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VIBRATIONAL CONTROL OF NONLINEAR TIME LAG SYSTEMS WITH ARBITRARILY LARGE BUT BOUNDED DELAY: AVERAGING THEORY, STABILIZABILITY, AND TRANSIENT BEHAVIOR

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ABSTRACT: This paper develops the theory of vibrational control of nonlinear time lag systems with arbitrarily large but bounded delay. Averaging theory for fast oscillating differential delay equations is presented and then applied to vibrational control. Conditions are given which ensure the existence of parametric vibrations that stabilize nonlinear time lag systems. Transient behavior is also discussed. Illustrative examples are given which show (1) the feasibility of the theory to important applications and (2) the differences in the theory presented and the existing known theory for vibrational control of ordinary differential equations.

I. INTRODUCTION

Vibrational control is a recently developed nonclassical control technique, that unlike feedback and feedforward, does not require measurements of states or disturbances. Instead, zero mean parametric excitation is used as the tool for open loop modification of the plant behavior.

Vibrational control of systems governed by linear and nonlinear ordinary differential equations has been thoroughly discussed [1-4]. Application of this theory has been experimentally verified for: (1) an exothermic irreversible chemical reaction in a continuous stirred reactor (CSTR) by [5], and (2) a laser illuminated thermochemical system [6]. In both cases, zero mean parametric oscillations were used to eliminate the significant expenses in feedback due to cooling.

A number of practically important systems, however, are best described by including time delays in their states. In particular, if the exothermic reaction vibrationally controlled in [5] includes a recycle stream, as in [7], the model must include state delays. Population models, combustion models, manufacturing systems, among many others (cf. [8]) also have state delays in their models. It is, therefore of interest to vibrationally control such systems.

However, vibrational control of time lag systems is a more complicated task, since the plant is infinite dimensional. For systems with small delays, finite dimensional approximations can be made to approximate the original oscillating system by an ordinary differential equation (o.d.e.) [9-11]. For systems with large delay, though, finite dimensional approximations are known to have significant errors associated with them, especially for time varying systems [8,12,13].

Additionally,the primary mathematical tool for analysis used in vibrational control is the method of averaging, which is well known for o.d.e.'s [14, page 186] and for time lag systems with small delays [15, page 215]. For fast oscillating systems with large bounded delay, however, the averaging technique until recently has not been developed. Therefore, in order to extend vibrational control to general time lag systems, it is first necessary to extend averaging techniques to a more general class of delay differential system.This is performed in [16] (entirely motivated to enable the development of vibrational control) and Chapter II, which together, provide the set of required mathematical tools needed for the extension.

The results given in Chapter II are significant mathematical contributions themselves since they extend the method of averaging to an extremely broad class of differential delay equations. The averaging technique of o.d.e.'s has found important applications in adaptive control algorithms, basic stability analysis, noise control, pulse width modulation, periodic control, as well as vibrational control, just to name a few. The results given in Chapter II should allow similar extensions to delay differential equations. However, in this paper, the applications of these new averaging results will be used to extend the vibrational control technique to a general class of time lag systems.

This is done in Chapter III where vibrational stabilizability of nonlinear delay differential equations is discussed, partial results (the linear case) of which are published in [17]. Example 3.1, dealing with population dynamics, typifies the differences between the theory of vibrational control in o.d.e.'s and the theory of vibrational control of time lag systems by showing that some systems are vibrationally stabilizable only when there is a delay in the state, i.e if the delay is assumed zero, vibrational stabilizability is not possible.

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Chapter IV discusses transient behavior of vibrationally controlled systems, and Chapter V proposes the vibrational control of an exothermic irreversible chemical reaction in a CSTR with delayed recycle stream. Chapter VI contains conclusions. All formal proofs are in Appendix I.

II. AVERAGING THEORY

In this chapter, the mathematical foundations of averaging of differential delay equations are presented. These techniques will be used in subsequent chapters to develop the theory of vibrational control. However, as the introduction suggests, the results of this chapter have broad applications to general control theory.

Suppose $f(s,x,y)$ is continuous function, $f: \mathbb{R} \times \Omega \times \Omega \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$. Let ε be a real parameter, and let $\varphi(t) \in \Omega$ be a continuous function for $t \in [-r,0]$. Consider the system of differential delay equations for $t \geq 0$

$$\dot{x}(t) = f\left(\frac{t}{\varepsilon}, x(t), x(t-r)\right); \qquad x(t) = \varphi(t), \text{ for } t \in [-r,0] \tag{2.1}$$

along with

$$\dot{y}(t) = f_0(y(t), y(t-r)); \qquad y(t) = \psi(t), \text{ for } t \in [-r,0], \tag{2.2}$$

$$f_0(z_1, z_2) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t, z_1, z_2)dt.$$

Let $x(t,\varepsilon;\varphi)$ denote the solution to (2.1), and let $y(t;\psi)$ denote the solution to (2.2).

In order to address the transient behavior of vibrationally controlled systems, it is necessary to consider the closeness of the solutions $x(t,\varepsilon;\varphi)$ and $y(t;\psi)$. The following two theorems address this problem:

Theorem 2.1: Assume that in (2.1): (1) $f(t,z_1,z_2)$ is a continuous function in t, z_1, z_2 for $(t, z_1,z_2) \in \mathbb{R} \times \Omega \times \Omega$; (2) the limit, $\lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} f(s,z_1,z_2)ds$, exists uniformly with respect to (t, z_1,z_2) in compact sets of $\mathbb{R} \times \Omega \times \Omega$ and is independent of t; (3) there exists a constant k such that for $(s, z_1,z_2) \in \mathbb{R} \times \Omega \times \Omega$

$$\|f(s,z_1,z_2) - f(s,z'_1,z'_2)\| \leq k \max_{i=1,2} \|z_i - z'_i\|. \tag{2.3}$$

Then for any $L > 0$ and any $\gamma > 0$ there exists an $\varepsilon_0 = \varepsilon_0(\gamma,L)$ such that for $0 < \varepsilon \leq \varepsilon_0$

$$\|x(t,\varepsilon;\varphi) - y(t;\psi)\| \leq \left(\gamma + \sup_{s \in [-r,0]} \|\varphi(s) - \psi(s)\|\right) e^{kt} \tag{2.4}$$

for any $t \in [0,L]$.

Remark 2.1: Let the assumptions of Theorem 2.1 be true. Suppose x and y have identical initial functions, i.e. $\varphi(t) = \psi(t)$ for $t \in [-r,0]$. Then for any $\gamma > 0$ and any $L > 0$ there exists an $\varepsilon_0 = \varepsilon_0(\gamma,L)$ such that, for $0 < \varepsilon \leq \varepsilon_0$, $\|x(t,\varepsilon;\varphi) - y(t,\varepsilon)\| \leq \gamma e^{kL}$ for any $t \in [0,L]$, where k is the Lipschitz constant defined in (2.3). Since e^{kL} is a constant, the bound on the inequality can be made arbitrarily small by choosing $\varepsilon_0(\gamma,L)$ sufficiently small.

Remark 2.2: As expected, if the initial time, $t_0 \neq 0$, then the conclusions of Theorem 2.1, given by (2.4), become

$$\| x(t,\varepsilon;t_0,\varphi) - y(t;t_0,\psi)\| \le \left(\gamma + \sup_{s\in[t_0-r,t_0]} \|\varphi(s)-\psi(s)\|\right) e^{k(t-t_0)} \tag{2.5}$$

Remark 2.3: It is the authors' of this paper belief that the proofs presented have a great number of benefits over the techniques of [16] because: (1) they are easier to understand since, in essence, the main idea of the proof is to apply a special form of the Fundamental Theorem of Calculus (2) all analysis is kept in $\mathbb{R}^n$ and therefore, the theorems permit synthesis of vibrational controllers which must be implemented in $\mathbb{R}^n$ and (3) the proofs accurately describe transient behavior of a general class of highly oscillating differential delay equations and give a bound on the closeness of the solutions of (2.1) and (2.2). Of course, since an o.d.e. is a special case of a delay differential equation, the proof of Theorem 2.1 provides a novel alternative proof to the classical techniques of averaging of o.d.e.'s found in references such as [14].

For vibrational control theory it is also of interest to examine the closeness of solutions $x(t,\varepsilon;\varphi)$ and $y(t;\psi)$ on infinite time intervals. In order to extend the results of Theorem 2.1 to an infinite time interval, which can be done if $y(t;\psi)$ approaches a hyperbolic equilibrium point, along with (2.1) consider the corresponding averaged delay differential equation, for $t \ge 0$,

$$\dot y(t) = f_0(y(t), y(t-r)); \qquad y(t) = \varphi(t), \text{ for } t\in[-r,0] \tag{2.6}$$

where f_0 is defined in (2.2). That is, consider (2.2) with $\psi(t) = \varphi(t)$, for $t\in[-r,0]$, i.e. $x(t)$ and $y(t)$ have the same initial functions. The solution to (2.6) is denoted as $y(t;\varphi)$.

Theorem 2.2: Assume that f satisfies conditions (1), (2), and (3) of Theorem 2.1, and that $f(s,z_1,z_2)$ has continuous Fréchet derivatives in (z_1,z_2) on $\mathbb{R}\times\Omega\times\Omega$. Suppose that $y(t;\varphi)$, the solution of (2.6), is defined for all $t\ge 0$ and is contained in Ω with its ρ neighborhood, $\rho > 0$.

If there is a point $y_s \in \Omega$ such that $\lim_{t\to\infty} y(t;\varphi) = y_s$ and

$$\mathrm{Det}\left[sI - \frac{\partial f_0(y_s,y_s)}{\partial y} - \frac{\partial f_0(y_s,y_s)}{\partial y(t-r)} e^{-rs} \right] = 0$$

has all solutions with real parts less than zero, then there are constants η, $\varepsilon_0 = \varepsilon_0(\eta)$, and a function $x(t,\varepsilon;\varphi)$ satisfying (2.1) such that, for any $\eta > 0$ and any ε, $0 < \varepsilon \le \varepsilon_0$,

$$\|x(t,\varepsilon;\varphi) - y(t;\varphi)\| < \eta \text{ for all } t \ge 0 . \tag{2.7}$$

III. VIBRATIONAL STABILIZABILITY

A. Problem Statement

Consider the delay differential equation

$$\dot x(t) = \widetilde{P}_1(x(t),x(t-r)) + \widetilde{P}_2(\lambda,x(t)) \tag{3.1}$$

where $\widetilde{P}_1: \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$, $\widetilde{P}_2: \mathbb{R}^d\times\mathbb{R}^n\to\mathbb{R}^n$ are continuously differentiable, $x(t)\in\mathbb{R}^n$, $\lambda = [\lambda_1,\lambda_2,\dots,\lambda_d]^T$ are parameters subject to vibrations, and r is the constant positive delay.

Introduce into (3.1) parametric vibrations according to the law $\lambda(t) = \lambda_0 + f(t)$, where λ_0 is a constant vector, and $f(t)$ is an almost periodic average zero (APAZ) vector. Then (3.1) becomes

$$\dot x(t) = \widetilde{P}_1(x(t),x(t-r)) + \widetilde{P}_2(\lambda_0 + f(t),x(t)) . \tag{3.2}$$

Assume that (3.1) has a fixed equilibrium point $x_s = x_s(\lambda_0)$ for a fixed λ_0 (note that $x_s(t) = x_s(t-r) = x_s(\lambda_0)$).

Definition 3.1: An equilibrium point $x_s(\lambda_0)$ of (3.1) is said to be vibrationally δ-stabilizable (vδ-stabilizable) if for a given fixed $\delta > 0$ there exists an APAZ vector f(t) such that (3.2) has an asymptotically stable almost periodic solution $x^s(t)$, $0 \le t < \infty$, characterized by

$$\|\bar{x}^s - x_s(\lambda_0)\| < \delta; \quad \bar{x}^s = \overline{x^s(t)} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^s(t)\,dt .$$

Definition 3.2: An equilibrium point $x_s(\lambda_0)$ of (3.1) is said to be totally vibrationally stabilizable (t-stabilizable) if it is vδ-stabilizable with $\delta = 0$, and moreover, $x^s(t) = x_s(\lambda_0)$, for $0 \le t < \infty$.

Often, only one component of $x_s(\lambda_0)$ requires or admits vibrational stabilization, such as in chemical reactors where only one of the two, the rate of conversion or the temperature, can be vibrationally stabilized. This practical situation is reflected in the following definition:

Definition 3.3: An equilibrium point $x_s(\lambda_0) = [x_{1s}(\lambda_0),\dots,x_{ns}(\lambda_0)]^T$, of (3.1) is said to be partially vibrationally δ-stabilizable with respect to component $x_{is}(\lambda_0)$ if for a given fixed $\delta > 0$ there exists an APAZ vector f(t) such that (3.2) has an asymptotically stable almost periodic solution $x^s(t) = [x_1^s(t),\dots,x_n^s(t)]^T$, $0 \le t < \infty$ the ith component of which is characterized by $\|\bar{x}_i^s - x_{is}(\lambda_0)\| < \delta$.

The problem of vibrational stabilization consists of (1) finding conditions for the existence of stabilizing vibrations, and (2) finding the actual parameters of vibrations that ensure the desired response.

This paper will always assume that $\widetilde{P}_2(\lambda_0 + f(t),x(t)) = P(\lambda_0,x(t)) + P_2(f(t),x(t))$, where $P_2(\cdot,\cdot)$ is a vector function linear with respect to its first argument. Then (3.2) can be rewritten as

$$\dot x(t) = P_1(x(t),x(t-r)) + P_2(f(t),x(t)) , \tag{3.3}$$

where $P_1(x(t),x(t-r)) = \widetilde{P}_1(x(t),x(t-r)) + P(\lambda_0,x(t))$, and P_1 and P_2 are assumed continuously differentiable in both arguments.

Following the terminology introduced in Bellman, et al. [2,3], if $P_2(f(t),x(t)) = L(t)$, where L(t) is an APAZ vector, the vibrations are referred to as vector additive. If $P_2(f(t),x(t)) = D(t)x(t)$, the vibrations are called linear multiplicative.

B. General Case

In order to formulate the conditions for vδ-stabilizability of (3.1), consider the equation

$$\dot x(t) = P_2(f(t),x(t)) . \tag{3.4}$$

Denote the general solution (3.4) as h(t,c) where each $c\in\mathbb{R}^n$ is a constant vector. Assume further that h(t,c) is almost periodic in t and define constant m, $m > 0$, such that for any c_1,c_2 $\|h(t,c_1) - h(t,c_2)\| \le m\|c_1 - c_2\|$.

Using the substitution $x(t) = h(t,y(t))$, $x(t-r) = h(t-r,y(t-r))$, (3.3) for $t \ge 0$ becomes

$$\dot y(t) = \left[\frac{\partial h(t,y(t))}{\partial y}\right]^{-1} P_1(h(t,y(t)),h(t-r,y(t-r))) \triangleq Y(t,r,y(t),y(t-r)) . \tag{3.5}$$

Introduce the equation

$$\dot z(t) = P_0(z(t),z(t-r), r) ;$$

$$P_0(\xi,\beta,r) \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T Y(\tau,r,\xi,\beta)\,d\tau . \tag{3.6}$$

Let z_s denote an equilibrium point of (3.6) and

$$\dot z(t) = M_0(r)z(t) + M_1(r)z(t-r) \tag{3.7}$$

be the linearization of (3.6) at z_s with

$$M_0(r) \triangleq \frac{\partial P_0(z_s,z_s,r)}{\partial z(t)}, \qquad M_1(r) \triangleq \frac{\partial P_0(z_s,z_s,r)}{\partial z(t-r)} .$$

Lemma 3.1: Let $z_s\in\Omega$ denote an equilibrium point of (3.6). Assume that (1) there exists an APAZ vector f(t) such that the general solution, h(t,c), of (3.4) is almost periodic for any $c\in\Omega$; (2) both $Y(t,r,\beta,\eta)$ and $P_0(\beta,\eta,r)$ are continuously differentiable for all $\beta,\eta\in\Omega$, $\Omega\subset\mathbb{R}^n$. Then

i) for any $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that for any $\varepsilon\in(0,\varepsilon_\delta]$, the equation

$$\dot{y}(t) = \left[\frac{\partial h(t/\varepsilon, y(t))}{\partial y}\right]^{-1} P_1\big(h(t/\varepsilon, y(t)), h(t/\varepsilon - \gamma, y(t-r))\big), \quad (3.8)$$

has an almost periodic solution $y^s(t)$ satisfying $\|y^s(t) - z_s\| < \delta/m$, $\forall \varepsilon \in (0, \varepsilon_\delta]$, $\forall t \geq 0$.

ii) If, in addition to assumptions (1) and (2), there exists constant $\gamma > 0$ such that the equation given by

$$\mathrm{Det}[sI - M_0(\gamma) - M_1(\gamma)e^{-sr}] = 0 \quad (3.9)$$

has all the roots with negative real parts (assumption (3)), then there exists an $\varepsilon_0 > 0$ such that $y^s(t)$ exists and is locally asymptotically stable for any $\varepsilon \in (0, \varepsilon_0]$.

Theorem 3.1: Let the assumptions (1), (2), and (3) of Lemma 3.1 hold. Then $x_s(\lambda_0)$ of (3.1) is

i) vδ-stabilizable by vibrations $f(t) = (1/\varepsilon_1)\, g(t/\varepsilon_1)$, $\varepsilon_1 \triangleq r/\gamma$ = constant if: (a) $0 < \varepsilon_1 \leq \min[\varepsilon_0, \varepsilon_\delta]$, where ε_0 and ε_δ are defined and are guaranteed to exist for system (3.8) by Lemma 3.1. (b) There exists an equilibrium point z_s of (3.6), $z_s \in \Omega$ characterized by $\overline{h(t,z_s)} = x_s(\lambda_0)$.

ii) t-stabilizable if it is vδ-stabilizable, and in addition, (3.8) has an equilibrium point $y_s \in \Omega$ characterized by $y_s = z_s$, and $h(t, y_s) = x_s(\lambda_0)$.

Remark 3.1: The condition that $\overline{h(t,z_s)} = x_s(\lambda_o)$ means that equilibria of (3.1) and (3.6) are related through the average value of the substitution $x(t) = h(t/\varepsilon, y(t))$. This is clearly not always the case. When this condition does not hold, however, vδ-stabilizability can still take place.

Remark 3.2: Theorem 3.1 reduces the problem of vδ-stabilizability of (3.1) to the following procedure. First, a search is made for an APAZ vector f(t) so that (3.4) generates an almost periodic general solution h(t,c) such that all roots of (3.9) have Re(s) < 0. Second, the existence of stabilizing vibrations is established if $r/\gamma \leq \min[\varepsilon_0, \varepsilon_\delta]$. The actual stabilizing vibrations are obtained by rescaling the magnitude and frequencies of vibrations f(t) as $f(t) = (1/\varepsilon_1)\, g(t/\varepsilon_1)$.

Remark 3.3: Analytical estimates of ε_0 and ε_δ are usually extremely conservative. Therefore, the values of ε_0 and ε_δ are best determined by numerical simulation of system (3.8).

Remark 3.4: In the case of vector additive vibrations, $P_2(f(t), x(t)) = L(t)$ and $h(t,c) = u(t) + c$, where $u(t) = \int L(t)dt$. Therefore, the substitution becomes $x(t) = u(t) + y(t)$ and $x(t-r) = u(t-r) + y(t-r)$. Vector additive vibrations are incapable of t-stabilizing a system, since $x^s(t) = u(t) + y_s$, i.e. $x^s(t)$ is always non constant and almost periodic.

Example 3.1 (Harvesting of a Single Natural Population): The problem of harvesting renewable resources (game, fish, plants, etc.) is to determine a harvesting strategy which maximizes a sustainable yield, that does not strain the population of resources to die out.

We discuss, here, the vibrational control of the classical one specie population model (discussed in [8], [15]) with a constant harvest (see [18], page 27):

$$\dot{N}(t) = \alpha N(t)\left[1 - \frac{N(t-r)}{K}\right] - Y. \quad (3.10)$$

Here, $N(t) \in \mathbb{R}$ is the population of a single specie, such as fish in a hatchery, α, K, r, Y are positive constants, where α represents birth rate, K represents the carrying capacity of the environment, r is the positive constant delay taking into account a finite gestation period, time to reach maturity, etc. and Y is the yield which is to be maximized (harvesting rate).

Obviously, if the harvesting yield, Y, is chosen too large when the population of the specie is low, the specie will die out (perhaps as the whale population in the 1970's). As a matter of fact, if Y is chosen sufficiently large, $N \to 0$ in finite time, even when r = 0, since N = 0 is not an equilibrium. The largest Y which does not cause the population to die out is called the maximum sustainable yield, and is denoted as Y_{max}.

Introduce zero mean oscillations into Y so that (3.10) becomes

$$\dot{N}(t) = \alpha N(t)\left[1 - \frac{N(t-r)}{K}\right] - \left(Y + \frac{\beta}{\varepsilon}\sin(t/\varepsilon)\right), \quad (3.11)$$

which simply means that species N is being harvested at a periodic rate instead of at a constant rate. The goal of vibrationally stabilizing (3.10) is to choose β and ε in such a manner that Y_{max} can be increased so that the population of the specie does not die out.

In this case, the following substitution is used

$$N(t) = y(t) + \beta\cos(t/\varepsilon), \qquad N(t-r) = y(t-r) + \beta\cos\left(\frac{t-r}{\varepsilon}\right) \quad (3.12)$$

which transforms (3.11) into

$$\dot{y}(t) = \alpha\left[y(t) + \beta\cos(t/\varepsilon)\right] - \frac{\alpha}{K}\left[y(t) + \beta\cos(t/\varepsilon)\right]\left[y(t-r) + \beta\cos\left(\frac{t-r}{\varepsilon}\right)\right] - Y. \quad (3.13)$$

The corresponding average of (3.13) is given by

$$\dot{z}(t) = \alpha z(t)\left[1 - \frac{z(t-r)}{K}\right] - \left(Y + \frac{\alpha\beta^2}{2K}\cos(r/\varepsilon)\right). \quad (3.14)$$

Suppose the delay, r, is equal to zero. Then it is easily seen by (3.14) that the maximum sustainable yield, Y_{max}, actually decreases as β increases, since $\cos(r/\varepsilon) = 1$ when r = 0. However, if r and ε are chosen in such a manner that $\cos(r/\varepsilon) < 0$, then it would appear that Y_{max} actually increases. This is verified by computer simulation.

For the purposes of simulation, let in (3.11) $\alpha = 1$, K = 4, r =1.4, $\varepsilon = 0.445$, and N(t) = 4 for $t \in [-1.4, 0]$. Since, $\cos(r/\varepsilon) \approx -1$ the theory suggests that increasing the amplitude of β should increase the average maximum yield, Y_{max}. Fig. 1 plots Y_{max} versus β. For $\beta = 0$, simulations show $Y_{max} = 0.939$, which is clearly marked by the bold horizontal line. Fig. 1 also shows that increasing β, increases maximum yield. For $\beta = 0.418$, Y_{max} is calculated to be 0.942, an increase in yield of 0.3%. This value is important since at $\beta = 0.418$, $(\beta/\varepsilon) \approx 0.942 = Y_{max}$, which indicates the maximum β permitted if there is a constraint on Y being positive, i.e. never adding population to the specie. For $\beta >$ 0.418, this simply means that species are at times being added to the population, instead of being removed. This may represent, for instance, periodically removing fish from one hatchery and placing into another.

When there are no constraints on β, Fig. 1 shows substantial gains of the maximum yield for $\beta > 0.418$. The shaded region in Fig. 1 gives the simulated stability area of (3.11), i.e. $Y \leq Y_{max}$. It is seen that for $\beta = 1.0$, $Y_{max} = 1.01$, which is a far more significant gain in yield.

Remark 3.5: For linear multiplicative vibrations $P_2(f(t), x(t)) = D(t)x(t)$, and $h(t,c) = \Phi(t)c$, where $\Phi(t)$ is the fundamental solution to $\dot{x}(t) = D(t)x(t)$. Therefore, the substitution is: $x(t) = \Phi(t)y(t)$ and $x(t-r) = \Phi(t-r)y(t-r)$. Linear multiplicative vibrations are capable of t-stabilizablity. This is shown in the following section.

C. Linear Multiplicative Vibrations: $P_2(f(t), x(t)) = D(t)x(t)$

Theorem 3.2: Assume that: (1) $\tilde{P}_1(0,0) = 0$ and $\tilde{P}_2(\lambda, 0) = 0$ in (3.1); (2) there exists a sufficiently large set $\Omega \subset \mathbb{R}^n (0 \in \Omega)$ such that $\tilde{P}_1(\beta, \eta)$ and $\tilde{P}_2(\lambda, \beta)$ are continuously differentiable for all $\beta, \eta \in \Omega$; (3) there exists an APAZ matrix D(t) such that a state transition matrix $\Phi(t)$ of $\dot{x}(t) = D(t)x(t)$ is almost periodic; (4) there exists a constant $\gamma > 0$ such that (3.6), with

$$P_1(z(t), z(t-r), \gamma) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi^{-1}(\tau)P_1(\Phi(\tau)z(t), \Phi(\tau-\gamma)z(t-r))d\tau$$

has linearization about $z_s = 0$ given by (3.7) with the property that $\mathrm{Det}[sI - M_0(\gamma) - M_1(\gamma)e^{-rs}] = 0$ has all solutions with Re(s) < 0.

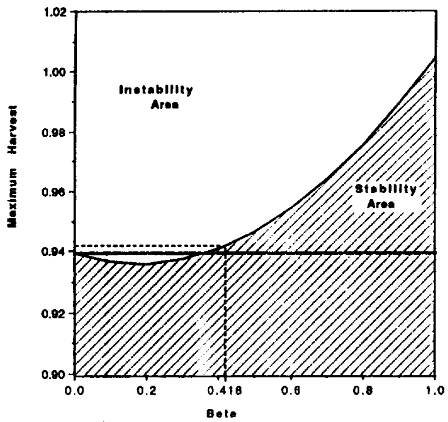

Fig. 1. Stability - instability boundary for system (3.10) with vibrations (3.11), r = 1.4 and ε = 0.445.

Then

i) there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, the trivial solution of the equation

$$\dot{y}(t) = \Phi^{-1}\left(\tfrac{t}{\varepsilon}\right)P_1\left(\Phi\left(\tfrac{t}{\varepsilon}\right)y(t), \Phi\left(\tfrac{t}{\varepsilon} - \gamma\right)y(t-r)\right) \tag{3.15}$$

is asymptotically stable;

ii) the trivial solution of (3.1) is t-stabilizable by linear multiplicative vibrations $\frac{1}{\varepsilon_1} D\left(\frac{t}{\varepsilon_1}\right)x(t)$, $\varepsilon_1 = \frac{r}{\gamma}$ if $\varepsilon_1 \le \varepsilon_0$.

Remark 3.7: An example using linear multiplicative vibrations will be discussed in Chapter V.

IV. TRANSIENT BEHAVIOR

A. Problem Statement

Chapter III describes the method of stabilizing equilibria of delay differential equations by introducing vibrations into parameters. Vibrational stabilizability describes changes in local attractivity in the vicinity of equilibria, i.e. local behavior as $t \to \infty$. For control purposes, it is also of interest to analyze the nonlocal system behavior at every time moment from t = 0, i.e. the transient behavior of the system. Analysis of such trajectories is a difficult task since vibrationally controlled systems are composed of a fast oscillatory trajectory superimposed on a slow trajectory. However, a comparison of the slow trajectory of the oscillatory system with the trajectory of the corresponding system without vibrations reveals the qualitative changes in the system behavior induced by vibrations.

Consider (3.1) with continuous initial function, $\varphi(t) \in \mathbb{R}^n$:

$$\dot{x}(t) = \tilde{P}_1(x(t), x(t-r)) + \tilde{P}_2(\lambda, x(t)); \quad x(t) = \varphi(t) \text{ for } t \in [-r, 0]. \tag{4.1}$$

Introducing into (4.1) parametric vibrations according to the law $\lambda(t) = \lambda_0 + f(t)$, for $t \ge 0$ and using the notation of Chapter III yields

$$\dot{x}(t) = P_1(x(t), x(t-r)) + P_2(f(t), x(t)), \tag{4.2}$$

with $x(t) = \varphi(t)$ for $t \in [-r, 0]$. Therefore, (3.5) is now considered with initial function $y(t) = \psi(t)$ for $t \in [-r, 0]$, where $\psi(t)$ is given by the relation $\varphi(t) = h(t; \psi(t))$ for $t \in [-r, 0]$. It is not too difficult to show that $\psi(t)$ is continuous and uniquely defined on $t \in [-r, 0]$.

The corresponding average of (3.5) now becomes

$$P_0(z(t), z(t-r), r); \qquad z(t) = \psi(t) \quad t \in [-r, 0]. \tag{4.3}$$

Let $y(t; \psi)$ and $z(t; \psi)$ denote the solution of (3.5) and (4.3), respectively, both with initial function ψ. Introduce

$$\bar{x}(y(t; \psi)) = \lim_{T \to \infty} \frac{1}{T} \int_0^T h(\tau; y(t; \psi))\, d\tau, \tag{4.4}$$

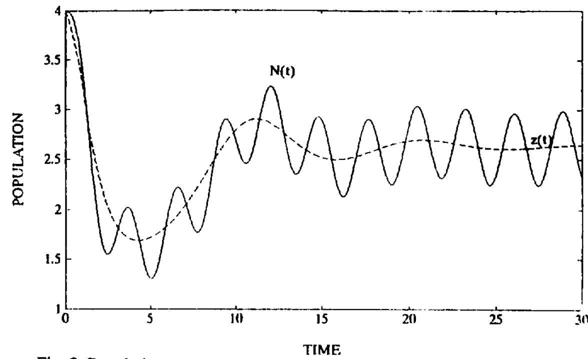

Fig. 2. Population N(t) of harvesting equation with vibrations along with its approximate average z(t).

which represents the averaged trajectory of vibrationally controlled system (4.2). If $y(t; \psi)$ and $z(t; \psi)$ are close to each other, $\bar{x}(y(t; \psi))$ can be approximated by

$$\bar{x}(z(t; \psi)) = \lim_{T \to \infty} \frac{1}{T} \int_0^T h(\tau; z(t; \psi))\, d\tau, \tag{4.5}$$

where $\bar{x}(z(t; \psi))$ will represent the approximate averaged transient behavior of the vibrationally controlled system (4.2). Comparison of $x(t; \varphi)$ of (4.1) with $\bar{x}(z(t; \psi))$ reveals the change of global transient behavior of (4.1) due to parametric oscillations.

Definition 4.1: For any given fixed $\delta > 0$ and any $L > 0$, an APAZ vector f(t) is said to induce a δ–global dynamic equivalence between systems (4.2) and (4.3) if

$$\|\bar{x}(y(t; \psi)) - \bar{x}(z(t; \psi))\| < \delta, \; t \in [0, L], \; \forall \psi(t) \in \Omega \subset \mathbb{R}^n$$

where Ω is an open subset of $\mathbb{R}^n$.

B. General Case

Let γ be a positive constant and let ε be a positive real parameter. Consider the delay differential equations

$$\dot{y}(t) = Y\left(\tfrac{t}{\varepsilon}, \gamma, y(t), y(t-r)\right); \; y(t) = \psi(t) \text{ for } t \in [-r, 0] \tag{4.6}$$

$$\dot{z}(t) = P_0(z(t), z(t-r), \gamma); \; z(t) = \psi(t) \text{ for } t \in [-r, 0] \tag{4.7}$$

where functions Y and P_0 are defined by (3.5) and (3.6), respectively, and $\psi(t) \in \Omega$, $\Omega \subset \mathbb{R}^n$, is continuous for $t \in [-r, 0]$. Denote the solution of (4.6) as $\tilde{y}_\gamma(t; \psi)$ and the solution of (4.7) as $\tilde{z}_\gamma(t; \psi)$.

Lemma 4.1: Let Ω be an open subset of $\mathbb{R}^n$, and let γ be a fixed positive constant. Assume that:

(1) h(t,c) is almost periodic in t;

(2) $Y(s, \gamma, y(t), y(t-r))$, defined in (3.5), is continuous in all arguments and there exists a positive constant k > 0 such that for $(s, y_1, y_2) \in \mathbb{R} \times \Omega \times \Omega$,

$$\| Y(s, \gamma, y_1, y_2) - Y(s, \gamma, y'_1, y'_2) \| \le k \max_{i=1,2} \| y_i - y'_i \|;$$

(3) The limit $P_0(y_1, y_2, \gamma) = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} Y(t, \gamma, y_1, y_2)\, dt$ exists uniformly with respect to (t, y_1, y_2) in compact sets of $\mathbb{R} \times \Omega \times \Omega$ and is independent of t.

Then for any $\sigma > 0$, $\rho > 0$, and $L > 0$, there exists an ε_σ such that, for $0 < \varepsilon \le \varepsilon_\sigma$,

i) $\|\tilde{y}_\gamma(t; \psi) - \tilde{z}_\gamma(t; \psi)\| < \sigma$ for $t \in [0, L]$, provided $\tilde{z}_\gamma(t; \psi)$ together with its ρ vicinity belongs to Ω for all $t \in [0, L]$;

ii) whenever the averaged equation (4.7) has an asymptotically stable equilibrium point $z_s \in \Omega$, then, $\|\tilde{y}_\gamma(t; \psi) - \tilde{z}_\gamma(t; \psi)\| < \sigma$ for $t \ge 0$ provided that $\tilde{z}_\gamma(t; \psi)$ together with its ρ vicinity belongs to Ω for all $t \ge 0$ and that $\tilde{z}_\gamma(t; \psi) \to z_s$ as $t \to \infty$.

Theorem 4.1: Let assumptions (1), (2), and (3) of Lemma 4.1 hold. Define $\varepsilon_1 \triangleq r/\gamma$, where γ and r are previously defined, and assume $0 < \varepsilon_1 \le \varepsilon_\sigma$ where ε_σ is defined and guaranteed to exist in Lemma 4.1. Then

i) vibrations $f(t) = 1/\varepsilon_1\, g(t/\varepsilon_1)$ induce a δ–global dynamic equivalence between (4.2) and (4.3) for $t \in [0, L]$, $L > 0$, provided that $z(t;\psi)$, the solution of (4.3), and its ρ vicinity belongs to $\Omega \subset \mathbb{R}^n$ for $t \in [-r, L]$;

ii) whenever (4.7) has an asymptotically stable equilibrium point, $z_s \in \Omega_1$, where $\Omega_1 \subset \mathbb{R}^n$ is the domain of attraction of z_s, vibrations $f(t) = 1/\varepsilon_1\, g(t/\varepsilon_1)$ induce a δ–global dynamic equivalence between (4.2) and (4.3) for $t \ge 0$, provided that $z(t;\psi)$ and its ρ vicinity belongs to Ω_1 for $t \ge -r$.

Remark 4.1: Theorem 4.1 reduces the problem of δ-global dynamic equivalence of (4.2) and (4.3) to the search for an APAZ vector $f(t)$ with sufficiently small period that induces a closeness of trajectories of (4.6) and (4.7). If $r/\gamma < \varepsilon_\sigma$, the search is complete. ε_σ is best determined by numerical simulation.

Remark 4.2: Since $\bar{x}(z(t;\psi))$ is easy to compute, Theorem 4.1 offers a constructive method for analysis of the transient behavior and the oscillation induced transitions of vibrationally controlled system (4.2).

Remark 4.3: In the case of vector additive vibrations, it is easy to show that $\bar{x}(z(t;\psi)) = z(t;\psi)$. In the case of linear multiplicative vibrations $\bar{x}(z(t;\psi)) = \bar{\Phi}(t)\, z(t;\psi)$ where $\bar{\Phi}(t) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi(t)dt$.

Example 4.1: Suppose in Example 3.1, $\alpha = 1$, $K = 4$, $r = 1.4$, $Y = 0.92$, $\beta = 0.4$, and $\varepsilon = 0.445$. Let $N(t,\varepsilon;\theta)$ denote the solution of (3.10) where $N(t) = \theta(t) = 4$ for $t \in [-1.4, 0]$. Let $z(t;\psi)$ denote the solution of (3.14) with $z(t) = \psi(t) = 4 - \beta\cos(t/\varepsilon)$ for $t \in [-1.4, 0]$. Fig. 2 plots $N(t,\varepsilon;\theta)$, the oscillating curve, and $\bar{N}(z(t;\psi)) = z(t;\psi)$ versus time for $t \ge 0$. It is seen that $\varepsilon = 0.445$ induces a δ–global dynamic equivalence between (3.11) and (3.14) for $\delta \ge 0.5$.

V. VIBRATIONAL CONTROL OF A CSTR WITH DELAY IN THE RECYCLE STREAM

This chapter discusses the vibrational control of a first order, irreversible, exothermic chemical reaction in a continuous stirred chemical reactor (CSTR) with delayed recycle stream. Vibrational control of such reactions when there is no recycle has been thoroughly investigated both theoretically [2,19] and experimentally [5]. However, it is important to consider the effects of recycle since it often times has noticable influence on the CSTR dynamics.

Reactor recycle not only increases the overall conversion, but also reduces the cost of a reaction, and is, therefore, very popular in industry. In order to recycle, the input specie must be separated from the yield, then travel through pipes after separation. This "total time" of recycle introduces delays in the states, and thus complicates the dynamics.

The benefits of vibrational control of exothermic reactions in a CSTR are given in [5,19]. Frequently, feedback in a CSTR is expensive [5] or slow [19]. Hence, "non - traditional" control techniques, such as vibrational control, are often used.

The purpose of this chapter is to give conditions under which a first order, irreversible reaction in a CSTR can be partially vibrationally stabilized and to show the inducement of a δ-global dynamic equivalence between the vibrationally controlled system and its corresponding average.

A. Model

Consider the first order, irreversible, exothermic reaction $\mathbf{A} \rightarrow \mathbf{B}$, carried out in a well mixed CSTR. Suppose, at the input, that the fresh feed of pure $\mathbf{A}$ is to be mixed with a recycle stream of unreacted $\mathbf{A}$ with recycle flow rate $(1-\lambda)\,q$. Let t be the instant of time the output exits the CSTR. Then according to [7] the material and energy balance equations are

$$V\frac{dA}{dt} = \lambda q A_o + q(1-\lambda)A(t-r) - qA(t) - VK_o\exp\left\{\frac{-E}{RT(t)}\right\}A(t)$$

$$VC\rho\frac{dT}{dt} = qC\rho[\lambda T_0 + (1-\lambda)T(t-\alpha) - T(t)] + V(-\Delta H)K_o\exp\left\{\frac{-E}{RT(t)}\right\}A(t) - U(T(t) - T_w), \qquad (5.1)$$

where $A(t) = \varphi_1(t)$ and $T(t) = \varphi_2(t)$ for $t \in [-r, 0]$, $A(t)$ is the concentration of chemical $\mathbf{A}$, $T(t)$ is temperature, and the remaining constants: α, λ, q, A_o, V, K_o, $-E/R$, C, ρ, $(-\Delta H)$, U, and T_w are all positive and defined in Appendix II. The constant λ varies from zero to one, with zero corresponding to total recycle and one corresponding to no recycle.

Typically, (5.1) is reduced to dimensionless form using the following notations:

$$x_1(t) = \frac{A_o - A(t)}{A_o}, \quad x_2(t) = \frac{T(t) - T_o}{T_o}\left(\frac{E}{RT_o}\right), \quad \theta_1(t) = \frac{A_o - \varphi_1(t)}{A_o},$$

$$\theta_2(t) = \frac{\varphi_2(t) - T_o}{T_o}\left(\frac{E}{RT_o}\right), \quad t_{new} = \frac{t}{\tau}, \quad \tau = \frac{V}{q\lambda},$$

$$D_a = K_o\tau\exp\{-\gamma_o\}, \quad B = \frac{(-\Delta H)A_oE}{C\rho T_o^2 R}, \quad r = \frac{\alpha}{\tau},$$

$$\beta = \frac{U\tau}{VC\rho}, \quad \gamma_o = \frac{E}{RT_o}. \qquad (5.2)$$

Without loss of generality, assume that $T_o = T_w$ and that $\gamma_o \rightarrow \infty$. The case when γ_o is finite poses no additional difficulties, but makes calculations tedious. The techniques are the same, as [5] shows.. Then (5.1) in dimensionless variables become:

$$\dot{x}_1(t) = \frac{-1}{\lambda}x_1(t) + \left(\frac{1}{\lambda} - 1\right)x_1(t-r) + D_a\exp\{x_2(t)\}(1 - x_1(t))$$

$$\dot{x}_2(t) = -\left(\frac{1}{\lambda} + \beta\right)x_2(t) + \left(\frac{1}{\lambda} - 1\right)x_2(t-r) + BD_a\exp\{x_2(t)\}(1 - x_1(t)), \qquad (5.3)$$

where $x_i(t) = \theta_i(t)$ for $t \in [-r, 0]$, $i = 1, 2$. The state $x_1(t)$ corresponds to the conversion rate of the reaction, $0 \le x_1(t) \le 1$, and $x_2(t)$ is the dimensionless temperature. Clearly, we must restrict $\lambda \neq 0$, or the right hand sides of (5.3) become invalid. Constants B, β, D_a, γ_o, and r are all positive.

Frequently, it is helpful to plot the locus of reactor steady states, x_{1s}, versus the Damköhler number D_a which is shown by the taller curve in Fig. 3. For fixed D_a, Fig. 3 shows how it is possible to have three steady states. In this case, [7] has shown that the upper and lower values of x_{1s} correspond to the steady states at upper and lower temperatures and are locally asymptotically stable, while the middle temperature gives an unstable steady state.

It is of interest to attempt to vibrationally control the middle steady states of x_{1s}. Usually, the upper steady states, which have the best conversion rate for the reaction, run at too high a temperature for a CSTR to operate at. If a middle steady state could be stabilized, it may produce the best conversion rate for the reaction under a temperature constraint.

B. Vibrational control of a CSTR

Introduce vibrations into (5.1) so that the input flow rate and output flow rate oscillate identically, i.e. consider (5.1) with vibrations:

$$V\frac{dA}{dt} = \lambda q A_o\left[1 + \frac{c}{\varepsilon}\sin\left(\frac{tq\lambda}{\varepsilon V}\right)\right] + q(1-\lambda)A(t-r) - q\left[1 + \frac{c}{\varepsilon}\sin\left(\frac{tq\lambda}{\varepsilon V}\right)\right]A(t) - VK_o\exp\left\{\frac{-E}{RT(t)}\right\}A(t)$$

$$VC\rho\frac{dT}{dt} = qC\rho\lambda T_o\left[1 + \frac{c}{\varepsilon}\sin\left(\frac{tq\lambda}{\varepsilon V}\right)\right] + q(1-\lambda)T(t-r) - qC\rho\left[1 + \frac{c}{\varepsilon}\sin\left(\frac{tq\lambda}{\varepsilon V}\right)\right]T(t) + V(-\Delta H)K_o\exp\left\{\frac{-E}{RT(t)}\right\}A(t) - U(T(t) - T_w). \qquad (5.4)$$

In dimensionless variables, (5.3) with oscillations as in (5.4) are written as

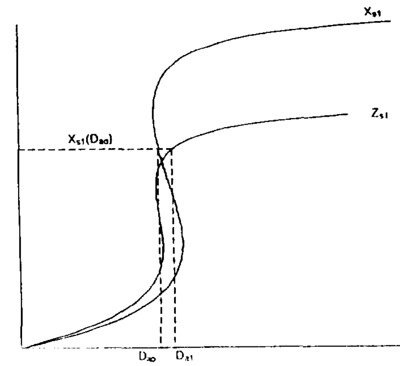

Fig.3. Steady state conversion of exothermic reaction vs. Damkohler number.

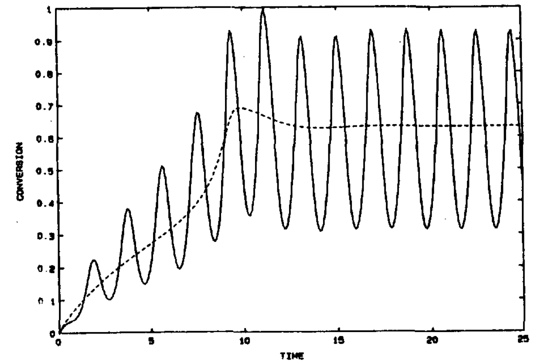

Fig.4. Conversion of exothermic reaction with vibrating flow rate and its approximate average for r = 0.9425 and ε = 0.3.

applying the previously described techniques the corresponding average is computed as, up to $O(c^4)$,

$$\dot{z}_1(t) = \frac{-1}{\lambda} z_1(t) + \left(\frac{1}{\lambda} - 1\right)\left(1 + \frac{c^2}{2}[1 - \cos(r/\varepsilon)]\right) z_1(t-r)$$

$$+ D_a \exp\{z_2(t)\}\left(1 - z_1(t) + \frac{c^2}{4}[1 - z_2(t) - z_1(t)z_2(t) + z_2^2(t) - z_1(t)z_2^2(t)]\right)$$

$$\dot{z}_2(t) = -\left(\frac{1}{\lambda} + \beta\right) z_2(t) + \left(\frac{1}{\lambda} - 1\right)\left(1 + \frac{c^2}{2}[1 - \cos(r/\varepsilon)]\right) z_2(t-r)$$

$$+BD_a \exp\{z_2(t)\}\{1 - z_1(t) + \frac{c^2}{4}[1 - z_2(t) - z_1(t)z_2(t) + z_2^2(t) - z_1(t)z_2^2(t)]\} \quad (5.6)$$

where $z_i(t) = \psi_i(t)$ for $t \in [-r, 0]$.

By the procedures of Chapter III, it is seen that for fixed $\varepsilon = \varepsilon_1$, sufficiently small, (5.1) is being partially vδ-stabilizable, since steady state x_{1s} is only being stabilized. Steady state characteristics of z_{1s} are shown in Fig. 3 for B = 7, β = 0.5, r = 0.9425, and c = 0.55. Fig. 3 shows the partial vδ-stabilization of an unstable equilibrium point $x_s(D_{a0} = 0.0908)$ with respect to its component $x_{1s}(D_{a0} = 0.0908) = 0.63$. Since a conversion of 0.63 corresponds to an upper steady state z_{1s} for the averaged equation (where $z_{1s} = x_s^1$), as shown in Fig.3, the conversion rate of 0.63 is an asymptotically stable steady state for $z_{1s}(D_{a1} = 0.10372)$. Fig. 4 verifies partial vδ-stabilization by plotting $x_1(t)$ and $z_1(t)$ versus time for $\varepsilon_1 = 0.3$, and λ = 0.8.

VI. CONCLUSIONS

This paper shows that vibrational control of nonlinear time lag systems is a feasible alternative to classical control techniques when measurements are either unavailable or expensive. Averaging theory is developed and then applied to vibrational control. Conditions for vδ-stabilizability and t-stabilizablity are discussed. In addition it is shown that for fixed, sufficiently small period of oscillation there exists a δ-global dynamic equivalence between the vibrating system and its corresponding average.

Two important applications of the theory are discussed. The example of harvesting a single natural population shows that vibrational control improves yield. It is also noted that if the delay in the state is assumed to be zero, vector additive vibrations have the effect of reducing the maximum yield. A second example is presented which shows that vibrating the input flow rate of the exothermic reaction described in Chapter V, stabilizes a previously unstable steady state. This steady state is often the preferable point of operation.

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APPENDIX I

Proof of Theorem 2.1: In order to prove Theorem 2.1, we must use the following two Lemmas, given without proof:

Lemma A.1 [20, page 461]: Assume that γ_1 and γ_2 are constants. If $\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\lambda,\gamma_1,\gamma_2)d\lambda \triangleq f_0(\gamma_1,\gamma_2)$, then for any piecewise constant function $\tilde{x}(t)$, any constants $L_1 > 0$ and $\beta > 0$, there exists an $\varepsilon_1 = \varepsilon_1(\beta, L_1)$ such that, for $0 < \varepsilon \le \varepsilon_1$ and any $t \in [0,L_1]$, $\|\int_0^t f(\frac{\tau}{\varepsilon},\tilde{x}(\tau),\tilde{x}(\tau-r))d\tau - \int_0^t f_0(\tilde{x}(\tau),\tilde{x}(\tau-r))d\tau\| \le \beta$.

Lemma A.2: Suppose conditions (2) and (3) of Theorem 2.1 are true. Then for every $(z_1,z_2) \in \Omega \times \Omega$

$$\|f_0(z_1,z_2) - f_0(z'_1,z'_2)\| \le k \max_{i=1,2}\|z_i - z'_i\| \tag{A.1}$$

where $f_0(z_1,z_2)$ is defined in (2.2) and k is the same Lipschitz constant as defined in (2.3).

We are now ready to prove Theorem 2.1:

The solutions to (2.1) and (2.2) are given respectively as

$$x(t,\varepsilon;\varphi) = \varphi(0) + \int_0^t f\left(\frac{\tau}{\varepsilon},x(\tau,\varepsilon;\varphi),x(\tau-r,\varepsilon;\varphi)\right)d\tau \tag{A.2}$$

$$y(t;\psi) = \psi(0) + \int_0^t f_0(y(\tau;\psi),y(\tau-r;\psi))d\tau. \tag{A.3}$$

Construct a piecewise constant function in Ω, $\tilde{x}(t)$, so that

$$\|x(t,\varepsilon;\varphi) - \tilde{x}(t)\| \le \frac{\gamma}{3kL};\ t\in[-r,L]. \tag{A.4}$$

Consider now the following equality which is always true:

$$\begin{aligned}\int_0^t \Big[f\left(\tfrac{\tau}{\varepsilon},x(\tau,\varepsilon;\varphi),x(\tau-r,\varepsilon;\varphi)\right) - f_0(x(\tau,\varepsilon;\varphi),x(\tau-r,\varepsilon;\varphi))\Big]d\tau =\\ = \int_0^t \Big[f\left(\tfrac{\tau}{\varepsilon},x(\tau,\varepsilon;\varphi),x(\tau-r,\varepsilon;\varphi)\right) - f\left(\tfrac{\tau}{\varepsilon},\tilde{x}(\tau),\tilde{x}(\tau-r)\right)\Big]d\tau\\ + \int_0^t \Big[f\left(\tfrac{\tau}{\varepsilon},\tilde{x}(\tau),\tilde{x}(\tau-r)\right) - f_0\left(\tfrac{\tau}{\varepsilon},\tilde{x}(\tau),\tilde{x}(\tau-r)\right)\Big]d\tau\\ + \int_0^t \big[f_0(\tilde{x}(\tau),\tilde{x}(\tau-r)) - f_0(x(\tau,\varepsilon;\varphi),x(\tau-r,\varepsilon;\varphi))\big]d\tau.\end{aligned} \tag{A.5}$$

By (2.3) and (A.4), for any $t\in[-r,L]$

$$\left\|f\left(\tfrac{t}{\varepsilon},x(t,\varepsilon;\varphi),x(t-r,\varepsilon;\varphi)\right) - f\left(\tfrac{t}{\varepsilon},\tilde{x}(t),\tilde{x}(t-r)\right)\right\| \le \frac{\gamma}{3L} \tag{A.6}$$

Likewise, by (A.1) of Lemma A.2 and by (A.4), for any $t \ge 0$

$$\|f_0(\tilde{x}(t),\tilde{x}(t-r)) - f_0(x(t,\varepsilon;\varphi),x(t-r,\varepsilon;\varphi))\| \le \frac{\gamma}{3L}. \tag{A.7}$$

Lemma A.1 guarantees that for any $t\in[0,L]$, there exists an $\varepsilon_0 = \varepsilon_0(\gamma/3,L)$ such that for $0 < \varepsilon \le \varepsilon_0$

$$\left\|\int_0^t f\left(\tfrac{t}{\varepsilon},\tilde{x}(\tau),\tilde{x}(\tau-r)\right) - f_0(\tilde{x}(\tau),\tilde{x}(\tau-r))d\tau\right\| \le \frac{\gamma}{3}. \tag{A.8}$$

Hence, for $0 < \varepsilon \le \varepsilon_0$ and $t\in[0,L]$

$$\begin{aligned}\left\|\int_0^t \Big[f\left(\tfrac{\tau}{\varepsilon},x(\tau,\varepsilon;\varphi),x(\tau-r,\varepsilon;\varphi)\right) - f_0(x(\tau,\varepsilon;\varphi),x(\tau-r,\varepsilon;\varphi))\Big]d\tau\right\|\\ \le \left(\frac{\gamma}{3L}\right)L + \left(\frac{\gamma}{3L}\right)L + \frac{\gamma}{3} = \gamma.\end{aligned} \tag{A.9}$$

By Lemma A.2, for any $t\in[0,L]$ and any $(x,y)\in\Omega\times\Omega$

$$\|f_0(x(t),x(t-r)) - f_0(y(t),y(t-r))\| \le k \sup_{s\in[-r,L]}\|x(s)-y(s)\|. \tag{A.10}$$

Using (A.2) and (A.3) the equality

$$\begin{aligned}x(t,\varepsilon;\varphi) - y(t;\psi) = \varphi(0) - \psi(0)\\ + \int_0^t \left[f\left(\frac{\tau}{\varepsilon},x(\tau,\varepsilon;\varphi),x(\tau-r,\varepsilon;\varphi)\right) - f_0(x(\tau,\varepsilon;\varphi),x(\tau-r,\varepsilon;\varphi))\right]d\tau\\ + \int_0^t [f_0(x(\tau,\varepsilon;\varphi),x(\tau-r,\varepsilon;\varphi)) - f_0(y(\tau;\psi),y(\tau-r;\psi))]d\tau\end{aligned} \tag{A.11}$$

is obtained. Taking the supreme norm of both sides of (A.11), and assuming $0 < \varepsilon \le \varepsilon_0$, we have

$$\begin{aligned}\sup_{s\in[-r,L]}\|x(s,\varepsilon;\varphi) - y(s;\psi)\| \le \sup_{s\in[-r,0]}\|\varphi(s)-\psi(s)\| + \gamma\\ + \int_0^t k \sup_{s\in[-r,L]}\|x(s,\varepsilon;\varphi) - y(s;\psi)\|ds.\end{aligned} \tag{A.12}$$

Applying Gronwall's inequality (see Hale [15]), the proof is complete. **Q.E.D.**

Proof of Theorem 2.2: For a fixed $R > 0$, let $B_R(y_s) = \{y : \|y - y_s\| < R\}$. Under the assumptions of Theorem 2.2, y_s is a locally uniformly asymptotically stable equilibrium point. By definition of uniform asymptotic stability, there exists an $\eta_0 > 0$ such that if

$$(y(t;\varphi_1),y(t;\varphi_2)) \in B_{\eta_0}(y_s)\times B_{\eta_0}(y_s) \text{ for } t\in[t_1-r,t_1],\ t_1>0,$$

then, for any $0 < \eta \le \eta_0 \le \rho$, there is a δ, $0<\delta<\eta$, and a $T_0(\delta)$ such that whenever

$$\|y(t;\varphi_1) - y(t;\varphi_2)\| < \delta,\ t\in[t_1-r,t_1] \tag{A.13}$$

where $t_1 \ge 0$ and $(\varphi_1,\varphi_2)\in\Omega\times\Omega$ are initial functions, possibly different, then

$$\|y(t;\varphi_1) - y(t;\varphi_2)\| < \frac{\eta}{2},\ t \ge t_1, \tag{A.14}$$

and further,

$$\|y(t;\varphi_1) - y(t;\varphi_2)\| < \frac{\delta}{2},\ t \ge t_1 + T_0\left(\frac{\delta}{2}\right). \tag{A.15}$$

Choose a time, $t = L_1$, large enough so that $y(L;\varphi)\in B_{\delta/2}(y_s)$. Then for $t \ge L_1$, $y(t;\varphi)\in B_{\eta/2}(y_s)$. By Theorem 2.1 and Remark 2.1, there exists an $\varepsilon_1 = \varepsilon_1(\frac{\delta}{2},L_1+r)$ such that if $0<\varepsilon\le\varepsilon_1$,

$$\|x(t,\varepsilon;\varphi) - y(t;\varphi)\| \le \frac{\delta}{2},\ t\in[0,L_1+r]. \tag{A.16}$$

Assume that there exists no range of ε, $0<\varepsilon\le\varepsilon_0$ such that

$$\|x(t,\varepsilon;\varphi) - y(t;\varphi)\| < \eta,\ t \ge 0. \tag{A.17}$$

Then there exists a nonempty set of positive numbers $\{m_i\}$, $m_i > L_1 + r, i = 1,2,\ldots,n$, such that $\|x(m_i,\varepsilon;\varphi) - y(m_i;\varphi)\| = \eta$. Choose $t_2 = \min_i \{m_i\}$. Clearly $t_2 > L_1 + r$.

By (A.16) it is also apparent that there exists another nonempty set $\{\alpha_j\}$, $j = 1, 2, \ldots, n$, $n < \infty$, such that $L_1 + r < \alpha_j < t_2$ and $\|x(\alpha_j,\varepsilon;\varphi) - y(\alpha_j;\varphi)\| = \delta$. Choose $t_3 = \min_j\{\alpha_j\}$ and $t_4 = \max_j\{\alpha_j\}$ (t_3 may equal t_4).

Therefore, we know

$$\|x(t,\varepsilon;\varphi) - y(t;\varphi)\| < \delta,\ t \in [L_1 + r, t_3) \tag{A.18}$$

and that

$$0 < \delta \le \|x(t,\varepsilon;\varphi) - y(t;\varphi)\| \le \eta,\ t \in [t_4, t_2]. \tag{A.19}$$

Let $t_5 = L_1 + r$, and redefine in (2.6) initial time $t_5 = t_0$. Let $\tilde{y}(t; t_5,\chi)$ denote the solution of (2.6) for initial function $\tilde{y}(t; t_5,\chi) = x(t,\varepsilon;\varphi)$ for $t \in [t_5 - r, t_5]$. Using (A.16) and the fact that $y(t;\varphi) \in B_{\delta/2}(y_s)$ for $t \in [t_5 - r, t_5]$, it is seen that for $0 < \varepsilon \le \varepsilon_1$

$$\|\tilde{y}(t; t_5,\chi) - y_s\| \le \|\tilde{y}(t; t_5,\chi) - y(t;\varphi)\| + \|y(t;\varphi) - y_s\|$$

$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad t \in [t_5 - r, t_5]. \tag{A.20}$$

Therefore, $\tilde{y}(t; t_5,\chi) \in B_\delta(y_s)$. Since by (A.16), $\|\tilde{y}(t; t_5,\chi) - y(t;\varphi)\| < \delta/2$ for $t \in [t_5 - r, t_5]$, (A.14) guarantees that

$$\|\tilde{y}(t; t_5,\chi) - y(t;\varphi)\| < \frac{\eta}{2},\ t \ge t_5, \tag{A.21}$$

and by (A.15), there exists a constant $T_0(\delta/2) > 0$ such that

$$\|\tilde{y}(t; t_5,\chi) - y(t;\varphi)\| < \frac{\delta}{2},\ t \ge t_5 + T_0\left(\frac{\delta}{2}\right). \tag{A.22}$$

Let $t_6 = t_4 + T_0(\delta/2)$. Then by Theorem 2.1 and Remarks 2.1 and 2.2, there exists an $\varepsilon_2(\delta/2, t_6 - t_5)$ such that for $0 < \varepsilon \le \varepsilon_2$

$$\|x(t,\varepsilon;\varphi) - \tilde{y}(t; t_5,\chi)\| \le \frac{\delta}{2},\ t \in [t_5, t_6]. \tag{A.23}$$

By using the inequality

$$\|x(t,\varepsilon;\varphi) - y(t;\varphi)\| \le \|x(t,\varepsilon;\varphi) - \tilde{y}(t; t_5,\chi)\| + \|\tilde{y}(t; t_5,\chi) - y(t;\varphi)\|, \tag{A.24}$$

and by using (A.23) and (A.21), if $0 < \varepsilon \le \varepsilon_2$,

$$\|x(t,\varepsilon;\varphi) - y(t;\varphi)\| < \frac{\delta}{2} + \frac{\eta}{2} < \eta,\ t \in [t_5, t_6]. \tag{A.25}$$

Therefore, $t_6 < t_2$ since $\|x(t_2,\varepsilon;\varphi) - y(t_2;\varphi)\| \triangleq \eta$.

However, if $t = t_6$, by (A.22), (A.23), and (A.24)

$$\|x(t_6,\varepsilon;\varphi) - y(t_6;\varphi)\| < \frac{\delta}{2} + \frac{\delta}{2} = \delta. \tag{A.26}$$

Choosing $\varepsilon_0 = \min[\varepsilon_1, \varepsilon_2]$, (A.26) contradicts (A.19) since $t_4 < t_6 < t_2$. **Q.E.D.**

Proof of Lemma 3.1: Noting that the average of (3.8) is given by

$$\dot{z}(t) = P_0(z(t), z(t-r), \gamma);$$

$$P_0(y(t), y(t-r), \gamma) = \lim_{T\to\infty} \frac{1}{T}\int_0^T Y(\tau,\gamma,y(t),y(t-r))d\tau, \tag{A.27}$$

the proof is immediate from [16]. **Q.E.D.**

Proof of Theorem 3.1: Consider (3.3) with vibrations $f(t) = (1/\varepsilon)g(t/\varepsilon)$, where ε is a positive constant:

$$\dot{x}(t) = P_1(x(t), x(t-r)) + P_2\left(\frac{1}{\varepsilon}g\left(\frac{t}{\varepsilon}\right), x(t)\right). \tag{A.28}$$

Since $P_2(\cdot,\cdot)$ has been assumed to be linear in its first argument, (A.28) may be rewritten as

$$\dot{x}(t) = P_1(x(t), x(t-r)) + \frac{1}{\varepsilon}P_2\left(g\left(\frac{t}{\varepsilon}\right), x(t)\right). \tag{A.29}$$

Introduce into (A.29) substitutions $x(t) = h(t/\varepsilon, y(t))$ and $x(t-r) = h(\frac{t-r}{\varepsilon}, y(t-r))$ where $h(t,c)$ is the general solution of (3.4), to obtain

$$\dot{y}(t) = \left[\frac{\partial h(t/\varepsilon, y(t))}{\partial y}\right]^{-1} P_1\left(h\left(\frac{t}{\varepsilon}, y(t)\right), h\left(\frac{t}{\varepsilon} - \frac{r}{\varepsilon}, y(t-r)\right)\right). \tag{A.30}$$

Replacing r/ε by γ yields system (3.8). Therefore, if vibrations $f(t)$ and a constant γ can be found such that all the assumptions of Theorem 3.1 hold, then by Lemma 3.1, system (3.5) has a solution $y^\gamma(t)$ with the properties satisfying assertions (i) and (ii) of Lemma 3.1. The remaining part of the proof follows from the same arguments as found in the proof of Lemma 1 of [2] **Q.E.D.**

Proof of Theorem 3.2: The proof follows almost directly from Lemma 3.1 and Theorem 3.1. **Q.E.D.**

Proof of Lemma 4.1: The proof of i) is immediate from Theorem 2.1, and the proof of ii) follows immediately from Theorem 2.2. **Q.E.D.**

Proof of Theorem 4.1: The proof follows the proof of Theorem 3.1, up to and including (A.29), where $y(t) = \psi(t)$ for $t \in [-r, 0]$.

Assuming $\gamma = r/\varepsilon_1$ and $0 < \varepsilon_1 \le \varepsilon_\sigma$, where ε_σ is defined and guaranteed by Lemma 4.1, then the solution of (A.29) with $\varepsilon = \varepsilon_1$, denoted as $y(t,\varepsilon_1;\psi)$ satisfies (by (i) of Lemma 4.1)

$$\|y(t,\varepsilon_1;\psi) - z(t;\psi)\| \le \sigma,\ t \in [0,L] \tag{A.30}$$

where σ is the arbitrarily small fixed positive constant defined in Lemma 4.1, $L > 0$, and $z(t;\psi)$ denotes the solution to (4.3).

By applying arguements similar to those of Theorem 4 in [3], the remaining part of the proof follows. **Q.E.D.**

APPENDIX II

$A(t)$ - concentration of **A**	ρ - density
$T(t)$ - reactor temperature	C - specific heat
α - recycle delay time	$-\Delta H$ - heat of reaction (positive)
V - reactor volume	U- product of heat transfer coefficient and surface area of reactor
λ - coefficient of recirculation	
q - feed flow rate	T_o - feed temperature
A_o - feed concentration	T_w - average coolant temperature

K_o - reaction velocity constant in reactor cooling coil
E/R - ratio of Arrhenius activation energy to the gas constant