be rewritten as

$$x'(t) = A(t)x(t) + [f(t, x(t), x(t - g_1(t)), \cdots, x(t - g_m(t))) - A(t)x(t)] \quad (3.21)$$

where $A(t) = (\partial f(t, 0, 0, \cdots, 0)/\partial x(t))$. By defining $F(t, x_t)$ as the term in the square brackets, system (3.21) is in the form such that the conditions of the theorem may be applied for some sufficiently small neighborhood of the origin where the above linearization is valid.

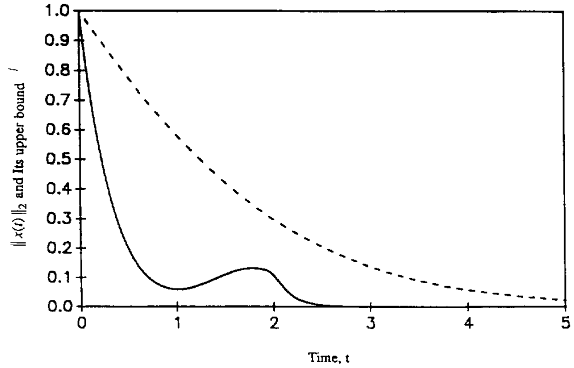

Fig. 1. $\|x(t)\|_2$ (solid curve) and its upper bound (dashed curve) versus time.

Example: Consider (2.1) with $x(t) = [x_1(t), x_2(t)]$

$$A(t) = \begin{bmatrix} -t-3 & e^t \\ -e^t & -t-4 \end{bmatrix},$$

$$f(t, x(t), x(t - g_1(t))) = \begin{bmatrix} \sin^2(t)x_1^2(t - \sin(t) - 1) \\ x_2^3(t) \end{bmatrix}. \quad (3.22)$$

Let $x_1(t) = c\cos t$ on $t \in [-2, 0]$, where $c =$ constant, and let $x_2(0) = 0$. Using the notation of the theorem, define $\Omega \subset \mathcal{R}^n$ as $\{x \in \Omega\colon \|x\|_k < 1\}$, and let $F(t, \xi) = f(t, \xi(t), \xi(t - \sin(t) - 1))$. Since $0 < \sin(t) + 1 \le 2$, we can define the constant r to be 2.

Using (2.4) we have

$$\mu_1(A(t)) = \mu_\infty(A(t)) = -t - 3 + e^t$$

$$\mu_2(A(t)) = -t - 3. \quad (3.23)$$

For sufficiently large t, $\mu_1 = \mu_\infty$ are positive functions, and therefore to apply the theorem, we must use $\mu_2(A(t))$ which is a nonincreasing function. Using $\|\cdot\|_2$, the condition of the theorem that $\|F(t, \xi)\|_2 < M\|\xi\|_{2r}$, for all $\xi \in \Omega$ is satisfied for $M = 1$. Since (3.11) is satisfied for all $t > 0$, i.e.,

$$-t - 3 < -1 < 0, \qquad t > 0 \quad (3.24)$$

the trivial solution of this system is uniformly asymptotically stable independent of delay.

Furthermore, in (3.12), $\|\psi\|_{2r} = \sup_{-2 \le \sigma \le 0} \|[c\cos\sigma, 0]^T\|_2 = c$. Hence, if $c < 1$, then an estimate of the transient decay is given by

$$\|x(t)\|_2 \le c\exp\left\{-\int_0^t \gamma_2(s)\,ds\right\} \quad (3.25)$$

where $0 < \gamma_2(t) < t + 3$ and $\gamma_2(t)$ is the solution to

$$\gamma_2(t) = t + 3 - e^{2\gamma_2(t)}, \qquad t \ge 0. \quad (3.26)$$

Fig. 1 plots both the left-hand side of (3.25), (solid curve) and the right-hand side of (3.25), (dashed curve) versus time for $c = 0.99$.

From this figure, it is verified that the right-hand side of (3.25) is an upper bound on $\|x(t)\|_2$ for all $t \geq 0$.

IV. Conclusion

This paper presents sufficient, easily calculable, delay independent stability conditions for time varying functional differential equations. An estimate on the bound of transient decay is also presented. It is shown that this bound depends exponentially on the area underneath the curve of the time varying function $\gamma_k(t)$. This function is found by solving a time varying nonlinear algebraic equation.

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Iterative Matrix Bounds and Computational Solutions to the Discrete Algebraic Riccati Equation

N. Komaroff

Abstract— **Bilateral matrix bounds for the solution of the discrete algebraic Riccati equation (DARE) are presented. They are new or tighter than the existing bound. Computational algorithms to solve the DARE follow.**

I. Introduction

Consider the discrete algebraic Riccati equation (DARE)

$$P = A'PA - A'PB(I + B'PB)^{-1}B'PA + Q, \qquad Q = Q' \geq 0 \tag{1}$$

where A, P, $Q \in R^{n\times n}$, $B \in R^{n\times r}$, $(')$, I and (≥ 0) denote the transpose, the unit matrix, and positive semidefiniteness, respectively. This equation plays a central role in various branches of engineering, including signal processing and control theory [1].

It is well known that for (1) a bounded positive definite solution P exists with (A, B) stabilizable, (A, Q) detectable.

Application of the matrix identity

$$(X^{-1} + YZ)^{-1} = X - XY(I + ZXY)^{-1}ZX$$

allows (1) to be written as

$$P = A'(P^{-1} + R)^{-1}A + Q \tag{2}$$

where $R = BB'$. This note deals with the version (2) of (1).

The computation of the positive definite solution P to (2) is of some difficulty especially when the dimension n of the matrices is high. The closer the initial estimate is to the actual solution, the less computer time is expected to be used in the solution algorithm. Therefore, it is important to obtain an accurate estimate of the solution. Only one matrix estimate of P, a lower bound [8], has been presented. The usual estimate is given by bounds on functions of the eigenvalues of P, such as summations that include $\operatorname{tr}(P)$, the trace of P, and products that include $|P|$, the determinant of P [5]–[11].

In this note we exploit Loewner's ordering for matrix-monotone and matrix-convex functions [2], [3] to

i) derive upper and lower matrix bounds for P of (2) which are new or are tighter than the one in the literature [8], and
ii) develop from these results convergent computational algorithms to obtain the positive definite solution matrix P to (2).

II. Preliminaries

The following notation and theorems shall be used.

Let $\lambda_i(X)$ denote the ith eigenvalue of a matrix X, $i = 1, 2, \cdots, n$. All $\lambda_i(X)$ are ordered such that their real parts are nonincreasing

$$\operatorname{Re}\lambda_1(X) \geq \operatorname{Re}\lambda_2(X) \geq \cdots \geq \operatorname{Re}\lambda_n(X).$$

We shall use the following results from the Loewner ordering for matrix valued functions of symmetric matrices $X, Y \in R^{n\times n}$ [2, pp. 462–466, 474–475], [3, pp. 469–471].

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