Vibrational Control of Nonlinear Time Lag Systems with Bounded Delay: Averaging Theory, Stabilizability, and Transient Behavior

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***Abstract*—This paper develops the theory of vibrational control of nonlinear time lag systems with arbitrarily large but bounded delay. Averaging theory for fast oscillating, differential delay equations is presented and then applied to vibrational control. Conditions are given which ensure the existence of parametric vibrations that stabilize nonlinear time lag systems. Transient behavior is also discussed. Illustrative examples are given which show 1) the feasibility of the theory to important applications and 2) the differences in the theory presented and the existing known theory for vibrational control of ordinary differential equations.**

I. Introduction

Vibrational control is a recently developed nonclassical control technique that, unlike feedback and feedforward, does not require measurements of states or disturbances. Instead, zero mean parametric excitation is used as the tool for open-loop modification of the plant behavior. For example, oscillations in an airplane wing can be introduced by tapping the wing in a described manner. To apply vibrational control to a combustion system, it may be possible to oscillate (open and shut quickly) an intake valve. Because no state measurements are required, vibrational control is a viable alternative to feedback and feedforward techniques when measurements are costly or, for some reason, unavailable.

Vibrational control of systems governed by linear and nonlinear ordinary differential equations has been thoroughly discussed [1]–[4]. Application of this theory has been experimentally verified for: 1) an exothermic irreversible chemical reaction in a continuous stirred tank reactor (CSTR) by [5], and 2) a laser illuminated thermochemical system [6]. For example, [5] showed that by vibrating the flow rates in a CSTR, it is possible to operate the reactor at (average) conversion rates which were previously unstable. This technique eliminated significant cooling expenses associated with feedback. Additionally, since the vibrations depended only on time (and not on the value of the state), there no longer was a need to take measurements of concentration, thus reducing the cost of the reaction even more. For similar reasons, the vibrational control described by [6] has many benefits.

A number of practical, important systems, however, are best described by including time delays in their states. In particular, if the exothermic reaction vibrationally controlled in [5] includes a recycle stream, as in [7], the model must include state delays. Population models, combustion models, and manufacturing systems, among many others (cf. [8]), also have state delays in their models. It is, therefore, of interest to vibrationally control such systems.

Vibrational control of time lag systems is a more complicated task since the plant is infinite dimensional. For systems with small delays, finite dimensional approximations were made to approximate the vibrationally controlled delay system by an ordinary differential equation (ODE) [9]–[11]. The methods of these papers, though, fail when vibrationally controlling general time lag systems, i.e., those systems with arbitrary but bounded delay.

The main difficulty in extending the results of [9]–[11] is the lack of mathematical stability theory for time-varying delay differential equations. In particular, the primary mathematical tool for analysis used in vibrational control is the method of averaging, which is well known for ODE's [12, p. 186] and for time lag systems with small delays [13, p. 215]. For fast oscillating systems with large bounded delay, however, the averaging technique, until recently, was not developed. Therefore, to extend vibrational control to general time lag systems, it is first necessary to extend averaging techniques to a more general class of delay differential systems. This is performed in [14] (entirely motivated to enable the development of vibrational control) and Section II, which together provide the set of required mathematical tools needed for the extension.

The results given in Section II are significant mathematical contributions themselves since they extend the method of averaging to an extremely broad class of differential delay equations. The averaging technique for ODE's has found important applications in adaptive control algorithms, basic stability analysis, noise control, pulse width modulation, periodic control, as well as vibrational control, just to name a few. The results given in Section II should allow similar extensions to delay differential equations. In this paper, however, the

Manuscript received March 2, 1992; revised May 10, 1993. Recommended by Associate Editor, J. S. Gibson. This work was supported in part by Alcoa Science Foundation Grant 30–021607847 and NSF Grant MSS–89–57198 PYI

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IEEE Log Number 9216622.

applications of these new averaging results will only be used to extend the vibrational control technique to a general class of time lag systems.

This is done in Section III where vibrational stabilizability of nonlinear differential equations is discussed, partial results (the linear case) of which are published in [15]. Example 3.1, dealing with population dynamics, typifies the differences between the theory of vibrational control ODE's and the theory of vibrational control of time lag systems by showing that some systems are vibrationally stabilizable only when there is a delay in the state, i.e., if the delay is assumed zero, vibrational stabilizability is not possible.

Section IV discusses transient behavior of vibrationally controlled systems, and Section V proposes the vibrational control of an exothermic irreversible chemical reaction in a CSTR with delayed recycle stream; Section VI contains conclusions. All forms proofs are in Appendix I.

II. Averaging Theory

In this section, the mathematical foundations of averaging differential delay equations are presented. These techniques will be used in subsequent sections to develop the theory of vibrational control. As the introduction suggests, however, the results of this chapter have broad applications to general control theory.

Suppose $f(s, x, y)$ is a continuous function, $f\colon \mathbb{R} \times \Omega \times \Omega \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$. Let ϵ be a real parameter, and let $\varphi(t) \in \Omega$ be a continuous function for $t \in [-r, 0]$. Consider the system of differential delay equations for $t \geq 0$

$$\dot{x}(t) = f\left(\frac{t}{\epsilon}, x(t), x(t-r)\right); \qquad x(t) = \varphi(t), \text{ for } t \in [-r, 0] \quad (2.1)$$

along with

$$\dot{y}(t) = f_0(y(t), y(t-r)); \qquad y(t) = \psi(t), \text{ for } t \in [-r, 0] \quad (2.2)$$

where

$$f_0(z_1, z_2) \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t, z_1, z_2)\, dt.$$

Let $x(t, \epsilon; \varphi)$ denote the solution to (2.1), and let $y(t; \psi)$ denote the solution to (2.2).

Theorem 2.1 [14]: Assume that $f(s, z_1, z_2)$ has continuous Fréchet derivatives in (z_1, z_2) on $\mathbb{R} \times \Omega \times \Omega$, $0 \in \Omega$ and that f is almost periodic in s uniformly with respect to (z_1, z_2) in compact subsets of $\Omega \times \Omega$. If $y = y_s$ is a hyperbolic equilibrium of (2.2), then there are constants $\rho > 0$ and $\epsilon_0 > 0$, such that for $0 < \epsilon \leq \epsilon_0$, there is a unique almost periodic solution $x^*(t, \epsilon)$ of (2.1) almost periodic in t uniformly with respect to ϵ, where $x^*(t, 0) = y_s$ and $x^*(t, \epsilon)$ is the unique solution of (2.1) defined on $\mathbb{R}$ and remaining within ρ of y_s. Furthermore, $x^*(t, \epsilon)$ has the same hyperbolic stability properties as the equilibrium y_s of (2.2).

Theorem 2.1 provides insight into stability properties of (2.1) and (2.2). In particular, if $y = y_s$ is hyperbolic and uniformly asymptotically stable, then the unique, almost periodic, solution $x^*(t, \epsilon)$ is also hyperbolic and uniformly asymptotically stable.

To address the transient behavior of vibrationally controlled systems, it is also necessary to consider the closeness of the solutions $x(t, \epsilon; \varphi)$ and $y(t; \psi)$ in neighborhoods outside y_s. The following two theorems address this problem.

Theorem 2.2: Assume that for every $(s, z_1, z_2) \in \mathbb{R} \times \Omega \times \Omega$:

1) $f(s, z_1, z_2)$ is continuous with respect to all its arguments;
2) the limit, $\lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} f(s, z_1, z_2)\, ds$, exists uniformly with respect to (t, z_1, z_2) in compact sets of $\mathbb{R} \times \Omega \times \Omega$;
3) there exists a constant k such that

$$\|f(s, z_1, z_2) - f(s, z_1', z_2')\| \leq k \max_{i=1,2} \|z_i - z_i'\|. \quad (2.3)$$

Then for any $L > 0$ and any $\gamma > 0$ there exists an $\epsilon_0 = \epsilon_0(\gamma, L)$ such that for, $0 < \epsilon \leq \epsilon_0$

$$\|x(t, \epsilon; \varphi) - y(t; \psi)\| \leq \left(\gamma + \sup_{s\in[-r, 0]} \|\varphi(s) - \psi(s)\|\right) e^{kt} \quad (2.4)$$

for any $t \in [0, L]$.

Remark 2.1: Let the assumptions of Theorem 2.2 be true. Suppose that $x(t)$ and $y(t)$ have identical initial functions, i.e., in (2.1) and (2.2) $\varphi(t) = \psi(t)$ for $t \in [-r, 0]$. Then for any $\gamma > 0$ and any $L > 0$ there exists an $\epsilon_0 = \epsilon_0(\gamma, L)$ such that, for $0 < \epsilon \leq \epsilon_0$

$$\|x(t, \epsilon; \varphi) - y(t, \epsilon)\| \leq \gamma e^{kL} \quad (2.5)$$

for any $t \in [0, L]$, where k is the Lipschitz constant defined in (2.3). Therefore, for any fixed L, the constant γ can be chosen to make the bound in inequality (2.5) arbitrarily small for all sufficiently small ϵ.

Remark 2.2: As expected, if the initial time $t_0 \neq 0$, then the conclusions of Theorem 2.2, given by (2.4), become

$$\|x(t, \epsilon; t_0, \varphi) - y(t; t_0, \psi)\| \leq \left(\gamma + \sup_{s\in[t_0-r, t_0]} \|\varphi(s) - \psi(s)\|\right) e^{k(t-t_0)}. \quad (2.6)$$

Remark 2.3: Instead of assuming that f is almost periodic, Theorem 2.2 assumes that the limit in condition 2) exists and that f is continuous and Lipschitz. Therefore, the conditions of Theorem 2.2 are less restrictive than those of Theorem 2.1. The method of [14] can, however, be extended to prove the results of Theorem 2.2 and Theorem 2.3. The approach of [14] is to use the variation of constants formula in an abstract infinite dimensional Banach space, the Sun–Star Banach space. Then [14] views the problem as a perturbation of the zero solution. We believe, however, that the new proofs presented have many benefits over those of [14] because: 1) they are easier to understand since, in essence, the main idea of the proof is to apply a special form of the Fundamental Theorem of Calculus, 2) all analysis is kept in $\mathbb{R}^n$ and therefore, the theorems permit synthesis of vibrational controllers which must be implemented in $\mathbf{R}^n$, and 3) the proofs accurately describe transient behavior of a general class of highly oscillating differential delay equations and give a bound on the closeness of the solutions of (2.1) and (2.2). Of course, since an ODE is a special case of a delay differential equation, the proof of Theorem 2.2 provides an alternative proof to the classical techniques of averaging ODE's found in references such as [12].

For vibrational control theory, it is also of interest to examine the closeness of solutions $x(t, \epsilon; \varphi)$ and $y(t; \psi)$ on infinite time intervals. To extend the results of Theorem 2.2 to an infinite time interval (which can be done if $y(t; \psi)$ approaches a hyperbolic equilibrium point) along with (2.1) consider the corresponding averaged delay differential equation, for $t \geq 0$

$$\dot{y}(t) = f_0(y(t), y(t-r)); \qquad y(t) = \varphi(t), \text{ for } t \in [-r, 0] \tag{2.7}$$

where f_0 is defined in (2.2). That is, consider (2.2) with $\psi(t) = \varphi(t)$, for $t \in [-r, 0]$, i.e., $x(t)$ and $y(t)$ have the same initial functions. The solution to (2.7) is denoted as $y(t; \varphi)$.

Theorem 2.3: Assume that f satisfies conditions 1), 2), and 3) of Theorem 2.2 and that $f(s, z_1, z_2)$ has continuous Fréchet derivatives in (z_1, z_2) on $\mathbf{R} \times \Omega \times \Omega$. Suppose that $y(t; \varphi)$, the solution of (2.7), is defined for all $t \geq 0$ and is contained in Ω with its ρ neighborhood, $\rho > 0$.

If there is a point $y_s \in \Omega$ such that $\lim_{t\to\infty} y(t, \varphi) = y_s$ and

$$\text{Det}\left[sI - \frac{\partial f_0(y_s, y_s)}{\partial y} - \frac{\partial f_0(y_s, y_s)}{\partial y(t-r)} e^{-rs}\right] = 0$$

has all solutions with real parts less than zero, then for any $\eta > 0$, there exists an $\epsilon_0 = \epsilon_0(\eta)$ such that, for all ϵ, $0 < \epsilon \leq \epsilon_0$

$$\|x(t, \epsilon; \varphi) - y(t; \varphi)\| < \eta \text{ for all } t \geq 0. \tag{2.8}$$

Remark 2.4: Suppose, in addition to the assumptions in Theorem 2.3, $f(s, z_1, z_2)$ is almost periodic in s, uniformly with respect to (z_1, z_2) in compact subsets of $\Omega \times \Omega$, $0 \in \Omega$. Then by Theorems 2.1, 2.2, and 2.3 it is easy to see that for $0 < \epsilon \leq \epsilon_0$, $\lim_{t\to\infty} x(t, \epsilon; \varphi) = x^*(t, \epsilon)$.

Remark 2.5: Suppose x and y have different initial functions such as (2.1) and (2.2). Then there exists a constant $\beta_0 = \beta_0(L, \eta) > 0$, sufficiently small such that if $\sup_{s\in[-r,0]} \|\varphi(s) - \psi(s)\| \leq \beta_0$, the conclusions of Theorem 2.3 remain valid. This is an immediate consequence of both continuity and the proof of Theorem 2.3.

III. Vibrational Stabilizability

A. Problem Statement

Consider the delay differential equation

$$\dot{x}(t) = \tilde{P}_1(x(t), x(t-r)) + \tilde{P}_2(\lambda, x(t))$$
$$\tilde{P}_1\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n, \; \tilde{P}_2\colon \mathbf{R}^d \times \mathbf{R}^n \to \mathbf{R}^n \tag{3.1}$$

where $x(t) \in \mathbf{R}^n$, $\tilde{P}_1$ and $\tilde{P}_2$ are continuously differentiable, $\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_d]^T$ are parameters subject to vibrations, and r is the constant positive delay.

Introduce into (3.1) parametric vibrations according to the law $\lambda(t) = \lambda_0 + f(t)$, where λ_0 is a constant vector, and $f(t)$ is an almost periodic average zero (APAZ) vector. Then (3.1) becomes

$$\dot{x}(t) = \tilde{P}_1(x(t), x(t-r)) + \tilde{P}_2(\lambda_0 + f(t), x(t)). \tag{3.2}$$

Assume that (3.1) has a fixed equilibrium point $x_s = x_s(\lambda_0)$ for a fixed λ_0 (note that $x(t) = x(t-r) = x_s(\lambda_0)$ at steady state).

Definition 3.1: An equilibrium point $x_s(\lambda_0)$ of (3.1) is said to be *vibrationally δ-stabilizable* (*$v\delta$-stabilizable*) if for a given fixed $\delta \geq 0$ there exists an APAZ vector $f(t)$ such that (3.2) has an asymptotically stable, almost periodic, solution $x^s(t)$, $0 \leq t < \infty$, characterized by

$$\|\overline{x}^s - x_s(\lambda_0)\| \leq \delta; \quad \overline{x}^s = \overline{x^s(t)} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^s(t)\, dt.$$

Definition 3.2: An equilibrium point $x_s(\lambda_0)$ of (3.1) is said to be *totally vibrationally stabilizable* (*t-stabilizable*) if it is $v\delta$-stabilizable with $\delta = 0$, and moreover, $x^s(t) = \text{const.} = x_s(\lambda_0)$, for $0 \leq t < \infty$.

Often, only one component of $x_s(\lambda_0)$ requires or admits vibrational stabilization, such as in chemical reactors where only one of the two, the rate of conversion or the temperature, can be vibrationally stabilized. This practical situation is reflected in the following definition.

Definition 3.3: An equilibrium point $x_s(\lambda_0) = [x_{1s}(\lambda_0), \cdots, x_{ns}(\lambda_0)]^T$, of (3.1) is said to be *partially vibrationally δ-stabilizable with respect to component $x_{is}(\lambda_0)$* if for a given fixed $\delta \geq 0$ there exists an APAZ vector $f(t)$ such that (3.2) has an asymptotically stable almost periodic solution $x^s(t) = [x_1^s(t), \cdots, x_n^s(t)]^T$, $0 \leq t < \infty$ the ith component of which is characterized by

$$\|\overline{x}_i^s - x_{is}(\lambda_0)\| \leq \delta.$$

Throughout this paper it will be assumed that $\tilde{P}_2(\lambda_0 + f(t), x(t)) \triangleq P(\lambda_0, x(t)) + P_2(f(t), x(t))$, where $P_2(\cdot, \cdot)$ is a vector function linear with respect to its first argument. Then (3.2) can be rewritten as

$$\dot{x}(t) = P_1(x(t), x(t-r)) + P_2(f(t), x(t)) \tag{3.3}$$

where $P_1(x(t), x(t-r)) = \tilde{P}_1(x(t), x(t-r)) + P(\lambda_0, x(t))$, and $P_1(z_1, z_2)$ and $P_2(\cdot, z_1)$ are assumed to have continuous Fréchet derivatives in (z_1, z_2).

Following the terminology introduced in Bellman *et al.* [2], [3], if $P_2(f(t), x(t)) = L(t)$, where $L(t)$ is an APAZ vector, the vibrations are referred to as *vector additive*. If $P_2(f(t), x(t)) = D(t)x(t)$, the vibrations are called *linear multiplicative*, and if $P_2(f(t), x(t)) = B(t)\Gamma(x)$ where $\Gamma\colon \mathbf{R}^n \to \mathbf{R}^n$, is a nonlinear function, the vibrations are called *nonlinear multiplicative*. In each one of these three cases, function $P_2(\cdot, \cdot)$ has zero average, and it is not obvious why (3.3) will have different stability properties than (3.1).

It is interesting to compare vibrational control to other types of known high frequency control techniques, such as sliding mode control [16] and dithering [17]–[19]. A main difference between vibrational control and these techniques is that in vibrational control, parameters, λ, are vibrated independent of $x(t)$. The control is, therefore, a function depending only on time and is implemented as an open-loop control technique. This is in direct contrast with sliding mode control, which is a closed-loop control technique that requires current knowledge of the state for implementation.

A dither is a high frequency input which, by sweeping back and forth across the domain of the nonlinear elements, produces the effect of vibrational linearization [17] and thereby modifies the performance of the closed-loop feedback system by making nonlinearities appear linear on average. Vibrational control, on the other hand, has the ability to maintain nonlinearities in the system. Additionally in [18], [19], dithering is thoroughly examined from an input/output point of view. These papers further explain how nonlinearities of a closed-loop system are attenuated by a dither; however, these papers clearly show that the dithering technique fails for linear systems. Vibrational control (which is applied to open loop systems, unlike a dither) *is* effective in the case of linear systems [15].

Qualitatively, vibrational control can be thought of as the introduction of zero mean parametric oscillations into a dynamical system to achieve a desired response (such as stabilizing effects). For example, (3.1) may have unstable equilibrium $x_s(\lambda_0)$, but (3.2) may have a hyperbolic uniformly asymptotically stable, almost periodic, orbit $x^s(t)$ which vibrates in the vicinity of $x_s(\lambda_0)$. Of course, it would be preferable that (3.2) have the same fixed equilibrium point, $x_s(\lambda_0)$, as (3.1) (this is t-stabilization). This is not always the case, however, since the right-hand side of (3.2) is time varying and almost periodic. Therefore, the idea of vibrational stabilization is to determine vibrations $f(t)$ such that unstable equilibrium point $x_s(\lambda_0)$ in (3.1) bifurcates into a stable almost periodic solution whose average is close to $x^s(t)$. The engineering aspects of the problem consist of: 1) finding the conditions for the existence of stabilizing vibrations, 2) determining which parameters, λ, are physically possible to vibrate, and 3) finding the actual parameters of vibrations that ensure the desired response.

B. General Case

To formulate the conditions for $v\delta$-stabilizability of (3.1), consider the equation

$$\dot{x}(t) = P_2(f(t), x(t)). \tag{3.4}$$

Assume that for each initial condition, $x(t_0) = x_0$, (3.4) has a unique almost periodic solution $h(t, c)$, where each $c = c(x_0)$ is a constant vector in R^n. Then, there will always exist a constant m, $m > 0$, such that $\|h(t, c_1) - h(t, c_2)\| \leq m\|c_1 - c_2\|$ for any c_1, c_2.

Introducing the substitution

$$x(t) = h(t, y(t)), \quad x(t-r) = h(t-r, y(t-r)) \tag{3.5}$$

into (3.3) for $t \geq 0$ yields

$$\begin{aligned}\dot{y}(t) &= \left[\frac{\partial h(t, y(t))}{\partial y}\right]^{-1} P_1(h(t, y(t)), h(t-r, y(t-r))) \\ &\triangleq Y(t, r, y(t), y(t-r)).\end{aligned} \tag{3.6}$$

Introduce the equation

$$\begin{aligned}&\dot{z}(t) = P_0(z(t), z(t-r), r);\\ &P_0(y(t), y(t-r), r)\\ &\quad \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T Y(\tau, r, y(t), y(t-r))\, d\tau.\end{aligned} \tag{3.7}$$

Let z_s denote an equilibrium point of (3.7) and

$$\dot{z}(t) = M_0(r)z(t) + M_1(r)z(t-r) \tag{3.8}$$

be the linearization of (3.7) at z_s with

$$M_0(r) \triangleq \frac{\partial P_0(z_s, z_s, r)}{\partial z(t)}, \qquad M_1(r) \triangleq \frac{\partial P_0(z_s, z_s, r)}{\partial z(t-r)}.$$

Lemma 3.1: Let $z_s \in \Omega$ denote an equilibrium point of (3.7). Assume that

1) there exists an APAZ vector $f(t)$ such that the general solution, $h(t, c)$, of (3.4) is almost periodic for any $c \in \Omega$;
2) both $Y(t, r, \beta, \eta)$ and $P_0(\beta, \eta, r)$ are continuously differentiable for all $(\beta, \eta) \in \Omega \times \Omega$.

i) Then for any $\delta > 0$ there exists $\epsilon_\delta > 0$ such that for any $\epsilon \in (0, \epsilon_\delta]$, the equation

$$\begin{aligned}\dot{y}(t) &= \left[\frac{\partial h(t/\epsilon, y(t))}{\partial y}\right]^{-1}\\ &\quad \cdot P_1(h(t/\epsilon, y(t)), h(t/\epsilon - \gamma, y(t-r)))\end{aligned} \tag{3.9}$$

has an almost periodic solution $y^s(t)$ satisfying

$$\|y^s(t) - z_s\| < \delta/m, \ \forall \epsilon \in (0, \epsilon_\delta], \ \forall t \geq 0.$$

ii) If, in addition to assumptions 1) and 2), there exists a constant $\gamma > 0$ such that the equation given by

$$\mathrm{Det}\,[sI - M_0(\gamma) - M_1(\gamma)e^{-sr}] = 0 \tag{3.10}$$

has all the roots with negative real parts (assumption 3)), then there exists an $\epsilon_0 > 0$ such that $y^s(t)$ exists and is locally asymptotically stable for any $\epsilon \in (0, \epsilon_0]$.

Theorem 3.1: Let the assumptions 1), 2), and 3) of Lemma 3.1 hold. Then $x_s(\lambda_0)$ of (3.1) is

i) $v\delta$-stabilizable by vibrations $f(t) = (1/\epsilon_1)g(t/\epsilon_1)$, $\epsilon_1 \triangleq r/\gamma =$ constant if

a) $0 < \epsilon_1 \le \min[\epsilon_0, \epsilon_\delta]$, where ϵ_0 and ϵ_δ are defined and are guaranteed to exist for system (3.9) by Lemma 3.1.

b) There exists an equilibrium point z_s of (3.7), $z_s \in \Omega$, characterized by $\overline{h(t, z_s)} = x_s(\lambda_0)$.

ii) t-stabilizable if it is $v\delta$-stabilizable, and in addition, (3.9) has an equilibrium point $y_s \in \Omega$ characterized by $y_s = z_s$, and $h(t, y_s) = \text{const.} = x_s(\lambda_0)$.

Remark 3.1: The condition that $\overline{h(t, z_s)} = x_s(\lambda_0)$ means that equilibria of (3.1) and (3.7) are related through the average value of substitution (3.5). This is clearly not always the case. When this condition does not hold, however, $v\delta$-stabilizability can still take place.

Remark 3.2: Theorem 3.1 reduces the problem of $v\delta$-stabilizability of (3.1) to the following procedure. First, a search is made for an APAZ vector $f(t)$ so that (3.4) generates an almost periodic general solution $h(t, c)$ such that all roots of (3.10) have $Re(s) < 0$. (Newton–Raphson techniques can be used to solve for the dominant roots of (3.10).) Second, the existence of stabilizing vibrations is established if $r/\gamma \le \min[\epsilon_0, \epsilon_\delta]$. The actual stabilizing vibrations are obtained by rescaling the magnitude and frequencies of vibrations $f(t)$ as $f(t) = (1/\epsilon_1)g(t/\epsilon_1)$.

A difficult part of this procedure is to implement the search for vector $f(t)$. For special classes of time lag systems, [15] shows the precise manner to introduce stabilizing vibrations. For general time lag systems, however, such as (3.1), *a priori* knowledge of how to determine $f(t)$ is not yet known. Additionally, in practice, the engineer is constrained by being able to insert vibrations only into some of the parameters of the system (see Example 3.1 and Section V). This makes the problem of solving for $f(t)$ even more difficult and, therefore, $f(t)$ is usually determined by trial and error.

Remark 3.3: Analytical estimates of ϵ_0 and ϵ_δ are usually extremely conservative. Therefore, the values of ϵ_0 and ϵ_δ are best determined by the numerical simulation of system (3.9).

Remark 3.4: For small delays, $r = \epsilon r_0$, $0 < \epsilon \le \epsilon_0$, the stability of (3.8) is not dependent on a transcendental characteristic equation, and Theorem 3.1 essentially reduces to Theorem 1 of Bentsman, *et al.* [11], i.e., $v\delta$-stabilizability of $x_s(\lambda_0)$ of (3.1) is guaranteed for any $\delta > 0$ if there exists APAZ vector $f(t) = (1/\epsilon)g(t/\epsilon)$, such that polynomial $\text{Det}[sI - M_0(r_0) - M_1(r_0)]$ is Hurwitz and $\overline{h(t/\epsilon, z_s)} = x_s(\lambda_0)$, since $e^{-\epsilon r_0 s} \approx 1$ — for a rigorous proof Rouche's Theorem would be applied. The delays $r = \epsilon r_0$ still cause changes of the order O(1) in the location of the roots of $\text{Det}[sI - M_0(r_0) - M_1(r_0)]$ via the elements of matrices M_0 and M_1.

Remark 3.5: It is not yet known whether vibrational control is applicable to systems with time varying or distributed delay. While it may be possible to extend the averaging theory of Section II to such systems, a difficulty arises in determining a substitution, similar to (3.5), which transforms the original system to another system whose right-hand side has an average which exists uniformly for all time. Unless such a substitution can be found, the stability analysis techniques of this paper are not valid.

Remark 3.6: In the case of vector additive vibrations, $P_2(f(t), x(t)) = L(t)$ and $h(t, c) = u(t) + c$, where $u(t) = \int L(t)\,dt$. Therefore, substitution (3.5) becomes $x(t) = u(t) + y(t)$ and $x(t-r) = u(t-r) + y(t-r)$. Vector additive vibrations are incapable of t-stabilizing a system, since $x^s(t) = u(t) + y_s$, i.e., $x^s(t)$ is always nonconstant and almost periodic.

Example 3.1 (Harvesting of a Single Natural Population): The problem of harvesting renewable resources (game, fish, plants, etc.) is to determine a harvesting strategy which maximizes a sustainable yield and does not cause the population of resources to become extinct.

Here, we discuss the vibrational control of the classical one specie population model (discussed in [8], [13]) with a constant harvest (see [20, page 27])

$$\dot{N}(t) = \alpha N(t)\left[1 - \frac{N(t-r)}{K}\right] - Y. \tag{3.11}$$

Here, $N(t) \in \mathbf{R}$ is the population of a single specie, such as fish in a hatchery, α, K, r, Y are positive constants, where α represents birth rate, K represents the carrying capacity of the environment, r is the positive constant delay taking into account a finite gestation period, time to reach maturity, etc. and Y is the yield which is to be maximized (harvesting rate).

Obviously, if the harvesting yield, Y, is chosen too large when the population of the specie is low, the specie will die out (perhaps as the whale population in the 1970's). As a matter of fact, if Y is chosen sufficiently large, $N \to 0$ in finite time, even when $r = 0$, since $N = 0$ is not an equilibrium. The largest Y which does not cause the population to die out is called the maximum sustainable yield and is denoted as $Y_{\max}$.

Introduce zero mean oscillations into Y so that (3.11) becomes

$$\dot{N}(t) = \alpha N(t)\left[1 - \frac{N(t-r)}{K}\right] - \left(Y + \frac{\beta}{\epsilon}\sin(t/\epsilon)\right) \tag{3.12}$$

which simply means that species N is being harvested at a periodic rate instead of at a constant rate. The goal of vibrationally stabilizing (3.11) is to choose β and ϵ in such a manner that $Y_{\max}$ can be increased so that the population of the specie does not die out. Parameter Y was chosen to vibrate because it is the most easily accessible parameter. Theoretically, the birth rate or the carrying capacity of the environment could be vibrated, only such vibrations would be extremely difficult to implement.

In this case, substitution (3.5) corresponding to substitution (A.35) in Appendix I becomes

$$N(t) = y(t) + \beta\cos(t/\epsilon), \qquad N(t-r) = y(t-r) + \beta\cos\left(\frac{t-r}{\epsilon}\right) \tag{3.13}$$

which transforms (3.12) into

$$\dot{y}(t) = \alpha[y(t) + \beta\cos(t/\epsilon)] - \frac{\alpha}{K}[y(t) + \beta\cos(t/\epsilon)] \cdot\left[y(t-r) + \beta\cos\left(\frac{t-r}{\epsilon}\right)\right] - Y. \tag{3.14}$$

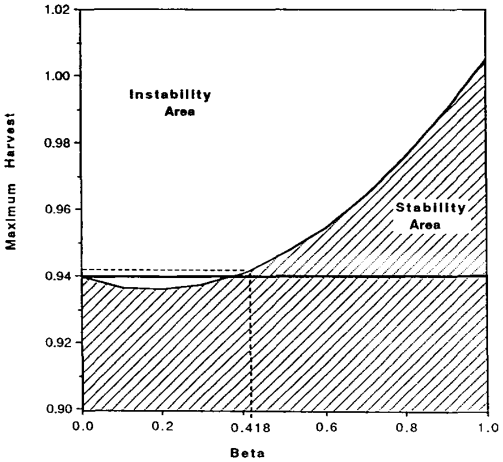

Fig. 1. Stability–instability boundary for system (3.11) with vibrations (3.12), $r = 1.4$ and $\epsilon = 0.445$.

The corresponding average of (3.14) is given by

$$\dot{z}(t) = \alpha z(t)\left[1 - \frac{z(t-r)}{K}\right] - \left(Y + \frac{\alpha\beta^2}{2K}\cos(r/\epsilon)\right). \tag{3.15}$$

Suppose the delay, r, is equal to zero. Then it is easily seen by (3.15) that the maximum sustainable yield, $Y_{\max}$, actually decreases as β increases, since $\cos(r/\epsilon) = 1$ when $r = 0$. However, if r and ϵ are chosen in such a manner that $\cos(r/\epsilon) < 0$, then it would appear that $Y_{\max}$ actually increases. This is verified by computer simulation.

For the purposes of simulation, in (3.12) let $\alpha = 1$, $K = 4$, $r = 1.4$, $\epsilon = 0.445$, and $N(t) = 4$ for $t \in [-1.4, 0]$. Since $\cos(r/\epsilon) \approx -1$, the theory suggests that increasing the amplitude, β, should increase the average maximum yield, $Y_{\max}$. Fig. 1 plots $Y_{\max}$ versus β. For $\beta = 0$, simulations show $Y_{\max} = 0.939$, which is clearly marked by the bold horizontal line. Fig. 1 also shows that increasing β, increases maximum yield. For $\beta = 0.418$, $Y_{\max}$ is calculated to be 0.942, an increase in yield of 0.3%. This value is important since at $\beta = 0.418$, $(\beta/\epsilon) \approx 0.942 = Y_{\max}$, which indicates the maximum β permitted if there is a constraint on Y being positive, i.e., never adding population to the specie. For $\beta > 0.418$, this simply means that species are at times being added to the population, instead of being removed. This may represent, for instance, periodically moving fish from one hatchery to another.

When there are no constraints on β, Fig. 1 shows substantial gains of the maximum yield for $\beta > 0.418$. The shaded region in Fig. 1 gives the simulated stability area of (3.12), i.e., $Y \le Y_{\max}$. It is seen that for $\beta = 1.0$, $Y_{\max} = 1.01$, which is a far more significant gain in yield.

Remark 3.7: For linear multiplicative vibrations, function $P_2(\cdot, \cdot)$ takes the form $P_2(f(t), x(t)) = D(t)x(t)$. In this case, function h, defined in (3.1), is given by $h(t, c) = \Phi(t)c$, where $\Phi(t)$ is any fundamental matrix solution to $\dot{x}(t) = D(t)x(t)$. Therefore, substitution (3.5) is $x(t) = \Phi(t)y(t)$ and $x(t-r) = \Phi(t-r)y(t-r)$. Linear multiplicative vibrations *are* capable of t-stabilizability. This is shown in the following section.

C. Linear Multiplicative Vibrations: $P_2(f(t), x(t)) = D(t)x(t)$

Theorem 3.2: Assume that:

1) $\tilde{P}_1(0, 0) = 0$ and $\tilde{P}_2(\lambda, 0) = 0$ in (3.1);
2) there exists a sufficiently large set $\Omega \subset \mathbb{R}^n (0 \in \Omega)$ such that $\tilde{P}_1(\beta, \eta)$ and $\tilde{P}_2(\lambda, \beta)$ are continuously differentiable for all $(\beta, \eta) \in \Omega \times \Omega$;
3) there exists an APAZ matrix $D(t)$ such that a fundamental solution matrix $\Phi(t)$ of $\dot{x}(t) = D(t)x(t)$ is almost periodic;
4) there exists a constant, $\gamma > 0$, such that (3.7), with

$$P_0(z(t), z(t-r), \gamma) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi^{-1}(\tau)P_1(\Phi(\tau)y(t), \Phi(\tau-\gamma)y(t-r))\,d\tau$$

has linearization about $z_s = 0$ given by (3.8) with the property that $\text{Det}[sI - M_0(\gamma) - M_1(\gamma)e^{-rs}] = 0$ has all solutions with $Re(s) < 0$.

Then

i) there exists an $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, the trivial solution of the equation

$$\dot{y}(t) = \Phi^{-1}\left(\frac{t}{\epsilon}\right)P_1\left(\Phi\left(\frac{t}{\epsilon}\right)y(t),\ \Phi\left(\frac{t}{\epsilon} - \gamma\right)y(t-r)\right) \tag{3.16}$$

is asymptotically stable;

ii) the trivial solution of (3.1) is t-stabilizable by linear multiplicative vibrations $1/\epsilon_1 D(t/\epsilon_1)x(t)$, $\epsilon_1 \triangleq r/\gamma$ if $\epsilon_1 \le \epsilon_0$.

Remark 3.8: An example using linear multiplicative vibrations will be discussed in Section V.

IV. Transient Behavior

A. Problem Statement

Section III describes the method of stabilizing equilibria of delay differential equations by introducing vibrations into parameters. Vibrational stabilizability describes changes in local attractivity in the vicinity of equilibria, i.e., local behavior as $t \to \infty$. For control purposes, it is also of interest to analyze the nonlocal system behavior at every time moment from $t = 0$, i.e., the transient behavior of the system. Analysis of such trajectories is a difficult task since vibrationally controlled systems are composed of a fast oscillatory trajectory superimposed on a slow trajectory. A comparison of the slow trajectory of the oscillatory system with the trajectory of the corresponding system without vibrations reveals the qualitative changes in the system behavior induced by vibrations.

Consider (3.1) with continuous initial function, $\varphi(t) = \mathbb{R}^n$;

$$\dot{x}(t) = \tilde{P}_1(x(t), x(t-r)) + \tilde{P}_2(\lambda, x(t)); \quad x(t) = \varphi(t) \text{ for } t \in [-r, 0]. \tag{4.1}$$

Introducing into (4.1) parametric vibrations according to the law $\lambda(t) = \lambda_0 + f(t)$, for $t \geq 0$ and using the notation of Section III yields

$$\dot{x}(t) = P_1(x(t), x(t-r)) + P_2(f(t), x(t)); \quad x(t) = \varphi(t) \text{ for } t \in [-r, 0] \tag{4.2}$$

i.e., (3.3) with initial function $\varphi(t)$.

Let $h(t, t_0; x(t_0))$ denote the unique solution to the ODE

$$\dot{x}(t) = P_2(f(t), x(t)); \qquad x(t) = x(t_0) \text{ at } t = t_0. \tag{4.3}$$

Assume h is almost periodic in t and define a constant m such that for any c_1 and c_2

$$\|h(t, t_1; c_1) - h(t, t_2; c_2)\| \leq m\|c_1 - c_2\|.$$

Introduce into (4.2) the substitution

$$x(t) = h(t, -r; y(t)), \qquad x(t-r) = h(t-r, -r; y(t-r)) \tag{4.4}$$

for $t \geq 0$. This is a well-defined substitution since h is continuous for all $t \in [-r, \infty)$. Then (4.2) becomes

$$\begin{aligned}\dot{y}(t) &= \left[\frac{\partial h(t, -r; y(t))}{\partial y}\right]^{-1} \\ &\quad \cdot P_1(h(t, -r; y(t)), h(t-r, -r; y(t-r))) \\ &\equiv Y_r(t, r, y(t), y(t-r))\end{aligned}$$

$$y(t) = \psi(t) \text{ for } t \in [-r, 0] \tag{4.5}$$

where $\psi(t)$ is given by the relationship $\varphi(t) = h(t, -r; \psi(t))$ for $t \in [-r, 0]$.

Since P_2 is continuous over all time and has continuous partial derivatives, it follows that, for any given pair $(t_0, x(t_0))$, the function $h(t, t_0; x(t_0))$ is uniquely defined for all $t \in (-\infty, \infty)$. Therefore, if constants t_0, t_1 and function $\varphi(t_1)$ are known quantities, the relationship $\varphi(t_1) = h(t_1, t_0; \psi(t_1))$ will uniquely define $\psi(t_1)$. This argument holds true for every $t_1 \in [-r, 0]$, and therefore the relationship $\varphi(t) = h(t, t_0; \psi(t))$ will always uniquely define the initial function $\psi(t)$ in (4.5). The continuity of $\psi(t)$ follows since it is known that $\varphi(t)$ is a continuous function and that $h(t, \cdot; \cdot)$ has continuous dependence on its initial conditions.

Starting time t_0 can be chosen arbitrarily since for any constant m, $\psi(t)$ will be uniquely defined by the relation $\varphi(t) = h(t, m; \psi(t))$ for $t \in [-r, 0]$. In this work, initial time is chosen to be $t_0 = -r$ for intuitive reasons (this notation clearly indicates that (4.3) needs to have a unique solution for $t \geq -r$, although by assumption it has a unique solution for any $t \in (-\infty, \infty)$).

Now consider the average of (4.5)

$$\dot{z}(t) = P_{0_r}(z(t), z(t-r), r); \quad z(t) = \psi(t) \text{ for } t \in [-r, 0],$$

$$P_{0_r}(y(t), y(t-r), r) \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T Y_r(\tau, r, y(t), y(t-r))\, d\tau. \tag{4.6}$$

Let $y(t; \psi)$ and $z(t; \psi)$ denote the solution of (4.5) and (4.6), respectively. Introduce

$$\overline{x}(y(t; \psi)) \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T h(\tau, -r; y(t; \psi))\, d\tau \tag{4.7}$$

which represents the averaged trajectory of vibrationally controlled system (4.2). If $y(t; \psi)$ and $z(t; \psi)$ are close to each other, $\overline{x}(y(t; \psi))$ can be approximated by

$$\overline{x}(z(t; \psi)) \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T h(\tau, -r; z(t; \psi))\, d\tau \tag{4.8}$$

where $\overline{x}(z(t; \psi))$ will represent the approximate averaged transient behavior of the vibrationally controlled system (4.2). Comparison of $x(t; \varphi)$ of (4.1) with $\overline{x}(z(t; \psi))$ reveals the change of global transient behavior of (4.1) due to parametric oscillations.

Definition 4.1: For any given fixed $\delta > 0$ and any $L > 0$, an APAZ vector $f(t)$ is said to introduce a δ-global dynamic equivalence between systems (4.2) and (4.6) if

$$\|\overline{x}(y(t; \psi)) - \overline{x}(z(t; \psi))\| < \delta, \ t \in [0, L], \quad \forall \psi(t) \in \Omega \subset \mathbf{R}^n$$

where Ω is an open subset of $\mathbf{R}^n$.

B. General Case

Let γ be a positive constant and let ϵ be a positive real parameter. Consider the delay differential equations

$$\dot{y}(t) = Y_r\left(\frac{t}{\epsilon}, \gamma, y(t), y(t-r)\right); \quad y(t) = \psi(t) \text{ for } t \in [-r, 0] \tag{4.9}$$

$$\dot{z}(t) = P_{0_r}(z(t), z(t-r), \gamma); \ z(t) = \psi(t) \text{ for } t \in [-r, 0] \tag{4.10}$$

where Y_r and P_{0_r} are defined by (4.5) and (4.6), respectively, and $\psi(t) \in \Omega$, $\Omega \subset \mathbf{R}^n$, is continuous for $t \in [-r, 0]$. Denote the solution of (4.9) as $\tilde{y}_\gamma(t; \psi)$ and the solution of (4.10) as $\tilde{z}_\gamma(t; \psi)$.

Lemma 4.1: Let Ω be an open subset of $\mathbf{R}^n$, and let γ be a fixed positive constant. Assume that

1) $h(t, t_0; x(t_0))$, as given in (4.3), is almost periodic in t;
2) $Y_r(s, \gamma, y(t), y(t-r))$, defined in (4.5), is continuous in all arguments and there exists a positive constant $k > 0$ such that for $(s, y_1, y_2) \in R \times \Omega \times \Omega$

$$\|Y_r(s, \gamma, y_1, y_2) - Y_r(s, \gamma, y_1', y_2')\| \leq k \max_{i=1,2} \|y_i - y_i'\|, \ \forall s \geq 0;$$

3) uniformly with respect to (t, y_1, y_2) in compact sets of $R \times \Omega \times \Omega$ there exists a limit

$$P_{0_r}(y_1, y_2, \gamma) \triangleq \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} Y_r(t, \gamma, y_1, y_2)\, dt.$$

Then for any $\sigma > 0$ and $\rho > 0$, however small, and any $L > 0$, however large, there exists an $\epsilon_\sigma > 0$ such that, for $0 < \epsilon \leq \epsilon_\sigma$,

1) $\|\tilde{y}_\gamma(t;\psi) - \tilde{z}_\gamma(t;\psi)\| < \sigma$ for $t \in [0, L]$, provided $\tilde{z}_\gamma(t;\psi)$ together with its ρ vicinity belongs to Ω for all $t \in [0, L]$;
2) whenever the averaged equation (4.10) has an asymptotically stable equilibrium point $z_s \in \Omega$, then $\|\tilde{y}_\gamma(t;\psi) - \tilde{z}_\gamma(t;\psi)\| < \sigma$ for $t \geq 0$, provided that $\tilde{z}_\gamma(t;\psi)$ together with its ρ vicinity belongs to Ω for all $t \geq 0$ and that $\lim_{t\to\infty} \tilde{z}_\gamma(t;\psi) = z_s$.

Theorem 4.1: Let assumptions 1), 2), and 3) of Lemma 4.1 hold. Define $\epsilon_1 \triangleq r/\gamma$, where γ and r are previously defined, and assume $0 < \epsilon_1 \leq \epsilon_\sigma$, where ϵ_σ is defined and guaranteed to exist in Lemma 4.1. Then

i) vibrations $f(t) = 1/\epsilon_1 g(t/\epsilon_1)$ induce a δ-global dynamic equivalence between (4.2) and (4.6) for $t \in [0, L]$, $L > 0$, provided that $z(t;\psi)$, the solution of (4.6), and its ρ vicinity belongs to $\Omega \subset \mathbb{R}^n$ for $t \in [-r, L]$;
ii) whenever (4.10) has an asymptotically stable equilibrium point, $z_s \in \Omega_1$, where $\Omega_1 \subset \mathbb{R}^n$ is the domain of attraction of z_s, vibrations $f(t) = 1/\epsilon_1 g(t/\epsilon_1)$ induce a δ-global dynamic equivalence between (4.2) and (4.6) for $t \geq 0$, provided that $z(t;\psi)$ and its ρ vicinity belongs to Ω_1 for $t \geq -r$.

Remark 4.1: Theorem 4.1 reduces the problem of δ-global dynamic equivalence of (4.2) and (4.6) to the search for an APAZ vector $f(t)$ with sufficiently small period that induces a closeness of trajectories of (4.9) and (4.10). If $r/\gamma < \epsilon_\sigma$, the search is complete; ϵ_σ is best determined by numerical simulation.

Remark 4.2: Since $\overline{x}(z(t;\psi))$ is easy to compute, Theorem 4.1 offers a constructive method for analysis of the transient behavior and the oscillation induced transitions of vibrationally controlled system (4.2).

Remark 4.3: In the case of vector additive vibrations, $h(t, -r; x(-r)) = u(t) - u(-r) + x(t)$, where $u(t)$ is defined by Remark 3.6. Substitution (4.4) becomes $x(t) = u(t) - u(-r) + y(t)$ and $x(t-r) = u(t-r) - u(-r) + y(t-r)$. Defining $w(t) = y(t) - u(-r)$ and $w(t-r) = y(t-r) - u(-r)$, the same transformation as described in Remark 3.6 is obtained. Clearly, when $u(t)$ has zero average, for vector additive vibration, $\overline{x(w(t;\psi))} = \overline{x}(z(t;\psi)) = z(t;\psi)$.

Example 4.1: Suppose in Example 3.1, $\alpha = 1$, $K = 4$, $r = 1.4$, $Y = 0.92$, $\beta = 0.4$, and $\epsilon = 0.445$. Let $N(t, \epsilon; \theta)$ denote the solution of (3.11) where $N(t) = \theta(t) = 4$ for $t \in [-1.4, 0]$. Let $z(t;\psi)$ denote the solution of (3.15) with $z(t) = \psi(t) = 4 - \beta\cos(t/\epsilon)$ for $t \in [-1.4, 0]$. Fig. 2 plots $N(t, \epsilon; \theta)$, the oscillating curve, and $\overline{N}(z(t;\psi)) = z(t;\psi)$ versus time for $t \geq 0$. It is seen that $\epsilon = 0.445$ induces a δ-global dynamic equivalence between (3.12) and (3.15) for $\delta \geq 0.5$.

Remark 4.4: In the case of linear multiplicative vibrations, $h(t, -r; x(-r)) = T(t, -r)x(-r)$, where $T(t, -r)$ is the state transition matrix given by $T(t, -r) = \Phi(t)\Phi^{-1}(-r)$, and $\Phi(t)$ is the fundamental matrix solution described by Theorem 3.2. Since there will always exist a constant $n \times n$ matrix, C, such that $T(t, -r)y(t) = \Phi(t)Cy(t) = \Phi(t)w(t)$, where $w(t) = Cy(t)$, substitution (4.4) can always be written as $x(t) = \Phi(t)w(t)$, which is the same transformation described in Remark 3.7 and Theorem 3.2. For linear multiplicative vibrations, if substitution $x(t) = \Phi(t)w(t)$ is used, then

$$\overline{x}(z(t;\psi)) = \overline{\Phi}(t)z(t;\psi) \text{ where } \overline{\Phi}(t) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \Phi(t)\,dt.$$

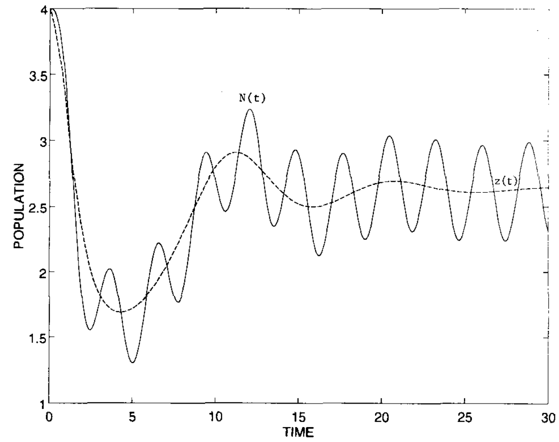

Fig. 2. Population $N(t)$ versus time of a single species harvesting equation with vibrations (3.12), $r = 1.4$ and $\epsilon = 0.445$, along with its approximate average $z(t)$ (3.15) versus time.

V. Vibrational Control of a CSTR with Delay in the Recycle Stream

This chapter discusses the vibrational control of a first order, irreversible, exothermic chemical reaction in a CSTR with delayed recycle stream. Vibrational control of such reactions when there is no recycle has been thoroughly investigated both theoretically [2], [21] and experimentally [5]. It is important to consider the effects of recycle since it often times has noticeable influence on the CSTR dynamics.

Reactor recycle not only increases the overall conversion, but also reduces the cost of a reaction and is, therefore, very popular in industry. To recycle, the input specie must be separated from the yield, then travel through pipes after separation. This "total time" of recycle introduces delays in the states and thus complicates the dynamics.

The benefits of vibrational control of exothermic reactions in a CSTR are given in [5], [21]. Frequently, feedback in a CSTR is expensive [5] or slow [21]. Hence, "nontraditional" control techniques, such as vibrational control, are often used.

The purpose of this section is to give conditions under which a first order, irreversible reaction in a CSTR can be partially vibrationally stabilized and to show the inducement of a δ-global dynamic equivalence between the vibrationally controlled system and its corresponding average.

A. Model

Consider the first order, irreversible, exothermic reaction $\mathbf{A} \to \mathbf{B}$, carried out in a well mixed CSTR. Suppose, at the input, that the fresh feed of pure $\mathbf{A}$ is to be mixed with a recycle stream of unreacted $\mathbf{A}$ with recycle flow rate $(1-\lambda)q$. Let t be the constant of time the output exits the CSTR. Then, according to [7], the material and energy balance equations are

$$V\frac{dA}{dt} = \lambda q A_0 + q(1-\lambda)A(t-\alpha) - qA(t) - VK_0 \exp\left\{\frac{-E}{RT(t)}\right\}A(t) \quad (5.1)$$

$$VC\rho\frac{dT}{dt} = qC\rho[\lambda T_0 + (1-\lambda)T(t-\alpha) - T(t)] + V(-\Delta H)K_0 \exp\left\{\frac{-E}{RT(t)}\right\}A(t) - U(T(t) - T_w) \quad (5.2)$$

where $A(t) = \varphi_1(t)$ and $T(t) = \varphi_2(t)$ for $t \in [-r, 0]$, $A(t)$ is the concentration of chemical **A**, $T(t)$ is temperature, and the remaining constants: α, λ, q, A_0, V, K_0, $-E/R$, C, ρ, $(-\Delta H)$, U, and T_w are all positive and defined in Appendix II. The constant λ varies from zero to one, with zero corresponding to total recycle and one corresponding to no recycle.

Typically, (5.1) is reduced to dimensionless form using the notation

$$x_1(t) = \frac{A_0 - A(t)}{A_0}, \qquad x_2(t) = \frac{T(t) - T_0}{T_0}\left(\frac{E}{RT_0}\right),$$
$$\theta_1(t) = \frac{A_0 - \varphi_1(t)}{A_0},$$
$$\theta_2(t) = \frac{\varphi_2(t) - T_0}{T_0}\left(\frac{E}{RT_0}\right), \quad t_{\text{new}} = \frac{t}{\tau}, \quad \tau = \frac{V}{q\lambda},$$
$$D_a = K_0\tau \exp\{-\gamma_0\}, \qquad B = \frac{(-\Delta H)A_0 E}{C\rho T_0^2 R}, \qquad r = \frac{\alpha}{\tau},$$
$$\beta = \frac{U\tau}{VC\rho}, \qquad \gamma_0 = \frac{E}{RT_0}. \quad (5.3)$$

Without loss of generality, assume that $T_0 = T_w$. Then (5.1) and (5.2) in dimensionless variables become

$$\dot{x}_1(t) = \frac{-1}{\lambda}x_1(t) + \left(\frac{1}{\lambda} - 1\right)x_1(t-r) + D_a \exp\left\{\frac{x_2(t)}{1 + x_2(t)/\gamma_0}\right\}(1 - x_1(t)) \quad (5.4)$$

$$\dot{x}_2(t) = -\left(\frac{1}{\lambda} + \beta\right)x_2(t) + \left(\frac{1}{\lambda} - 1\right)x_2(t-r) + BD_a \exp\left\{\frac{x_2(t)}{1 + x_2(t)/\gamma_0}\right\}(1 - x_1(t)) \quad (5.5)$$

where $x_i(t) = \theta_i(t)$ for $t \in [-r, 0]$, $i = 1, 2$. The state $x_1(t)$ corresponds to the conversion rate of the reaction, $0 \le x_1(t) \le 1$, and $x_2(t)$ is the dimensionless temperature. Clearly, we must restrict $\lambda \neq 0$, or the right-hand side of both (5.4) and (5.5) become invalid. Constants B, β, D_a, γ_0, and r are all positive.

To further simplify calculations, assume that γ_0 is large ($\gamma_0 \to \infty$). Computer simulations show that this is an accurate approximation when γ_0 is about 20 times greater than $x_2(t)$,

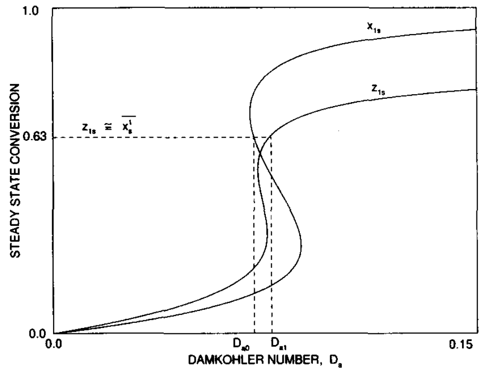

Fig. 3. Steady-state conversion of an exothermic reaction versus Damköhler number.

which is often the case. The case when γ_0 is finite poses no additional difficulties, but makes calculations tedious. The techniques are the same, as [5] shows.

Under these assumptions, (5.4) and (5.5) become

$$\dot{x}_1(t) = \frac{-1}{\lambda}x_1(t) + \left(\frac{1}{\lambda} - 1\right)x_1(t-r) + D_a \exp\{x_2(t)\}(1 - x_1(t)) \quad (5.6)$$

$$\dot{x}_2(t) = -\left(\frac{1}{\lambda} + \beta\right)x_2(t) + \left(\frac{1}{\lambda} - 1\right)x_2(t-r) + BD_a \exp\{x_2(t)\}(1 - x_1(t)). \quad (5.7)$$

Frequently, it is helpful to plot the locus of reactor steady states, x_{1s}, versus the Damköhler number D_a which is shown by the taller curve in Fig. 3. For fixed D_a, Fig. 3 shows how it is possible to have three steady states. In this case, [7] has shown that the upper and lower values of x_{1s} correspond to the steady states at upper and lower temperatures and are locally asymptotically stable, while the middle temperature gives an unstable steady state.

It is of interest to attempt to vibrationally control the middle steady states of x_{1s}. Often, the upper steady states, which have the best conversion rate for the reaction, run at a temperature too high for a CSTR to operate. If a middle steady state could be stabilized, it may produce the best conversion rate for the reaction under a temperature constraint.

B. Vibrational Control of a CSTR

Introduce vibrations into (5.1) and (5.2) so that the input flow rate and output flow rate oscillate identically, i.e., consider (5.1) and (5.2) with vibrations

$$V\frac{dA}{dt} = \lambda q A_0\left[1 + \frac{c}{\epsilon}\sin\left(\frac{tq\lambda}{\epsilon V}\right)\right] + q(1-\lambda)A(t-\alpha) - q\left[1 + \frac{c\lambda}{\epsilon}\sin\left(\frac{tq\lambda}{\epsilon V}\right)\right]A(t) - VK_0\exp\left\{\frac{-E}{RT(t)}\right\}A(t) \quad (5.8)$$

$$V C\rho\frac{dT}{dt} = qC\rho\lambda T_0\left[1+\frac{c}{\epsilon}\sin\left(\frac{tq\lambda}{\epsilon V}\right)\right] + qC\rho(1-\lambda)T(t-\alpha) - qC\rho \cdot\left[1+\frac{c\lambda}{\epsilon}\sin\left(\frac{tq\lambda}{\epsilon V}\right)\right]T(t) + V(-\Delta H)K_0 \cdot \exp\left\{\frac{-E}{RT(t)}\right\}A(t) - U(T(t)-T_w). \quad (5.9)$$

The vibrating of flow rates is commonly practiced in industry, so the oscillations are technologically feasible. Both the input flow rate and the output flow rate are identically vibrated so that the volume of the CSTR remains constant, which is a requirement of using model (5.1) and (5.2). (As in Example 3.1, it may be possible to insert vibrations into parameters other than the flow rates, such as A_0, T_0, K_0, λ, etc.; these methods are not examined, however, since the vibrating of flow rates is the most technologically feasible and since, as will be shown, the vibrating of flow rates successfully stabilizes middle steady states.)

In dimensionless variables, (5.6) and (5.7) with oscillations as in (5.8) and (5.9) are written as

$$\dot{x}_1(t) = -\left[\frac{1}{\lambda}+\frac{c}{\epsilon}\sin(t/\epsilon)\right]x_1(t) + \left(\frac{1}{\lambda}-1\right)x_1(t-r) + D_a\exp\{x_2(t)\}(1-x_1(t)) \quad (5.10)$$

$$\dot{x}_2(t) = -\left(\frac{1}{\lambda}+\beta+\frac{c}{\epsilon}\sin(t/\epsilon)\right)x_2(t) + \left(\frac{1}{\lambda}-1\right)x_2(t-r) + BD_a\exp\{x_2(t)\}(1-x_1(t)) \quad (5.11)$$

with $x_i(t) = \theta_i(t)$ for $t \in [-r, 0]$, $i = 1, 2$.

Equation (3.4) corresponding to this case is

$$\dot{x}_i(t) = -c\sin(t)x_i(t) \quad (5.12)$$

and, therefore, the substitution given in Appendix I by (A.42) is

$$x_i(t) = \exp\{c\cos(t/\epsilon)\}\exp\{-c\cos(r/\epsilon)\}w_i(t). \quad (5.13)$$

As Remark 4.4 suggests, define $y_i(t) \equiv \exp\{-c\cos(r/\epsilon)\}$ $w_i(t)$ and substitute (5.13) into (5.10) and (5.11) to obtain

$$\dot{y}_1(t) = \frac{-1}{\lambda}y_1(t) + \left(\frac{1}{\lambda}-1\right)\exp\left\{c\cos\left(\frac{t-r}{\epsilon}\right) - c\cos\left(\frac{t}{\epsilon}\right)\right\}y_1(t-r) + D_a\exp\{\exp\{c\cos(t/\epsilon)\}y_2(t)\} \cdot(\exp\{-c\cos(t/\epsilon)\} - y_1(t)) \quad (5.14)$$

$$\dot{y}_2(t) = \left(\frac{1}{\lambda}+\beta\right)y_2(t) + \left(\frac{1}{\lambda}-1\right)\exp\left\{c\cos\left(\frac{t-r}{\epsilon}\right) - c\cos\left(\frac{t}{\epsilon}\right)\right\}y_2(t-r) + BD_a\exp\{\exp\{c\cos(t/\epsilon)\}y_2(t)\} \cdot(\exp\{-c\cos(t/\epsilon)\} - y_1(t)) \quad (5.15)$$

with $y_i(t) = \psi_i(t) = \exp\{-c\cos(t/\epsilon)\}\theta_i(t)$, $i = 1, 2$ for $t \in [-r, 0]$. Let $\overline{f(t)}$ denote the average of $f(t)$, i.e.,

$$\overline{f(t)} = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(t)\,dt.$$

Noting that up to $O(c^4)$:

1) $\overline{\exp\{c\cos(\frac{t-r}{\epsilon}) - c\cos(t/\epsilon)\}} = 1 + \frac{c^2}{2}[1-\cos(r/\epsilon)]$,

2) $\overline{\exp\{\exp\{c\cos(\frac{t}{\epsilon})\}y_2(t)\}} = \exp\{y_2(t)\}\left[1+\frac{c^2}{4}(y_2(t)+y_2^2(t))\right]$,

3) $\overline{\exp\{\exp\{c\cos(\frac{t}{\epsilon})\}y_2(t)\}\exp\{-c\cos(t/\epsilon)\}} = \exp\{y_2(t)\}\left[1+\frac{c^2}{4}(1-y_2(t)+y_2^2(t))\right]$, (5.16)

the corresponding average of (5.14) and (5.15) is given as, up to $O(c^4)$

$$\dot{z}_1(t) = \frac{-1}{\lambda}z_1(t) + \left(\frac{1}{\lambda}-1\right) \cdot\left(1+\frac{c^2}{2}[1-\cos(r/\epsilon)]\right)z_1(t-r) + D_a\exp\{z_2(t)\} \cdot\left(1 - z_1(t) + \frac{c^2}{4}[1 - z_2(t) - z_1(t)z_2(t) + z_2^2(t) - z_1(t)z_2^2(t)]\right) \quad (5.17)$$

$$\dot{z}_2(t) = -\left(\frac{1}{\lambda}+\beta\right)z_2(t) + \left(\frac{1}{\lambda}-1\right) \cdot\left(1+\frac{c^2}{2}[1-\cos(r/\epsilon)]\right)z_2(t-r) + BD_a\exp\{z_2(t)\} \cdot\left(1 - z_1(t) + \frac{c^2}{4}[1 - z_2(t) - z_1(t)z_2(t) + z_2^2(t) - z_1(t)z_2^2(t)]\right) \quad (5.18)$$

where $z_i(t) = \psi_i(t)$ for $t \in [-r, 0]$.

By the procedures of Section III, it is seen that for fixed $\epsilon = \epsilon_1$, sufficiently small, (5.10) and (5.11) are partially $v\delta$-stabilizable since steady state x_{1s} is only being stabilized. Steady-state characteristics of z_{1s} are shown in Fig. 3 for $B = 7$, $\beta = 0.5$, $r = 0.9425$, and $c = 0.55$. Fig. 3 shows the partial $v\delta$-stabilization of an unstable equilibrium point $x_s(D_{a0} = 0.0908)$ with respect to its component $x_{1s}(D_{a0} = 0.0908) = 0.63$. Since a conversion of 0.63 corresponds to an upper steady state z_{1s} for the averaged equation (where $z_{1s} \approx \overline{x_s^1}$), as shown in Fig. 3, the conversion rate of 0.63 is an asymptotically stable steady state for $z_{1s}(D_{a1} = 0.10372)$. Fig. 4 verifies partial $v\delta$-stabilization by plotting $x_1(t)$ and $z_1(t)$ versus time for $\epsilon_1 = 0.3$ and $\lambda = 0.8$.

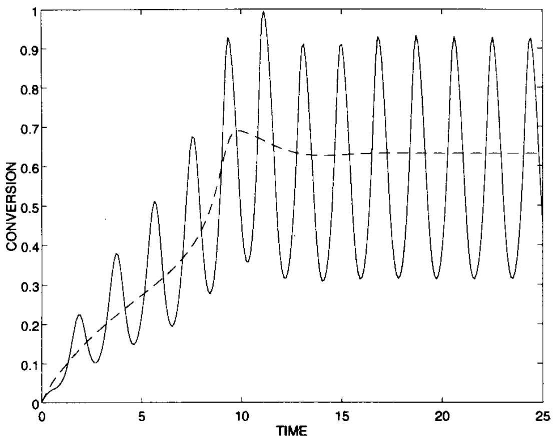

Fig. 4. Conversion versus time for an exothermic reaction with vibrating input flow rate and its approximate average for $r = 0.9425$ and $\epsilon = 0.3$.

VI. Conclusion

This paper shows that vibrational control of nonlinear time lag systems is a feasible alternative to classical control techniques when measurements are either unavailable or expensive. Averaging theory is developed and then applied to vibrational control. Conditions for $v\delta$-stabilizability and t-stabilizability are discussed. In addition, it is shown that for a fixed, sufficiently small period of oscillation there exists a δ-global dynamic equivalence between the vibrating system and its corresponding average.

Two important applications of the theory are discussed. The example of harvesting a single natural population shows that vibrational control improves yield. It is also noted that if the delay in the state is assumed to be zero, vector additive vibrations have the effect of reducing the maximum yield. A second example is presented which shows that vibrating the input flow rate of the exothermic reaction, described in Section V, stabilizes a previously unstable steady state. This steady state is often the preferable point of operation.

Appendix I

Proof of Theorem 2.2: To prove Theorem 2.2, we must use the following two lemmas.

Lemma A.1: Assume γ_1 and γ_2 are constants. If

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\lambda, \gamma_1, \gamma_2)\,d\lambda \triangleq f_0(\gamma_1, \gamma_2) \quad \text{(A.1)}$$

then for any piecewise constant function $\tilde{x}(t)$, any constant $L_1 > 0$, and any constant $\beta > 0$, there exists an $\epsilon_1 = \epsilon_1(\beta, L_1)$ such that for $0 < \epsilon \le \epsilon_1$

$$\left\|\int_0^t f\left(\frac{\tau}{\epsilon}, \tilde{x}(\tau), \tilde{x}(\tau - r)\right) d\tau - \int_0^t f_0(\tilde{x}(\tau), \tilde{x}(\tau - r))\,d\tau\right\| \le \beta \quad \text{(A.2)}$$

for any $t \in [0, L_1]$.

Proof of Lemma A.1: Immediate from Lemma 1, page 461 of [22], upon noting that f is assumed continuous in its first argument. Q.E.D.

Lemma A.2: Suppose conditions 2) and 3) of Theorem 2.2 are true. Then for every $(t, z_1, z_2) \in \mathbf{R} \times \Omega \times \Omega$

$$\|f_0(z_1, z_2) - f_0(z_1', z_2')\| \le k \max_{i=1,2}\|z_i - z_i'\| \quad \text{(A.3)}$$

where $f_0(z_1, z_2)$ is defined in (2.2) and k is the same Lipschitz constant as defined in (2.3).

Proof of Lemma A.2: Using (2.2), (2.3), and basic inequalities

$$\begin{aligned}
&\|f_0(z_1, z_2) - f_0(z_1', z_2')\| \\
&= \left\|\lim_{T\to\infty}\frac{1}{T}\int_0^T [f(\lambda, z_1, z_2) - f(\lambda, z_1', z_2')\,d\lambda]\right\| \\
&\le \lim_{T\to\infty}\frac{1}{T}\int_0^T \|f(\lambda, z_1, z_2) - f(\lambda, z_1', z_2')\|\,d\lambda \\
&\le \lim_{T\to\infty}\frac{1}{T}\int_0^T k\max_{i=1,2}\|z_i - z_i'\|d\lambda = k\max_{i=1,2}\|z_i - z_i'\|.
\end{aligned}$$

Q.E.D.

We are now ready to prove Theorem 2.2:

The solutions to (2.1) and (2.2) are given respectively as

$$x(t, \epsilon; \varphi) = \varphi(0) + \int_0^t f\left(\frac{\tau}{\epsilon}, x(\tau, \epsilon; \varphi), x(\tau - r, \epsilon; \varphi)\right) d\tau \quad \text{(A.4)}$$

$$y(t; \psi) = \psi(0) + \int_0^t f_0(y(\tau; \psi), y(\tau - r; \psi))\,d\tau. \quad \text{(A.5)}$$

Construct a piecewise constant function in Ω, $\tilde{x}(t)$, so that

$$\|x(t, \epsilon; \varphi) - \tilde{x}(t)\| \le \frac{\gamma}{3kL};\ t \in [-r, L]. \quad \text{(A.6)}$$

Consider now the following equality which is always true

$$\begin{aligned}
&\int_0^t \left[f\left(\frac{\tau}{\epsilon}, x(\tau, \epsilon; \varphi), x(\tau - r, \epsilon; \varphi)\right) - f_0(x(\tau, \epsilon; \varphi), x(\tau - r, \epsilon; \varphi))\right] d\tau \\
&= \int_0^t \left[f\left(\frac{\tau}{\epsilon}, x(\tau, \epsilon; \varphi), x(\tau - r, \epsilon; \varphi)\right) - f\left(\frac{\tau}{\epsilon}, \tilde{x}(\tau), \tilde{x}(\tau - r)\right)\right] d\tau \\
&+ \int_0^t \left[f\left(\frac{\tau}{\epsilon}, \tilde{x}(\tau), \tilde{x}(\tau - r)\right) - f_0(\tilde{x}(\tau), \tilde{x}(\tau - r))\right] d\tau \\
&+ \int_0^t [f_0(\tilde{x}(\tau), \tilde{x}(\tau - r)) - f_0(x(\tau, \epsilon; \varphi), x(\tau - r, \epsilon; \varphi))]\,d\tau.
\end{aligned} \quad \text{(A.7)}$$

By (2.3) and (A.6), for any $t \in [-r, L]$

$$\left\|f\left(\frac{t}{\epsilon}, x(t, \epsilon; \varphi), x(t - r, \epsilon; \varphi)\right) - f\left(\frac{t}{\epsilon}, \tilde{x}(t), \tilde{x}(t - r)\right)\right\| \le \frac{\gamma}{3L}. \quad \text{(A.8)}$$

Likewise, by (A.3) of Lemma A.2 and by (A.6), for any $t \geq 0$

$$\|f_0(\tilde{x}(t), \tilde{x}(t-r)) - f_0(x(t, \epsilon; \varphi), x(t-r, \epsilon; \varphi))\| \leq \frac{\gamma}{3L}. \quad \text{(A.9)}$$

Lemma A.1 guarantees that for any $t \in [0, L]$, there exists an $\epsilon_0 = \epsilon_0(\gamma/3, L)$ such that for $0 < \epsilon \leq \epsilon_0$

$$\left\|\int_0^t f\left(\frac{t}{\epsilon}, \tilde{x}(\tau), \tilde{x}(\tau - r)\right) - f_0(\tilde{x}(\tau), \tilde{x}(\tau - r))\, d\tau\right\| \leq \frac{\gamma}{3}. \quad \text{(A.10)}$$

Hence, for $0 < \epsilon \leq \epsilon_0$ and $t \in [0, L]$

$$\begin{aligned}&\left\|\int_0^t \left[f\left(\frac{\tau}{\epsilon}, x(\tau, \epsilon; \varphi), x(\tau - r, \epsilon; \varphi)\right)\right.\right.\\ &\left.\left. - f_0(x(\tau, \epsilon; \varphi), x(\tau - r, \epsilon; \varphi))\right] d\tau\right\|\\ &\leq \left(\frac{\gamma}{3L}\right)L + \left(\frac{\gamma}{3L}\right)L + \frac{\gamma}{3} = \gamma. \end{aligned} \quad \text{(A.11)}$$

By Lemma A.2, for any $t \in [0, L]$ and any $(x, y) \in \Omega \times \Omega$

$$\|f_0(x(t), x(t-r)) - f_0(y(t), y(t-r))\| \leq k \sup_{s \in [-r, L]} \|x(s) - y(s)\|. \quad \text{(A.12)}$$

Using (A.4) and (A.5), the equality

$$\begin{aligned}x(t, \epsilon; \varphi) - y(t; \psi) = \varphi(0) - \psi(0) &+ \int_0^t \left[f\left(\frac{\tau}{\epsilon}, x(\tau, \epsilon; \varphi), x(\tau - r, \epsilon; \varphi)\right)\right.\\ &\left. - f_0(x(\tau, \epsilon; \varphi), x(\tau - r, \epsilon; \varphi))\right] d\tau\\ &+ \int_0^t [f_0(x(\tau, \epsilon; \varphi), x(\tau - r, \epsilon; \varphi))\\ &- f_0(y(\tau; \psi), y(\tau - r; \psi))]\, d\tau \end{aligned} \quad \text{(A.13)}$$

is obtained. Taking the supreme norm of both sides of (A.13), and assuming $0 < \epsilon \leq \epsilon_0$, we have

$$\begin{aligned}\sup_{s \in [-r, L]} \|x(s, \epsilon; \varphi) - y(s; \psi)\| &\leq \sup_{s \in [-r, 0]} \|\varphi(s) - \psi(s)\| + \gamma\\ &+ \int_0^t k \sup_{s \in [-r, L]} \|x(s, \epsilon; \varphi) - y(s; \psi)\|\, ds \end{aligned} \quad \text{(A.14)}$$

where in (A.14) we used the fact that $\|\varphi(0) - \psi(0)\| \leq \sup_{s \in [-r, 0]} \|\varphi(s) - \psi(s)\|$. By Gronwall's inequality (see [13])

$$\sup_{s \in [-r, L]} \|x(s, \epsilon; \varphi) - y(s; \psi)\| \leq \left(\gamma + \sup_{s \in [-r, 0]} \|\varphi(s) - \psi(s)\|\right) e^{kt} \quad \text{(A.15)}$$

which clearly implies (2.4). Q.E.D.

Proof of Theorem 2.3: For a fixed $R > 0$, let $B_R(y_s) = \{y: \|y - y_s\| < R\}$. Under the assumptions of Theorem 2.3, y_s is a locally uniformly asymptotically stable equilibrium point.

By definition of uniform asymptotic stability, there exists an $\eta_0 > 0$ such that if

$$(y(t; \varphi_1), y(t; \varphi_2)) \in B_{\eta_0}(y_s) B_{\eta_0}(y_s) \quad \text{for } t \in [t_1 - r, t_1],\ t_1 > 0$$

then, for any $0 < \eta \leq \eta_0 \leq \rho$, there is a δ, $0 < \delta < \eta$, and a $T_0 \frac{\delta}{2}$ such that whenever

$$\|y(t; \varphi_1) - y(t; \varphi_2)\| < \delta \qquad t \in [t_1 - r, t_1] \quad \text{(A.16)}$$

where $t_1 \geq 0$ and $(\varphi_1, \varphi_2) \in \Omega \times \Omega$ are initial functions, possibly different, then

$$\|y(t; \varphi_1) - y(t; \varphi_2)\| < \frac{\eta}{2}, \qquad t \geq t_1 \quad \text{(A.17)}$$

and further

$$\|y(t; \varphi_1) - y(t; \varphi_2)\| < \frac{\delta}{2}, \quad t \geq t_1 + T_0\left(\frac{\delta}{2}\right). \quad \text{(A.18)}$$

Choose a time, $t = L_1$, large enough so that $y(L; \varphi) \in B_{\delta/2}(y_s)$. Then for $t \geq L_1$, $y(t; \varphi) \in B_{\eta/2}(y_s)$. By Theorem 2.2 and Remark 2.1, there exists an $\epsilon_1 = \epsilon_1(\delta/2, L_1 + r)$ such that if $0 < \epsilon \leq \epsilon_1$

$$\|x(t, \epsilon; \phi) - y(t; \varphi)\| \leq \frac{\delta}{2}, \qquad t \in [0, L_1 + r]. \quad \text{(A.19)}$$

Assume that there exists no range of ϵ, $0 < \epsilon \leq \epsilon_0$ such that

$$\|x(t, \epsilon; \varphi) - y(t; \varphi)\| < \eta, \qquad t \geq 0. \quad \text{(A.20)}$$

Then there exists at least one fixed $\tilde{\epsilon}$, $0 < \tilde{\epsilon} \leq \epsilon_0$, and a nonempty set of positive numbers $\{m_i\}$, where $m_i = m_i(\tilde{\epsilon})$ and $m_i > L_1 + r$, $i = 1, 2, \cdots, N$, such that

$$\|x(m_i, \tilde{\epsilon}; \varphi) - y(m_i; \varphi)\| = \eta. \quad \text{(A.21)}$$

Choose $t_2 = \min\{m_i\} \forall i$. Clearly $t_2 > L_1 + r$.

By (A.19) and (A.21), it is also apparent that there exists another nonempty set $\{\alpha_j\}$, $j = 1, 2, \cdots, n$, such that $L_1 + r < \alpha_j < t_2$ and

$$\|x(\alpha_j, \tilde{\epsilon}; \varphi) - y(\alpha_j; \varphi)\| = \delta. \quad \text{(A.22)}$$

Choose $t_3 = \min\{\alpha_j\}$ and $t_4 = \max\{\alpha_j\} \forall j$ (t_3 may equal t_4).

Since t_3 is the first instant in time when (A.22) is true, by (A.19) and (A.22)

$$\|x(t, \tilde{\epsilon}; \varphi) - y(t; \varphi)\| < \delta, \qquad t \in [L_1 + r, t_3). \quad \text{(A.23)}$$

Likewise, since t_2 is the first instant of time when (A.21) is true and t_4 is the previous value of time closest to t_2 such that (A.22) holds, the following inequality will always be true:

$$0 < \delta \leq \|x(t, \tilde{\epsilon}; \varphi) - y(t; \varphi)\| \leq \eta \qquad t \in [t_4, t_2]. \quad \text{(A.24)}$$

Let $t_5 = L_1 + r$, and redefine in (2.7) initial time $t_5 = t_0$. Let $\tilde{y}(t; t_5, x)$ denote the solution of (2.7) for initial function $\tilde{y}(t; t_5, x) = x(t, \tilde{\epsilon}; \varphi)$ for $t \in [t_5 - r, t_5]$. Using (A.19) and the fact that $y(t;\varphi) \in B_{\delta/2}(y_s)$ for $t \in [t_5 - r, t_5]$, it is seen that for all ϵ, $0 < \epsilon \le \epsilon_1$

$$\begin{aligned}\|\tilde{y}(t; t_5, x) - y_s\| &\le \|\tilde{y}(t; t_5, x) - y(t;\varphi)\| \\ &\quad + \|y(t;\varphi) - y_s\| \qquad t \in [t_5 - r, t_5] \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta \qquad t \in [t_5 - r, t_5]. \end{aligned} \tag{A.25}$$

Therefore, $\tilde{y}(t; t_5, x) \in B_\delta(y_s)$. Since by (A.19), $\|\tilde{y}(t; t_5, x) - y(t;\varphi)\| \le \delta/2 < \delta$ for $t \in [t_5 - r, t_5]$, (A.17) guarantees that

$$\|\tilde{y}(t; t_5, x) - y(t;\varphi)\| < \frac{\eta}{2}, \qquad t \ge t_5 \tag{A.26}$$

and by (A.18), there exists a constant $T_0(\delta/2) > 0$ such that

$$\|\tilde{y}(t; t_5, x) - y(t;\varphi)\| < \frac{\delta}{2}, \qquad t \ge t_5 + T_0\left(\frac{\delta}{2}\right). \tag{A.27}$$

Let $t_6 = t_4 + T_0(\delta/2)$. Then by Theorem 2.2 and Remarks 2.1 and 2.2, there exists an $\epsilon_2 = \epsilon_2(\delta/2, t_6 - t_5)$ such that, for all ϵ, $0 < \epsilon \le \epsilon_2$

$$\|x(t,\epsilon;\varphi) - \tilde{y}(t; t_5, x)\| \le \frac{\delta}{2}, \qquad t \in [t_5, t_6]. \tag{A.28}$$

By using the inequality

$$\begin{aligned}\|x(t,\epsilon;\varphi) - y(t;\varphi)\| &\le \|x(t,\epsilon;\varphi) - \tilde{y}(t;t_5,x)\| \\ &\quad + \|\tilde{y}(t;t_5,x) - y(t;\varphi)\| \end{aligned} \tag{A.29}$$

and by using (A.28) and (A.26), if $0 < \epsilon \le \min[\epsilon_1, \epsilon_2]$,

$$\|x(t,\epsilon;\varphi) - y(t;\varphi)\| < \frac{\delta}{2} + \frac{\eta}{2} < \eta, \; t \in [t_5, t_6]. \tag{A.30}$$

Therefore, $t_6 < t_2$ since $\|x(t_2,\epsilon;\varphi) - y(t_2;\varphi)\| \triangleq \eta$.

If $t = t_6$, by (A.27), (A.28), and (A.29), for any ϵ, $0 < \epsilon \le \min[\epsilon_1, \epsilon_2]$

$$\|x(t_6,\epsilon;\varphi) - y(t_6;\varphi)\| < \frac{\delta}{2} + \frac{\delta}{2} = \delta. \tag{A.31}$$

Choosing $\epsilon_0 = \min[\epsilon_1, \epsilon_2]$, (A.31) contradicts (A.24) since $t_4 < t_6 < t_2$, i.e., $\bar{\epsilon}$ will never exist if $\epsilon_0 = \min[\epsilon_1, \epsilon_2]$. Q.E.D.

Proof of Lemma 3.1: Noting that the average of (3.9) is given by

$$\begin{aligned}\dot{z}(t) &= P_0(z(t), z(t-r), \gamma); \\ P_0(y(t), y(t-r), \gamma) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T Y(\tau, \gamma, y(t), y(t-r))\, d\tau \end{aligned} \tag{A.32}$$

Theorem 2.1 guarantees that a) for any $\delta > 0$ there exists ϵ_δ such that the almost periodic solution $y^s(t)$ of equation

$$\dot{y}(t) = Y\left(\frac{t}{\epsilon}, \gamma, y(t), y(t-r)\right)$$

satisfies

$$\|y^s(t) - z_s\| < \delta, \; \forall \epsilon \in (0, \epsilon_\delta], \; \forall t \ge 0$$

and b) for sufficiently small ϵ, $0 < \epsilon \le \epsilon_0$, hyperbolic stability properties of $y(t) = y^s(t)$ of (A.32) and $z(t) = z_s$ of (3.7) are the same. Since assumption 3) guarantees local stability of (3.7), local stability of (3.9) is also guaranteed. Thus, setting $\sigma = \delta/m$, the proof of Lemma 3.1 follows immediately. Q.E.D.

Proof of Theorem 3.1: Consider (3.3) with vibrations $f(t) = (1/\epsilon)g(t/\epsilon)$, where ϵ is a positive constant

$$\dot{x}(t) = P_1(x(t), x(t-r)) + P_2\left(\frac{1}{\epsilon}g\left(\frac{t}{\epsilon}\right), x(t)\right). \tag{A.33}$$

Since $P_2(\cdot,\cdot)$ has been assumed to be linear in its first argument, (A.33) may be rewritten as

$$\dot{x}(t) = P_1(x(t), x(t-r)) + \frac{1}{\epsilon}P_2\left(g\left(\frac{t}{\epsilon}\right), x(t)\right). \tag{A.34}$$

Introduce into (A.34) substitutions

$$x(t) = h\left(\frac{t}{\epsilon}, y(t)\right), \qquad x(t-r) = h\left(\frac{t-r}{\epsilon}, y(t-r)\right) \tag{A.35}$$

where $h(t, c)$ is the general solution of (3.4), to obtain

$$\begin{aligned}\dot{y}(t) = \left[\frac{\partial h(t/\epsilon, y(t))}{\partial y}\right]^{-1} P_1\bigg(h\left(\frac{t}{\epsilon}, y(t)\right), \\ \cdot\, h\left(\frac{t}{\epsilon} - \frac{r}{\epsilon}, y(t-r)\right)\bigg). \end{aligned} \tag{A.36}$$

Replacing r/ϵ by γ yields system (3.9). Therefore, if vibrations $f(t)$ and a constant γ can be found such that all the assumptions of Theorem 3.1 hold, then by Lemma 3.1, system (3.6) has a solution $y^s(t)$ with the properties satisfying assertions i) and ii) of Lemma 3.1. Further, if $r/\gamma = \epsilon_1 \le \epsilon_0$ and $\epsilon_1 \le \epsilon_\delta$ also, then replacing in (3.6) γ by r/ϵ_1 and setting $\epsilon = \epsilon_1$, yields system (A.36) with $\epsilon = \epsilon_1$ which by Theorem 2.1 has asymptotically stable almost periodic solution $y^s(t) \in \Omega$ with

$$\|y^s(t) - z_s\| < \frac{\delta}{m}, \qquad t \ge 0, \qquad z_s \in \Omega. \tag{A.37}$$

Since $h(t/\epsilon, c)$ is an almost periodic function defined for all $t \ge 0$, there exists a constant $m > 0$ such that

$$\left\|h\left(\frac{t}{\epsilon}, y^s(t)\right) - h\left(\frac{t}{\epsilon}, z_s\right)\right\| \le m\|y^s(t) - z_s\| \tag{A.38}$$

for $y^s(t)$, $z_s \in \Omega$. By taking the time average of (A.38), we obtain

$$\left\|\overline{h\left(\frac{t}{\epsilon}, y^s(t)\right)} - \overline{h\left(\frac{t}{\epsilon}, z_s\right)}\right\| \le m\left\|\overline{y^s(t)} - z_s\right\| < \delta. \tag{A.39}$$

Due to almost periodicity of $h(t/\epsilon, c)$ and $y^s(t)$, and using the fact that $\overline{h(t/\epsilon, z_s)} = x_s$, (A.39) becomes

$$\left\|\overline{x^s(t)} - x_s(\lambda_0)\right\| < \delta \tag{A.40}$$

for $0 \le t < \infty$, which proves i).

Suppose now that $y^s(t) = y_s = z_s =$ constant and $h(t/\epsilon, y_s) = h(t/\epsilon, z_s) = x_s(\lambda_0)$. Then $m = 0$ in (A.38) and the conditions of Definition 3.2 hold and (3.1) is t-stabilizable. Q.E.D.

Proof of Theorem 3.2: The proof follows almost directly from Lemma 3.1 and Theorem 3.1. Substitution (A.35) in the proof of Theorem 3.1 which transforms (A.34) into (A.36) is given by

$$x(t) = \Phi\left(\frac{t}{\epsilon}\right)y(t), \qquad x(t-r) = \Phi\left(\frac{t}{\epsilon} - \frac{r}{\epsilon}\right)y(t-r). \tag{A.41}$$

Now, Lemma 3.1 and Theorem 3.1 can be applied and, noting that $P_0(0, 0, \gamma) = 0$ since $\tilde{P}_1(0, 0) = 0$, the proof is complete. Q.E.D.

Proof of Lemma 4.1: The proof of i) is immediate from Theorem 2.2, and the proof of ii) follows immediately from Theorem 2.3. Q.E.D.

Proof of Theorem 4.1: The proof follows the proof of Theorem 3.1, up to and including (A.34). Introduce into (A.34) substitutions

$$\begin{aligned} x(t) =& h\left(\frac{t}{\epsilon}, \frac{-r}{\epsilon}; y(t)\right), \\ x(t-r) =& h\left(\frac{t-r}{\epsilon}, \frac{-r}{\epsilon}; y(t-r)\right) \end{aligned} \tag{A.42}$$

where $h(t, t_0; x(t_0))$ is defined in (4.3), to obtain

$$\begin{aligned} \dot{y}(t) = & \left[\frac{\partial h(\frac{t}{\epsilon}, \frac{-r}{\epsilon}; y(t))}{\partial y}\right]^{-1} \\ & \cdot P_1\left(h\left(\frac{t}{\epsilon}, \frac{-r}{\epsilon}; y(t)\right) h\left(\frac{t}{\epsilon} - \frac{r}{\epsilon}, \frac{-r}{\epsilon}; y(t-r)\right)\right) \\ & y(t) = \psi(t) \text{ for } t \in [-r, 0] \end{aligned} \tag{A.43}$$

where $\psi(t)$ is assumed to be uniquely defined by the relation $\varphi(t) = h(t/\epsilon, -r/\epsilon; \psi(t))$ for $t \in [-r, 0]$.

Assuming $\gamma = r/\epsilon_1$ and $0 < \epsilon_1 < \epsilon_\sigma$, where ϵ_σ is defined and guaranteed by Lemma 4.1, then the solution of (A.43) with $\epsilon = \epsilon_1$, denoted as $y(t, \epsilon_1; \psi)$ satisfies (by i) of Lemma 4.1)

$$\|y(t, \epsilon_1; \psi) - z(t; \psi)\| \le \sigma, \; t \in [0, L] \tag{A.44}$$

where σ is the arbitrarily small fixed positive constant defined in Lemma 4.1, L is an arbitrarily large constant, and $z(t; \psi)$ denotes the solution to (4.6).

Introduce inequalities

$$\begin{aligned} &\|\overline{x}(y(t, \epsilon_1; \psi)) - \overline{x}(z(t; \psi))\| \\ &= \|\lim_{T\to\infty} \frac{1}{T}\int_0^T h\left(\lambda, \frac{-r}{\epsilon_1}; y(t, \epsilon_1; \psi)\right) d\lambda \\ &- \lim_{T\to\infty} \frac{1}{T}\int_0^T h\left(\lambda, \frac{-r}{\epsilon_1}; z(t; \psi)\right) d\lambda\| \\ &\le \lim_{T\to\infty} \frac{1}{T}\int_0^T \|h\left(\lambda, \frac{-r}{\epsilon_1}; y(t, \epsilon_1; \psi)\right) \\ &- h\left(\lambda, \frac{-r}{\epsilon_1}; z(t; \psi)\right)\| d\lambda \end{aligned} \tag{A.45}$$

and

$$\|h(t, t_0; y(t, \epsilon_1; \psi)) - h(t, t_0; z(t; \psi))\| < m\|y(t, \epsilon_1; \psi) - z(t; \psi)\| \tag{A.46}$$

for $(y(t), z(t)) \in \Omega \times \Omega$, where m is the positive constant guaranteed to exist since $h(t, t_0; x(t_0))$ is almost periodic. Now choose γ, ϵ_1, and ϵ_σ such that $\sigma = \delta/m$, then by (A.45) and (A.46)

$$\|\overline{x}(y(t, \epsilon_1; \psi)) - \tilde{x}(z(t; \psi))\| < \delta, \; t \in [0, L] \tag{A.47}$$

which proves statement i).

When the condition of ii) holds, equilibrium z_s must have a domain of attraction Ω_1 so that $\psi \in \Omega_1$ implies $z(t; \psi) \in D$, $\forall t \ge 0$, where $D \supset \Omega_1$ is some bounded region in $\mathbb{R}^n$, and $\lim_{t\to\infty} z(t; \psi) = z_s$.

Under these conditions, Lemma 4.1 implies (A.44) is valid for $L = \infty$, and the proof of ii) follows. Q.E.D.

Appendix II

$A(t)$	–chemical concentration of chemical specie **A**.
$T(t)$	–reactor temperature.
α	–recycle delay time.
V	–reactor volume.
λ	–coefficient of recirculation.
q	–feed flow rate.
A_0	–feed concentration.
K_0	–reaction velocity constant.
E/R	–ratio of Arrhenius activation energy to the gas constant.
ρ	–density.
C	–specific heat.
$-\Delta H$	–heat of reaction (positive).
U	–heat transfer coefficient times the surface area of reactor.
T_0	–feed temperature.
T_w	–average coolant temperature in reactor cooling coil.

Acknowledgment

The authors gratefully acknowledge Prof. Jack K. Hale of the Center for Dynamical Systems and Nonlinear Studies, Georgia Institute of Technology, for his help in developing the new averaging theory presented in Section II. This paper is based on the Ph.D. dissertation of Brad Lehman; Prof. Hale served as Brad's thesis co-advisor from 1988–1992.

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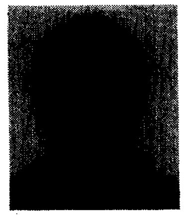

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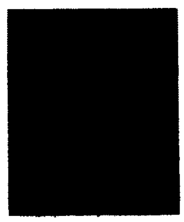

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