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Vibrational Feedback Control of Time Delay Systems

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Abstract— **This paper applies vibrational feedback control to time lag systems. Both stabilization and transient issues are discussed. An illustrative example is given which demonstrates that the proposed controller: 1) provides superior system gain and phase margin in comparison to time invariant controllers when applied correctly, 2) is not robust with respect to unknown delays and 3) does not have zero placement capabilities, in the sense defined in the paper.**

I. INTRODUCTION

Recently, there has been a great deal of research interest in showing that systems with time varying periodic controllers can have superior robustness properties in comparison to time invariant controllers [1-3], [9]. In particular, periodic controllers, for both continuous and discrete systems, have demonstrated capabilities of arbitrarily improving the gain margins for classes of LTI plants [2-4], [9]. Likewise, periodic controllers have been shown to stabilize systems with decentralized fixed modes [1], [5].

This paper proposes to use the techniques of vibrational feedback control, introduced in [6], to design periodic controllers for time delay systems. The results of this paper can be obtained due to recent advances in open loop vibrational control [7] and new stability results for delay equations [8].

It should be noted that most of the literature for periodic controllers centers around stabilization issues for finite dimensional plants [2], [4-6], [9]. This research differs from these types of results in two significant ways. First, the possibility that the plant includes a measurement and/or actuator delay is examined. Hence, the problem becomes infinite dimensional. Second, new methods to control the transient behavior of the system are introduced. For example, using the proposed techniques, it is now possible to control the rise time (in the sense described below) of a periodically controlled delayed plant with a step input. Of course, all results in this paper can be applied to the ODE case by setting the delay equal to zero. Hence, the results of this paper represent new techniques in the control of ODE's as well.

The formulation and the design of vibrational feedback control for time lag systems is presented in Sections II and III. The results obtained are, at times, surprising. For example, much of the research on periodic controllers centers around apparent zero placement capabilities [2-4, 6]. However, this research demonstrates that vibrational feedback controllers do not have zero placement capabilities, in the sense discussed in this paper, even for the ODE case.

Another important, and perhaps surprising, result is the lack of robustness that the vibrational feedback controllers proposed in [6] have with respect to time delays. In Section IV, an example from [4], [6] is reworked and shown to be unstable with delay of $\tau = 0.02$, even though the system has gain margin over 20 for zero delay. Perhaps more unusual is the fact that for larger delay, e.g., $\tau = 0.12566$, the system becomes stable again. New control algorithms are presented in this paper to compensate for the delay, and applications to robustness problems are also introduced in Section III and IV. An example is given to demonstrate that the algorithms in this paper can have superior performance over known finite dimensional time invariant controllers.

II. Controllers and Problem Formulation

Consider a SISO time-invariant plant with time delay having open loop transfer function $G_p(s)e^{-s\tau}$. In state space form, this system can be written as

$$\dot{x}(t) = Ax(t) + Bu(t)$$

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$$y(t) = Cx(t-\tau) + du(t-\tau), \qquad (2.1)$$

where $x \in \Re^n$ is the state, $u \in \Re$ is the input, and $y \in \Re$ is the output. In this research, all delays are lumped together in the output equation as above. That is, the sum of the measurement, computation, and actuator delay is equal to τ in (2.1).

A. *Controller*

Consider a periodic controller with unity feedback in the form of [6]

$$\begin{aligned} \dot{x}_c(t) &= [F + \frac{1}{\varepsilon}F_0(t/\varepsilon)]x_c(t) + Ge(t) \\ u(t) &= [K + \frac{1}{\varepsilon^r}K_r(t/\varepsilon)]x_c(t) \qquad (2.2) \\ e(t) &= l(t) - y(t) \end{aligned}$$

where $l \in \Re$ is the reference input, $F_0(t/\varepsilon)$ and $K_r(t/\varepsilon)$ are periodic zero average matrices, $x_c \in \Re^n$, $0 < \varepsilon \ll 1$, and r is the relative degree of the system defined as

$$r = \begin{cases} 0 & \text{if } d \neq 0 \\ \min\{k : CA^{k-1}B \neq 0, k = 1, \cdots, n\} & \text{if } d = 0. \end{cases} \qquad (2.3)$$

Additionally,

$$F = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ f_1 & f_2 & \cdots & f_n \end{bmatrix}, \qquad (2.4)$$

$$F_0(t/\varepsilon) = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \alpha(\frac{t}{\varepsilon}) \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T,$$

and

$$\begin{aligned} K &= [k_1 \; k_2 \cdots k_n], \qquad (2.5) \\ K_r(\frac{t}{\varepsilon}) &= [\beta_1(t/\varepsilon) \; \ldots \; \beta_{n-r}(t/\varepsilon) \quad 0 \ldots 0]. \end{aligned}$$

Note that ε is proportional to the period of $F_0(t/\varepsilon)$ and $K_r(t/\varepsilon)$.

It is assumed that $\alpha(t)$ and $\beta(t)$ as well as $\gamma_i(t) \equiv \int \cdots \int \beta_i(t)(dt)^r$ and $p(t) \equiv \exp\{\int \alpha(t)dt\}$ are all scalar functions satisfying the following conditions:

$$\begin{aligned} \overline{\alpha(t)} &= \overline{\beta_i(t)} = \overline{\gamma_i(t)} = 0, \overline{p(t)} = \overline{p^{-1}(t)} \neq 0, \\ \overline{p(t)\beta_i(t)} &= \overline{-p^{-1}(t)\beta_i(t)} \neq 0, \\ \overline{p(t)\gamma_i(t)} &= \overline{-p^{-1}(t)\gamma_i(t)} \neq 0, \quad i = 1, \cdots, n-r, \end{aligned}$$

where $\overline{q(t)} \equiv \lim_{T \longrightarrow \infty} \frac{1}{T} \int_t^{T+t} q(s)ds$. The above conditions are used for technical reasons in order to simplify the presentation. One set of $\alpha(t)$ and $\beta_i(t)$ that satisfy the conditions are [6]

$$\alpha(t) = \cos(t)$$

$$\beta_i(t) = \begin{cases} k_i^{(r)} \sin(t) & \text{if } r \text{ is even} \\ k_i^{(r)} \cos(t) & \text{if } r \text{ is odd.} \end{cases}$$

In this case, the closed loop equation of (3.1) and controller (3.2) is given as

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix} &= \begin{bmatrix} A & B[K + \frac{1}{\varepsilon^r}K_r(t/\varepsilon)] \\ 0 & F + \frac{1}{\varepsilon}F_0(t/\varepsilon) \end{bmatrix} \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ -GC & -dG[K + \frac{1}{\varepsilon^r}K_r(\frac{t-\tau}{\varepsilon})] \end{bmatrix} \begin{bmatrix} x(t-\tau) \\ x_c(t-\tau) \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ G \end{bmatrix} l(t) \\ y(t) &= \begin{bmatrix} C & d[K + K_0(\frac{t-\tau}{\varepsilon})] \end{bmatrix} \begin{bmatrix} x(t-\tau) \\ x_c(t-\tau) \end{bmatrix}. \end{aligned} \qquad (2.6)$$

Definition 3.1: Open loop system (2.1) with the above controller (2.2) will be referred to as $\sum_1$.

B. *Time Invariant System*

Along with $\sum_1$ and the closed loop equation (2.6), introduce the SISO LTI closed loop delay system with unity feedback given by

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}\hat{u}(t) \\ \hat{y}(t) &= \hat{C}\hat{x}(t-\tau) \qquad (2.7) \\ \hat{u}(t) &= \hat{l}(t) - \hat{K}\hat{x}(t-\tau), \end{aligned}$$

where $\hat{x} \in \Re^{2n}$, $\hat{u} \in \Re$, $\hat{l} \in \Re$ is the reference, $\hat{y} \in \Re$ is the output, and τ is as defined in (2.1). Assume that $\hat{x}(t) = [x^T(t),\ x_c^T(t)]^T = \theta(t)$ for $t \in [-\tau, 0]$, where x and x_c are as defined in (2.1) and θ is a continuous function.

Definition 3.2. Closed loop system (2.7) will be referred to as $\sum_2$.

Utilizing the technique of averaging, this research demonstrates that the output of $\sum_1$ can be "approximated" by the output of a time invariant system in the form $\sum_2$. In this manner, robustness properties of $\sum_2$ can be used as a measure of robustness of corresponding time varying systems.

Definition 3.3. Let $Q(t, \varepsilon)$ be a $2n \times 2n$ periodic matrix with bounded inverse, and let, in (2.6), $[x^T(t), x_c^T(t)]^T = Q(t, \varepsilon)\varsigma(t)$. Assume that this change of variables transforms (2.6) into an algebraically equivalent dynamical system

$$
\begin{aligned}
\dot{\varsigma}(t) &= A_0(t,\varepsilon)\varsigma(t) + A_1(t,\varepsilon)\varsigma(t-\tau) + G_0(t,\varepsilon)l(t) \\
y(t) &= C_0(t,\varepsilon)\varsigma(t-\tau),
\end{aligned}
$$

where A_0, A_1, G_0 and C_0 are of appropriate dimensions.

For any given fixed $\delta > 0$ and any $L > 0$, $\sum_1$ and $\sum_2$ are said to be *δ–equivalent* if $|\overline{y}(t) - \hat{y}(t)| < \delta$, $t \in [0, L]$, where $\hat{y}(t)$ is the output of $\sum_2$ and

$$
\overline{y}(t) = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} C_0(s,\varepsilon)\varsigma(t-\tau)ds.
$$

When $L = \infty$, $\sum_1$ and $\sum_2$ are said to be globally δ–equivalent.

Remark 3.1. The concept of δ–equivalence gives a measure of how closely the *moving averaged output* of the time varying system $\sum_1$ is approximated by the output of the time invariant system $\sum_2$. There are few general classes of time varying systems in which substantial information on dynamic behavior can be gained. By approximating the behavior of a time varying system to the behavior of a time invariant system, controller parameters can be designed based on the time invariant system. This simplifies the problem. Comparisons between $y(t)$ of (2.1) with $\overline{y}(t)$ and $\hat{y}(t)$ reveal the change of transient behavior due to periodic control.

In an effort to develop concepts of performance for time varying system $\sum_1$, consider the following definitions.

Definition 3.4. $\sum_1$ is said to have *δ–equivalent rise time*, $t_{\delta r}$, if: 1) $\sum_1$ is δ–equivalent to $\sum_2$ and 2) $\sum_2$ has a rise time t_r.

Definition 3.5. $\sum_1$ is said to have a *δ–equivalent zero* at $z \in C$ if: 1) $\sum_1$ is δ–equivalent to $\sum_2$ and 2) $\sum_2$ has a zero at z. (In this paper, we say that $\sum_2$ has a zero at z when its transfer function $\frac{n(s)}{d(s)}e^{-s\tau}$ has $n(z) = 0$.)

III. δ–Equivalence

This section will demonstrate how it is possible to relate properties of $\sum_1$to a time invariant system given by $\sum_1$, in the sense of δ–equivalence. Controller parameters can then be chosen through the analysis of the time invariant systems.

A. *Controller With Relative Degree Zero*

For simplicity, first consider $\sum_1$ when $d \neq 0$ and $r = 0$.

Theorem 3.1 : *Assume $l(t)$ is a step input, and $d \neq 0$. Suppose that $\Phi(t)$ is a fundamental matrix for the ODE $\dot{x}(t) = F_0(t)x(t)$, where $F_0(t)$ is defined in (2.4). Then, for any $L \geq 0$ and any $\delta > 0$, there exists an $\varepsilon_0 = \varepsilon_0(\delta, L) > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, $\sum_1$ is δ–equivalent to $\sum_2$ as given by (2.6) with*

$$
\begin{aligned}
\hat{A} &= \begin{bmatrix} A & B[K\overline{\Phi(t/\varepsilon)} + \overline{K_0(t/\varepsilon)\Phi(t/\varepsilon)}] \\ 0 & \overline{\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)} \end{bmatrix} \\
\hat{K} &= \overline{p^{-1}(t/\varepsilon)}C \quad d[\overline{Kp^{-1}(t/\varepsilon)\Phi(\tfrac{t-\tau}{\varepsilon})} \\
&\quad \overline{+K_0(\frac{t-\tau}{\varepsilon})p^{-1}(t/\varepsilon)\Phi(\frac{t-\tau}{\varepsilon})}] \\
\hat{C} &= \begin{bmatrix} C & d[\overline{K\Phi(\tfrac{t-\tau}{\varepsilon})} + \overline{K_0(\tfrac{t-\tau}{\varepsilon})\Phi(\tfrac{t-\tau}{\varepsilon})}] \end{bmatrix} \\
\hat{B} &= \begin{bmatrix} 0 \\ G \end{bmatrix}, \quad \hat{l}(t) = \overline{p^{-1}(t/\varepsilon)}l(t). \qquad (3.1)
\end{aligned}
$$

B. *Controller With Nonzero Relative Degree*

Theorem 3.2 : *Assume $l(t)$ is a step input, and $d = 0$. Suppose that $\Phi(t)$ is a fundamental matrix for the ODE $\dot{x}(t) = F_0(t)x(t)$, where $F_0(t)$ is defined in (2.4). For any $L \geq 0$ and any $\delta > 0$ there exists an $\varepsilon_0 = \varepsilon_0(\delta, L) > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, $\sum_1$ is δ–equivalent to $\sum_2$ as given by (2.7) with*

$$
\hat{A} = \begin{bmatrix} A & BK\overline{\Phi(t/\varepsilon)}) + V \\ 0 & \overline{\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)} \end{bmatrix}
$$

$$
\hat{K} = \begin{bmatrix} \overline{p^{-1}(t/\varepsilon)}C & \overline{p^{-1}(t/\varepsilon)CA^{r-1}BL_r(\tfrac{t-\tau}{\varepsilon})} \end{bmatrix}
$$

$$
\begin{aligned}
\hat{B} &= \begin{bmatrix} 0 \\ G \end{bmatrix}, \hat{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \\
\hat{l}(t) &= \overline{p^{-1}(t/\varepsilon)}l(t). \qquad (3.5)
\end{aligned}
$$

where $V = (-1)^{r-1}B\overline{L_r(t/\varepsilon)F^{r-1}\Phi^{-1}(t/\varepsilon)F\Phi(t/\varepsilon)}$ and $L_{i+1}(t) = \int L_i(t)dt$, $i = 1, 2, \ldots, r-2$, *$L_0(t) = K_r(t)$, and $p(t)$ is given in (2.5).*

C. *δ–Equivalent Zeros*

In the results of [6], it is proposed to use vibrational feedback control to move the open loop zeros of a corresponding averaged equation. In this sense, vibrational feedback control (when $\tau = 0$) was demonstrated to be helpful in problems of finite gain margin and decentralized fixed modes. However, in this subsection it is shown that the δ–equivalent zeros of $\sum_1$ always contain the open loop zeros of (2.1).

This implies that vibrational periodic control does not actually have zero placement capabilities in the sense of $\delta-$equivalence.

Theorem 3.3: *The zeros of the open loop system (2.1) are contained in the $\delta-$equivalent zeros of the closed loop time varying system $\sum_1$.*

Remark 3.1: The results of Theorem 3.3 are surprising. Several authors have claimed that periodic controllers have arbitrary zero placement capabilities [2, 3, 6]. In fact, such a claim was made in [6] for vibrational feedback control when $\tau = 0$. Clearly, such claims rely heavily on what is meant by zeros of a time-varying system.

We suggest that the averaged output of the periodic controlled system should be closely approximated by the output of a time invariant system in order to introduce a notion of a "zero" of a time-varying periodic system. In this manner, this research is proposing a new definition for zeros of periodic systems.

D. Controller Design

In this section, techniques are proposed that may be used to select the parameters in (2.2)-(2.5) in order to obtain a desired response for $\sum_1$. First, conditions for global $\delta-$equivalence between $\sum_1$ and $\sum_2$ are presented. The controller gains are designed based on the response of $\sum_2$ which is time invariant. In this way it is possible to control $\delta-$ equivalent time domain specifications for $\sum_2$.

Theorem 3.4: *Assume $l(t)$ is a step input and $d \neq 0$. Let $\hat{A}, \hat{B}, \hat{K}$, and $\hat{l}$ be as given in (3.1) and assume $\tau > 0$ is fixed. Suppose that there exist constants η_1 and η_2, $0 < \eta_1 < \eta_2$, such that for any $\varepsilon \in [\eta_1, \eta_2]$, $\det[sI - \hat{A} + \hat{B}\hat{K}e^{-\tau s}] = 0$ has all solutions with negative real parts. Then for any fixed $\delta > 0$ there exists an $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that, when $\varepsilon \in [\eta_1, \eta_2]$ and $\varepsilon < \varepsilon_0$, $\sum_1$ and $\sum_2$ are globally $\delta-$equivalent.*

Theorem 3.5: *Assume $l(t)$ is a step input and $d = 0$. Let $\hat{A}, \hat{B}, \hat{K}$, and $\hat{l}$ be as given in (3.2) and assume $\tau > 0$ is fixed. Suppose that there exist constants η_1 and η_2, $0 < \eta_1 < \eta_2$, such that for any for $\varepsilon \in [\eta_1, \eta_2]$, $\det[sI - \hat{A} + \hat{B}\hat{K}e^{-\tau s}] = 0$ has all solutions with negative real parts. Then for any fixed $\delta > 0$, there exists an $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that, when $\varepsilon \in [\eta_1, \eta_2]$ and $\varepsilon < \varepsilon_0$, $\sum_1$ and $\sum_2$ are globally $\delta-$equivalent.*

Remark 3.2: The above two theorems allow for the control of both transient response and stability of $\sum_1$ by examining $\sum_2$ and either (3.1) or (3.5) (depending on the relative degree). For example, it is possible to design a $\delta-$equivalent rise time by controlling the rise time of $\sum_2$. Asymptotic stability of $\sum_1$ is equivalent to designing a controller so that the conditions in Theorem 4.4 and Theorem 3.5 are satisfied. In fact, under the conditions of Theorem 3.4 and 3.2, $y(t)$ of $\sum_1$ will approach a periodic orbit in the vicinity of the steady state value of $\hat{y}(t)$. In terms of $\overline{y}(t)$, this implies that $\lim_{t\to\infty} \overline{y}(t) \approx \hat{C}(-\hat{A} + \hat{B}\hat{K})^{-1}\hat{B}\hat{l}(t)$, where $\hat{l}(t) =$ constant.

Alternatively, the technique [6] can be modified to design a stabilizing controller.

Theorem 3.6: *Assume $l(t)$ is a step input, and define $\hat{G}_{\hat{K}}(s) = \hat{K}(sI - \hat{A})\hat{B}$, where $\hat{A}, \hat{B}$ and $\hat{K}$ are as given in (3.1) or (3.2) (depending on relative degree). Suppose that there exist positive constants β, η_1 and η_2, $0 < \eta_1 \le \eta_2$, such that for any $\varepsilon \in [\eta_1, \eta_2]$, the plant $\hat{G}_{\hat{K}}(s)$ has positive phase margin, ϕ_m, satisfying $\phi_m > \beta$. Finally suppose that $0 \le \tau < \beta/\omega_\phi$, where ω_ϕ is the smallest frequency satisfying $\left|\hat{G}_{\hat{K}}(j\omega_\phi)\right| = 1$, and τ is the delay of the system.*

Then for any $\delta > 0$, there exists an $\varepsilon_0 = \varepsilon_0(\delta)$ such that, for $\varepsilon \in [\eta_1, \eta_2]$ and $\varepsilon < \varepsilon_0, \sum_1$ is globally $\delta-$equivalent to $\sum_2$.

Remark 3.4: An algorithm for controlling (2.1) can now be derived. Parameters for the periodic compensation can be selected so that the conditions of Theorem 3.6 are true. It should be noted that when $\tau = 0$, [6] has shown that the zeros of $\hat{G}_{\hat{K}}(s)$ can be arbitrarily placed provided that (A, B, C) in (2.1) is controllable and observable. This in no way implies that $\delta-$equivalent zeros of $\sum_1$ can be arbitrarily placed, as we previously showed. However, zero placement of $\hat{G}_{\hat{K}}(s)$ does allow for the phase margin of $\hat{G}_{\hat{K}}(s)$ to increase, making global $\delta-$equivalence possible when $\tau = 0$. When $\tau \neq 0$, this statement is not necessarily true unless the controller is "tuned" in a special manner, as below. In order to apply these methods, it is beneficial to fix ε to be a specific value, related to the period and delay. This is demonstrated in the following theorem.

Theorem 3.7: *Let $\hat{G}_{\hat{K}}(s)$ be as in Theorem 3.6. Assume that $F_0(t)$ and $K_r(t)$ are $T-$periodic, and suppose that (A, B, C) in (2.1) is controllable and observable when $\tau = 0$. Then the zeros of $\hat{G}_{\hat{K}}(s)$ can be arbitrarily placed provided that $\tau/\varepsilon = nT$, where n is any non-negative integer and τ is the fixed non-negative constant delay.*

Remark 3.5: When $\tau/\varepsilon \neq nT$, zero placement capabilities of $\hat{G}_{\hat{K}}(s)$ may not be possible. However, by tuning ε in a sufficient manner, the zeros of $\hat{G}_{\hat{K}}(s)$ can be moved arbitrarily. Often this will permit an increase in ϕ_m so that $\phi_m > \omega_\phi \tau$ in Theorem 3.7. While there are no guarantees (necessary and sufficient algorithms) that this procedure always works for general systems, in many applications it is successful. To summarize, one possible design procedure is:

Step1: Assume $\tau = 0$, and select controller gains so that $\hat{G}_{\hat{K}}(s)$ has large phase margin and small ω_ϕ. This can usually be accomplished by using the techniques of zero placement proposed in [6].

Step 2: Verify that $\phi_m > \omega_\phi \tau$ where τ is the fixed positive delay. If not, change the controller parameters and repeat step 1.

Step 3: Select $\varepsilon = \varepsilon_1$ such that $\tau/\varepsilon_1 = nT$, where ε_1 is sufficiently small and n is a positive integer.

IV. Example And Robustness

A. *Example*

Consider the system

$$\begin{aligned} \dot{x}(t) &= x(t) + u(t) \\ y(t) &= -2x(t-\tau) + u(t-\tau), \end{aligned}$$

which has transfer function given by $G(s) = G_p(s)e^{-\tau s} = \frac{(s-3)e^{-\tau s}}{(s-1)}$.

For $\tau = 0$, this plant was examined in [6]. Since the plant has a delay, a zero in the right half plane, and a pole in the right half plane, time invariant controllers have limited capabilities [4], [7]. In fact, the delay in the system makes stabilization of this plant by previously known finite dimensional techniques extremely difficult (we could find no previously published techniques in the literature that could adequately stabilize and control this plant with finite dimensional unity output feedback). We will now show that the proposed vibrational feedback controllers can robustly stabilize this system. Employ periodic controller (2.2) with $r = 0$ in the form of

$$\begin{aligned} \dot{x}_c(t) &= [f + \frac{a}{\varepsilon}\cos(t/\varepsilon)]x_c(t) + e(t) \\ u(t) &= [k + k^{(0)}\sin(t/\varepsilon)]x_c(t), \\ e(t) &= l(t) - y(t), \end{aligned}$$

where it is assumed that $l(t)$ is a step input. The following closed loop averaged equation is obtained a

$$\begin{aligned} \dot{\hat{x}}(t) &= \begin{bmatrix} 1 & \chi \\ 0 & f \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}(t) \\ \hat{y}(t) &= \begin{bmatrix} -2 & -[k\overline{p(t/\varepsilon)} + k^{(0)}\overline{q(t/\varepsilon)}] \end{bmatrix} \hat{x}(t-\tau) \end{aligned}$$

and $\hat{u}(t) = \hat{l}(t) - [-2\overline{p(t/\varepsilon)} \quad \overline{[kp^{-1}(t/\varepsilon)p(\frac{t-\tau}{\varepsilon}) + +k^{(0)}\sin(\frac{t-\tau}{\varepsilon})p^{-1}(t/\varepsilon)p(\frac{t-\tau}{\varepsilon})}]]\hat{x}(t - \tau)$ where $p(t/\varepsilon) = e^{\alpha\sin(t/\varepsilon)}$, $q(t/\varepsilon) = \sin(t/\varepsilon)e^{\alpha\sin(t/\varepsilon)}$, $\hat{l}(t) = \overline{p^{-1}(t/\varepsilon)}l(t)$, and $\chi = k\overline{p(t/\varepsilon)} + k^{(0)}\overline{q(t/\varepsilon)}$ The closed loop transfer function of $\sum_2$ is given by

$$\hat{H}(s,\tau) = \frac{[k\overline{p(t/\varepsilon)} + k^{(0)}\overline{q(t/\varepsilon)}](s-3)e^{-\tau s}}{(s-1)(s-f-Me^{-\tau s}) - 2\overline{p(t/\varepsilon)}Ne^{-\tau s}}$$

where $N = [k\overline{p(t/\varepsilon)} + k^{(0)}\overline{q(t/\varepsilon)}]$ and $M = \overline{[kp^{-1}(t/\varepsilon)p(\frac{t-\tau}{\varepsilon}) + k^{(0)}\sin(\frac{t-\tau}{\varepsilon})p^{-1}(t/\varepsilon)p(\frac{t-\tau}{\varepsilon})}]$. Notice that the zero at $s = 3$ has not been moved. This is consistent with Theorem 3.3 which states that the $\delta-$equivalent zeros contain the open loop plant zeros.

In order to select proper parameters for a stabilizing controller, the steps of Remark 3.5 can be followed directly.

STEP 1. Assume $\tau = 0$. Then

$$\begin{aligned} \hat{G}_{\hat{K}}(s) &= \begin{bmatrix} -2\overline{p(t/\varepsilon)} & k \end{bmatrix} \\ &\times \begin{bmatrix} s-1 & -[\chi] \\ 0 & s-f \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (4.1) \\ &= \frac{-[2\overline{p(t/\varepsilon)}(\chi) - k(s-1)]}{(s-1)(s-f)}. \end{aligned}$$

The goal, now, is to select the pole and zero of $\hat{G}_{\hat{K}}(s)$ in such a manner as to maximize the ratio ϕ_m/ω_ϕ. Only the pole at s = -1 in (4.1) is fixed. After a few trial and error iterations on MATLAB, it is found that choosing a pole at s = -3 and a zero at s= -4 has high ratio ϕ_m/ω_ϕ. Then straightforward calculations, utilizing (5.1), yields the parameters $k = 1$, $k^{(0)} = -5.743271, \alpha = 1$, $f = -3$, which, in turn, yields $\overline{p} = 1.26607$ and $\overline{q} = 0.56515$. This gives

$$\hat{G}_{\hat{K}}(s) = \frac{s+4}{(s-1)(s+3)},$$

as desired.

STEP 2. Calculate ϕ_m and ω_ϕ. In this case $\phi_m = 0.637$ (rad) and $\omega_\phi = 0.8486$. Therefore, $\phi_m > \tau\omega_\phi$ for any fixed τ satisfying $0 < \tau < 0.75$.

STEP 3. Choose $\varepsilon = \varepsilon_1$ fixed and sufficiently small to satisfy $\frac{\tau}{\varepsilon_1} = 2\pi n$, where n is a sufficiently large positive integer. Then, $y(t)$ is globally $\delta-$equivalent to $\hat{y}(t)$ for fixed positive $\tau < 0.75$, i.e., the system output is bounded, and $\overline{y}(t)$ can be approximated by $\hat{y}(t)$. Hence, choosing the controller parameters as above will stabilize the system.

B. *Lack of Robustness with Respect to Delay*

When the delay in the above system is assumed to be zero and the reference input is zero, then the trivial solution in $\sum_1$ is asymptotically stable for $\varepsilon = \frac{1}{50}$. This has been shown in [6] for similar parameters. However, for the case when $\tau \neq 0$, the transfer function for $\hat{G}_{\hat{K}}(s)$ is given by

$$\hat{G}_{\hat{K}}(s) = \frac{-[2\overline{p(t/\varepsilon)}(\chi) - M(s-1)]}{(s-1)(s-f)},$$

where M and χ are defined previously in this section. Consider the case when $\tau = 0.02$. Then, for the controller parameters above, i.e., $k = 1$, $k^{(0)} = -5.734271$, $f = -3$, and $\varepsilon = \frac{1}{50}$, the transfer function is

$$\hat{G}_{\hat{K}}(s) = \frac{[5 - 0.232(s-1)]}{(s-1)(s+3)}.$$

There is both an open-loop zero and open-loop pole in the right half plane, obviously implying that the output of the system is unstable. However, for $\tau = 2\pi n\varepsilon \leq 0.75$, $n = 0, 1, \ldots$, the output, $y(t)$, approaches an asymptotically stable periodic orbit for fixed $\varepsilon > 0$ sufficiently small. That is, the system can be stable for fixed delays larger than 0.02, even though it is unstable for $\tau = 0.02$. (As previously mentioned, the output is stable for $\tau = 0$.) This type of stability-instability sensitivity to the delay appears to be unique to fast time varying systems with delay and illustrates importance of exact modelling of the delay when employing vibrational feedback control. In order to guarantee that the output remains bounded for $\tau = 0.02$ and a step input, the parameter ε should be tuned so that $\varepsilon = \frac{\tau}{2\pi n}$. For example, when $\tau = 0.02$, one value of ε which guarantees stability is $\varepsilon = 0.003183$. In this case, $\hat{G}_{\hat{K}}(s)$ is as given in (4.1) and $y(t)$ becomes bounded.

V. Conclusions

This paper extends the technique of vibrational feedback control to systems with delay. Averaging theory for differential delay equations is presented and then applied to aid in the controller design. The method of control design is introduced and examples are presented to illustrate issues such as stability, gain margin, and robustness with respect to delay. The results of this research indicate that vibrational feedback control for delay systems can substantially improve performance when applied correctly.

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