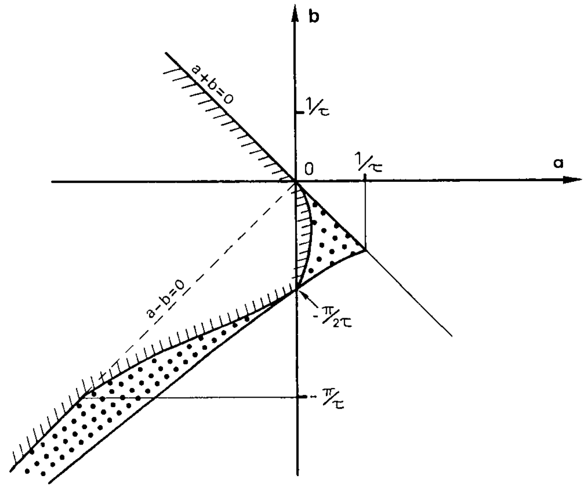

Fig. 3. Stability region in the parameter plane of the system (17).

The result obtained may include information on the delay τ and complements the existing delay-independent stability criteria. Its relation to the criteria is also discussed. The point of the theorem is twofold: the facts that for time-delay systems unstable characteristic roots exist only on some restricted bounded region of the complex plane and that to exclude the roots from that region it is sufficient to impose certain conditions on its boundaries. An interesting open problem is to consider situations where the condition (7) for s on some edge among (8)–(10) can be dispensed with.

APPENDIX

Definition of the Matrix Measure [11]: The matrix measure of a matrix X derived from the given norm $\|\cdot\|_\cdot$ is defined as follows:

$$\mu_\cdot(X) := \lim_{\epsilon\to 0} \frac{\|I+\epsilon X\|_\cdot - 1}{\epsilon}. \tag{A.0}$$

Lemma 3 [11]: For any matrices X, $Y \in C^{n\times n}$, the following inequalities hold:

$$\text{Re } \lambda_i(X) \leqq \mu(X) \tag{A.1}$$

$$-\mu(jX) \leqq \text{Im } \lambda_i(X) \leqq \mu(-jX) \tag{A.2}$$

$$\mu(X+Y) \leqq \mu(X)+\mu(Y) \tag{A.3}$$

$$\mu(X) \leqq \|X\| \tag{A.4}$$

$$\mu(\epsilon X) = \epsilon\mu(X), \qquad \forall \epsilon \geqq 0. \tag{A.5}$$

Computation of the Matrix Measure [11]: For a matrix $X = \{x_{ik}\} \in C^{n\times n}$, the matrix measures $\mu_\cdot(X)$, $\cdot = 1, 2, \infty$ are calculated as follows:

$$\mu_1(X) = \max_k \left(\text{Re } (x_{kk}) + \sum_{\substack{i=1\\ \neq k}}^{n} |x_{ik}| \right) \tag{A.6}$$

$$\mu_2(X) = \frac{1}{2} \max_i \lambda_i(X^* + X) \tag{A.7}$$

$$\mu_\infty(X) = \max_i \left(\text{Re } (x_{ii}) + \sum_{\substack{k=1\\ \neq i}}^{n} |x_{ik}| \right). \tag{A.8}$$

Here, (*) denotes the conjuate transpose symbol.

ACKNOWLEDGMENT

The first author is indebted to Prof. B. D. O. Anderson for scrutinizing the draft.

REFERENCES

[1] T. Mori, "Criteria for asymptotic stability of linear time-delay systems," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 158–161, 1985.

[2] E. W. Kamen, "On the relationship between zero criteria for two-variable polynomials and asymptotic stability of delay differential equations," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 983–984, 1980.

[3] H. Bourlés, "α-stability of systems governed by a functional differential equation-extension of results concerning linear delay systems," *Int. J. Contr.*, vol. 45, no. 6, pp. 2233–2234, 1987.

[4] T. Mori *et al.*, "On an estimate of the decay rate for stable linear delay systems," *Int. J. Contr.*, vol. 36, no. 1, pp. 95–97, 1982.

[5] R. M. Lewis and B. D. O. Anderson, "Necessary and sufficient conditions for delay-independent stability of linear autonomous systems," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 735–739, 1980.

[6] E. W. Kamen, "Correction to 'Linear systems with commensurate time delays: stability and stabilization independent of delay,' " *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 248–249, 1983.

[7] J. Hale, *Theory of Functional Differential Equations.* New York: Academic, 1977.

[8] L. V. Ahlfors, *Complex Analysis.* New York: McGraw-Hill, 1953.

[9] I. Postlethwaite and A. G. J. MacFarlane, *A Complex Variable Approach to the Analysis of Linear Multivariable Feedback Systems.* New York: Springer-Verlag, 1979.

[10] Y. S. Hung and A. G. J. MacFarlane, *Multivariable Feedback: A Quasi-Classical Approach.* New York: Springer-Verlag, 1982.

[11] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties.* New York: Academic, 1975.

[12] M. Malek-Zavarei and M. Jamshidi, *Time-Delay Systems: Analysis, Optimization and Applications.* Amsterdam, The Netherlands: North-Holland, 1987.

[13] V. B. Kolmanovskii and V. R. Nosov, *Stability of Functional Differential Equations.* New York: Academic, 1986.

[14] J. Chiasson, "A method for computing the interval of the delay values for which a differential-delay system is stable," in *Proc. 1986 Amer. Contr. Conf.*, Seattle, WA, 1986, pp. 324–325.

[15] S. D. Brierley, J. N. Chiasson, E. B. Lee, and S. H. Zak, "On stability independent of delay for linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 252–254, 1982.

Stability of Fast Periodic Systems with Time Lags

JOSEPH BENTSMAN, JAMEL FAKHFAKH, HARRY HVOSTOV, AND BRADLEY LEHMAN

Abstract—**This note extends the stability criterion of [1] for fast periodic linear systems to systems with time delays. Applying the results of Hale [5, 3.3], the asymptotic stability criterion for a class of time lag systems is derived. A comparison to a nondelay case is made, and it is demonstrated that small delays can significantly affect the stability properties of this class of systems.**

I. INTRODUCTION

A recent paper [1] presented a stability criterion for the system

$$\dot{x}(t) = \left(A + \frac{1}{\epsilon} D\left(\frac{t}{\epsilon}\right) \right) x(t), \qquad x: R_+ \to R^n \tag{1}$$

where A is a constant matrix, t is the dimensionless time, $D(t/\epsilon)$ is a zero

Manuscript received September 11, 1987; revised April 4, 1988. This work was supported by the Engineering Foundation under Grant RI-A-87-6.

J. Bentsman, J. Fakhfakh, and H. Hvostov are with the Department of Mechanical and Industrial Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801.

B. Lehman is with the Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801.

IEEE Log number 8825861.

average periodic matrix with the period of the order ϵ, and ϵ is a positive constant. Specifically, in [1] it was shown that there exists ϵ_o such that for any $0 < \epsilon < \epsilon_o$ the asymptotic stability and instability properties of (1) are identical to those of the time-invariant system $\dot{x}(t) = \bar{A}x(t)$, where $\bar{A}$ is a constant matrix, calculated as

$$\bar{A} = \lim_{T\to\infty} \frac{1}{T} \int_o^T \Phi^{-1}(\tau, 0) A \Phi(\tau, 0)\, d\tau \tag{2}$$

and $\Phi(\tau, 0)$ is the almost periodic state transition matrix of

$$dy(\tau)/d\tau = D(\tau)y(\tau), \ \tau = t/\epsilon \tag{3}$$

provided $\bar{A}$ has no eigenvalues with zero real parts. Whenever the state transition matrix of (3) can be found, this result becomes a constructive tool for the stability analysis of (1).

Since this stability condition has been utilized in the design of vibrational and periodic feedback controllers for finite-dimensional systems (see [2]–[4]), and since time lags are commonly encountered in practice, it seems desirable to obtain a similar condition for the time lag systems as well.

Using the approach of [1] we demonstrate that the stability properties of a linear system with a finite number of constant delays of the form

$$\dot{x}(t) = \left(A + \frac{1}{\epsilon} D\left(\frac{t}{\epsilon}\right)\right) x(t) + \sum_{i=1}^{m} B_i x(t - \epsilon r_i) \tag{4}$$

where each B_i is a constant matrix, and each r_i is a positive constant in $(0, r]$, $r = \text{const} > 0$ are also determined by the eigenvalues of a constant matrix

$$C \triangleq \bar{A} + \sum_{i=1}^{m} \bar{B}_i \tag{5}$$

where $\bar{A}$ is given by (2), and each $\bar{B}_i$ is calculated as

$$\bar{B}_i \triangleq \lim_{T\to\infty} \frac{1}{T} \int_o^T \Phi^{-1}(\tau, 0) B_i \Phi(\tau - r_i, 0)\, d\tau. \tag{6}$$

II. Preliminary Lemma

Consider a linear almost periodic system

$$\dot{y}(t) = \epsilon \sum_{i=1}^{m} F_i(t) y(t - r_i), \qquad r_i \in [0, r], \ \forall i = 1, \cdots, m,$$

$$y: [-r, \infty) \to R^n, \ F_i: R_+ \to R^{n\times n}, \qquad 0 < \epsilon \ll 1, \ t \in [0, \infty). \tag{7}$$

Define the averaged equation corresponding to (7) as the ordinary differential equation

$$\dot{z}(t) = \epsilon \sum_{i=1}^{m} \bar{F}_i z(t), \ z: R_+ \to R^n$$

$$\bar{F}_i = \lim_{T\to\infty} (1/T) \int_o^T F_i(t)\, dt, \qquad i = 1, \cdots, m. \tag{8}$$

Lemma: Assume that matrix $\Sigma^m_{i=1} \bar{F}_i$ has no eigenvalues with zero real parts. Then there exists ϵ_o such that for any $0 < \epsilon \le \epsilon_o$:

i) the trivial solution $y(t) = 0$ of (7) is asymptotically stable if $z(t) = 0$ of (8) is asymptotically stable;

ii) the trivial solution $y(t) = 0$ of (7) is unstable if $z(t) = 0$ of (8) is unstable.

Proof: Noting that $y(t) = 0$ is an *a priori* known almost periodic solution of (7) in the neighborhood of $z(t) = 0$, the proof of the lemma directly follows from [5, 3.3].

Remark 1: A solution of every delay equation in this note is interpreted in the sense of [6, p. 257] as a continuous function $x: [-r, \infty) \to R^n$ that reproduces the initial data on $[-r, 0]$ and satisfies the equation considered for $t \ge 0$, with $\dot{x}(0)$ being understood as the right-hand derivative.

Remark 2: Since analytical estimates of ϵ_o are usually extremely conservative (e.g., see [1]), the value of ϵ_o is best determined via numerical simulations.

III. The Main Result

Theorem: Assume that the state transition matrix $\Phi(\tau, 0)$, $\tau \in (-\infty, \infty)$ of (3) is almost periodic. Then there exists ϵ_o such that for any $0 < \epsilon \le \epsilon_o$ the trivial solution of (4) is:

i) asymptotically stable if C defined in (5) is a Hurwitz matrix;

ii) unstable if C has at least one right half-plane eigenvalue and has no eigenvalues on the imaginary axis.

Proof: In time $\tau = t/\epsilon$, (4) takes the form

$$dx(\tau)/d\tau = (\epsilon A + D(\tau))x(\tau) + \epsilon \sum_{i=1}^{m} B_i x(\tau - r_i). \tag{9}$$

For the purposes of the proof introduce a substitution

$$x(\tau) = \Phi(\tau, -r)y(\tau) \tag{10}$$

which for the delayed states with $\tau \to \tau - r_i$ takes the form

$$x(\tau - r_i) = \Phi(\tau - r_i, -r)y(\tau - r_i), \qquad i = 1, \cdots, m. \tag{11}$$

Since $r = \max_i r_i$, substitution (11) is well defined on $\tau \in [0, \infty)$. Using (10) and (11), equation (9) can be transformed into the equation

$$dy(\tau)/d\tau = \epsilon \left[\Phi^{-1}(\tau, -r) A \Phi(\tau, -r) y(\tau) + \sum_{i=1}^{m} \Phi^{-1}(\tau, -r) B_i \Phi(\tau - r_i, -r) y(\tau - r_i) \right]. \tag{12}$$

Since (12) is of the form (7), the lemma of the previous section can be used for the stability analysis of (12). Averaging the right-hand side of (12) with respect to τ and dropping delays in the argument of a state variable, we obtain

$$dv(\tau)/d\tau = \epsilon \left[\overline{\Phi^{-1}(\tau, -r) A \Phi(\tau, -r)} + \sum_{i=1}^{m} \overline{\Phi^{-1}(\tau, -r) B_i \Phi(\tau - r_i, -r)} \right] v(\tau)$$

$$= \epsilon \Phi^{-1}(o, -r) \left[\overline{\Phi^{-1}(\tau, o) A \Phi(\tau, o)} + \sum_{i=1}^{m} \overline{\Phi^{-1}(\tau, o) B_i \Phi(\tau - r_i, o)} \right] \Phi(o, -r) v(\tau)$$

where $\overline{u(\tau)} = \lim_{T\to\infty} (1/T) \int_o^T u(\tau)\, d\tau$ and $\Phi(o, -r)$ is a constant matrix, nonsingular as the state transition matrix of (3). Consequently, noting that the averaged equation corresponding to (12) with $z(\tau) \triangleq \Phi(o, -r)v(\tau)$ is given by $\dot{z}(\tau) = \epsilon C z(\tau)$, where matrix C is defined in (5), and that if $\Phi(\tau, o)$ is almost periodic, then (10) and (11) are stability preserving, the assertions of the theorem directly follow from the lemma. Q.E.D.

Remark 3: One can regard delays $d_i = \epsilon r_i$ in (4) as fixed numbers independent of ϵ. Then parameterization ϵr_i and boundedness of r_i, $i = 1, \cdots, m$, simply mean that the result presented above applies to systems with fixed delays of the same order of magnitude as the period of oscillations, $T = 2\pi\epsilon$, or lower. If $d_i = 0(\epsilon)$, $\forall i$, then from (5) and (6) it follows that parameters $r_i = d_i/\epsilon = 0(1)$ appear explicitly on the right-hand side of the averaged equation $\dot{z}(\tau) = \epsilon C z(\tau)$ that governs the stability properties of (4). The values of $r_i = 0(1)$ significantly affect the spectrum of matrix C, giving rise to the added difficulty in stability analysis of (4) in comparison to systems considered by Hale in [5] as well as those discussed in [1]. If $d_i = 0(\epsilon^2)$, $\forall i$, then $r_i = 0(\epsilon)$ and for sufficiently small ϵ the spectrum of C by continuity is close to that of the corresponding matrix for a system with no delays. Thus, delays $d_i = 0(\epsilon^2)$ can be neglected and the above theorem with $r_i \approx 0$, $\forall i$, reduces to the theorem of [1].

IV. Example

Suppose that in (4) $n = 2$, $m = 1$, $r_i \equiv r$,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \; B_1 \equiv B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \; D(\tau) = \alpha \cos \tau \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{13}$$

Then, the state transition matrix of (3) is given by

$$\Phi(\tau, 0) = \begin{bmatrix} 1 & \alpha \sin \tau \\ 0 & 1 \end{bmatrix}$$

the delayed matrix is

$$\Phi(\tau - r, 0) = \begin{bmatrix} 1 & \alpha \sin (\tau - r) \\ 0 & 1 \end{bmatrix}$$

and matrices $\bar{A}$, $\bar{B}$, and C take the following form:

$$\bar{A} = \begin{bmatrix} a_{11} & a_{12} - (\alpha^2/2) a_{21} \\ a_{12} & a_{22} \end{bmatrix}, \; \bar{B} = \begin{bmatrix} b_{11} & b_{12} - (\alpha^2/2) b_{21} \cos r \\ b_{21} & b_{22} \end{bmatrix}$$

and

$$C = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} - (\alpha^2/2)(a_{21} + b_{21} \cos r) \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}. \tag{14}$$

Thus, according to the theorem the eigenvalues of matrix C define the stability properties of (4) for any $0 < \epsilon \leq \epsilon_o$, where ϵ_o is a sufficiently small number.

Let us examine the effect of a fixed delay, $d = \epsilon r = $ const, on the stability of the system and determine the size of ϵ_o for $r = $ const. For this purpose, consider (4) with

$$A = \begin{bmatrix} 0.3 & 1 \\ 0.6 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0.3 & 0.3 \\ 0.2 & -0.6 \end{bmatrix}. \tag{15}$$

Then the asymptotic stability of (4) is guaranteed for a sufficiently small ϵ_o, $0 < \epsilon \leq \epsilon_o$, whenever Det $C > 0$ and Tr $C < 0$, where C is given in (14). With A and B of (15) this yields

$$\alpha > \left[\frac{5}{0.6 + 0.2 \cos r} \right]^{1/2}. \tag{16}$$

Instability takes place when the inequality sign in (16) is reversed. Setting in (16) $r = 0$ we recover the stability bound $\alpha > 2.5$ derived in [1] for the system without delay. The sets of the values of α that satisfy (16) and $\alpha > 2.5$ will be further referred to as the admissible stability regions for α for systems with and without delay, respectively.

Figs. 1 and 2 show the asymptotic stability and instability regions (shaded and clear areas, respectively) of a system without delay and with the delay $d = 1.6\,\pi$ in the parameter plane α versus $1/\epsilon$. From Fig. 1 it is seen that for any fixed value of $\alpha > 2.5$, say α_1, the value of ϵ_o that ensures stability is given by the first and the only point of the intersection of the horizontal line $\alpha = \alpha_1$ with the stability-instability boundary (the dashed curve in Fig. 1). Fig. 2 shows that the introduction of the delay d drastically changes the relation between the stability and instability areas in the same region of the parameter plane. Since in this case the value of r in (16) is varying as a function of ϵ, $r = 1.6\,\pi/\epsilon$, the inequality (16) yields the lower bound of the admissible stability region for α given in Fig. 2 by the curve periodic in $1/\epsilon$.

The actual stability-instability boundary that separates shaded and clear areas in Fig. 2, given by the dashed curve, has been determined with the help of the numerical functional equation solver of [7].

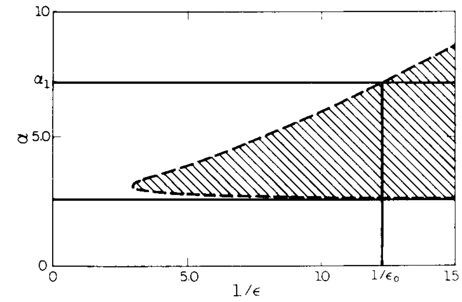

Fig. 1. Superposition of admissible and actual (simulated) regions of stability of a system without delay.

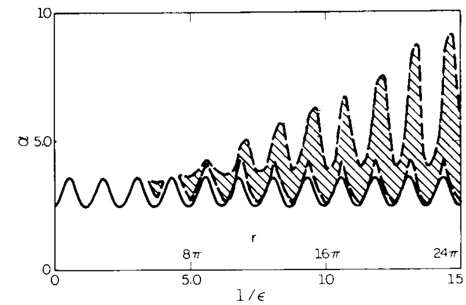

Fig. 2. Superposition of admissible and actual (simulated) regions of stability of a system with delay equal to 1.6π.

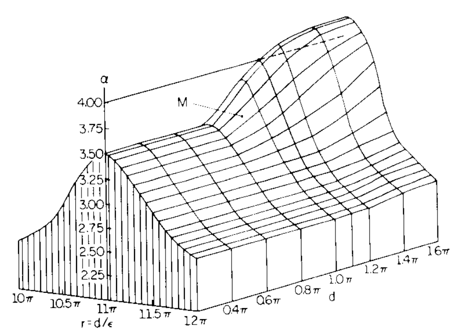

Fig. 3. Stability-instability boundary in the (α, r, d) parameter space of a system with delay $d \in [0.2\pi, 1.6\pi]$

Let us now note the following important difference between Figs. 1 and 2. Unlike Fig. 1, for most values of α in its admissible stability region in Fig. 2 there is no ϵ_o that ensures stability for *any* ϵ in $(0, \epsilon_o]$ under the presence of a fixed delay d. This is, of course, in no contradiction with the theorem which guarantees the existence of ϵ_o for a fixed ratio $r = d/\epsilon$ rather than for a fixed delay d itself. Thus, unlike the no delay case where the stability area is bounded by the curves monotonic in $1/\epsilon$, the presence of a fixed delay introduces the sequence of stable and unstable intervals in $1/\epsilon$ for most values of α in its admissible stability region.

Keeping r fixed and decreasing delay d, the value of ϵ_o can be found which ensures that stability properties of system (4) with A and B given by (15) and $D(t/\epsilon)$ given by (13) with $\tau = t/\epsilon$ are governed by inequality (16). Fig. 3 shows part of the actual (simulated) stability-instability boundary for the system with the delay $d = \epsilon r$ given by the surface M in the (α, r, d)-space. Every point in the (r, d)-plane corresponds to ϵ given by d/r. It is seen that for every fixed value of r in the interval $r \in [10\pi, 12\pi]$ and for all delays $d \leq 0.8\pi$, the value of ϵ is small enough for a simulated boundary to coincide with the theoretically predicted one. This indicates that for this system $\epsilon_o \approx 0.08$, which agrees with $\epsilon \approx 0.1$ found in [1] for the corresponding system without delay. Note that for many values of r and α, the stability takes place for ϵ higher than 0.08. For example, the point of the intersection of line ($\alpha = 4.0$, $r = 11\pi$) with surface M corresponds to $d \approx 1.1\pi$ and, hence, $\epsilon \approx 0.1$. Thus, system (4) with A and B given by (15), $r = 11\pi$, and $D(t/\epsilon)$ given by (13) with $\alpha = 4.0$ and $\tau = t/\epsilon$ is asymptotically stable for any positive $\epsilon \le 0.1$.

V. Conclusions

This note presents a constructive tool for the stability analysis of periodic linear time lag systems of the form (4). It shows that stability properties of this class of systems are sensitive to small delays, and therefore caution should be exercised in applying vibrational and fast periodic feedback controllers designed under the no delay assumption via results of [2]–[4], to systems with small delays.

Acknowledgment

The authors are grateful to the reviewers of a number of insightful comments that led to a substantial improvement of this note. Computer time was provided by the National Center for Supercomputing Applications of the University of Illinois at Urbana-Champaign.

References

[1] R. Bellman, J. Bentsman, and S. M. Meerkov, "Stability of fast periodic systems," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 289–291, Mar. 1985.
[2] ——, "Vibrational control of nonlinear systems: Vibrational stabilizability," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 710–716, Aug. 1986.
[3] ——, "Vibrational control of nonlinear systems: Vibrational controllability and transient behavior," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 717–724, Aug. 1986.
[4] S. Lee, S. M. Meerkov, and T. Runolfsson, "Vibrational feedback control: Zeros placement capabilities," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 604–611, July 1987.
[5] J. Hale, "Averaging methods for differential equations with retarded arguments and a small parameter," *J. Diff. Equations*, vol. 2, pp. 57–73, 1966.
[6] R. D. Driver, *Ordinary and Delay Differential Equations.* New York: Springer-Verlag, 1977.
[7] K. W. Neves, "Automatic integration of functional differential equations: An approach," *ACM Trans. Math. Soft.*, vol. 1, no. 4, pp. 357–368, 1975.

The Recursive Algorithm for the Optimal Static Output Feedback Control Problem of Linear Singularly Perturbed Systems

Z. GAJIC, DJ. PETKOVSKI, AND N. HARKARA

Abstract—**The recursive algorithm is developed for solving the algebraic equations comprising the solution of the optimal static output feedback control problem of singularly perturbed linear systems. The proposed algorithm is very efficient from the numerical point of view, since only low-order systems are involved in algebraic calculations and the required solution can be easily obtained up to an arbitrary order of accuracy, that is, $O(\epsilon^k)$ where ϵ is a small perturbation parameter. The real world example demonstrates the failure of $O(\epsilon)$ theory—used so far in the study of this problem, and the necessity for the existence of $O(\epsilon^k)$ theory.**

Manuscript received November 30, 1987; revised April 11, 1988. Paper recommended by Past Associate Editor, J. H. Chow. The work of Dj. Petkovski was supported in part by the U.S.-Yugoslav Scientific and Technological Cooperation under Grants PP-727 (DOE) and PP-736 (NSF).
Z. Gajic and N. Harkara are with the Department of Electrical and Computer Engineering, Rutgers University, Piscataway, NJ 08855.
Dj. Petkovski is with the Faculty of Technical Sciences, University of Novi Sad, 21000 Novi Sad, Yugoslavia.
IEEE Log Number 8826374.

I. Introduction

In the early 1970's, increasing attention was given to the problem of designing output constrained regulators where a very limited number of state measurements are available for control implementation (e.g., [1]–[4]). The optimal solution for this control problem is obtained in terms of high-order nonlinear matrix algebraic equations. The convergence complexities of the algorithms proposed for the solution of these equations have hindered for quite a long time a wider application of this technique. Recently, the convergence problem was solved in [5].

The output feedback control problem attracted the attention of the researchers from the field of singular perturbations in the 1980's [6]–[10], [14]. It is well known that the singularly perturbed systems belong to the class of systems with ill-conditioned dynamics which makes corresponding numerical problems stiff. Thus, in addition to the high-order nonlinear matrix algebraic equations, one is faced with the ill-defined numerical problems also.

Motivated by the results of [11]–[13] and [5], we have developed the well-defined recursive numerical technique for the solution of nonlinear algebraic matrix equations associated with the output feedback control problem of linear-quadratic singularly perturbed systems. Moreover, the numerical slow-fast decomposition has been achieved so that only low-order systems are involved in algebraic computations. It is shown that each iteration step of the proposed algorithm improves the accuracy by an order of magnitude, that is, the accuracy of $O(\epsilon^k)$ where ϵ is a small perturbation parameter, can be obtained by performing only k iterations. This represents a significant improvement since all results on the output feedback control problems for the singularly perturbed systems have been obtained so far with an accuracy of $O(\epsilon)$ only.

The real world example, an industrially important reactor, which demonstrates the efficiency of the proposed algorithm and the failure of $O(\epsilon)$ theory is included in the note.

II. Output Feedback Control for Singularly Perturbed Linear Systems

Consider the singularly perturbed linear system [15]

$$\dot{x}_1 = A_1x_1 + A_2x_2 + B_1u, \qquad x_1(t_0) = x_{10} \tag{1}$$

$$\epsilon\dot{x}_2 = A_3x_1 + A_4x_2 + B_2u, \qquad x_2(t_0) = x_{20} \tag{2}$$

$$y = C_1x_1 + C_2x_2 \tag{3}$$

where $x_1 \in R^{n_1}$ and $x_2 \in R^{n_2}$ are state vectors, $u \in R^m$ is a control input, and $y \in R^r$ is a measured output. In the following, A_i, B_j, and C_j, $i = 1, \cdots, 4$, $j = 1, 2$ are constant matrices of compatible dimensions; in general, they are continuous functions of a small positive parameter ϵ [11]. With (1)–(3), consider the performance criterion

$$J = \int_0^\infty \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T Q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u^TRu \right\} dt \tag{4}$$

with positive definite R and positive semidefinite Q, which has to be minimized. In addition, the control input $u(t)$ is constrained to

$$u(t) = Fy(t). \tag{5}$$

The optimal constant output feedback gain F is given by [1]

$$F = R^{-1}B^TKLC^T(CLC^T)^{-1} \tag{6}$$

where matrices K and L satisfy high-order nonlinear coupled algebraic equations

$$(A - BFC)L + L(A - BFC)^T + x_0x_0^T = 0 \tag{7}$$

$$(A - BFC)^TK + K(A - BFC) + Q + C^TF^TRFC = 0 \tag{8}$$