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Extensions of Classical Averaging Techniques to Delay Differential Equations

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ABSTRACT

This paper extends the KBM method of averaging to delay differential equations. Near identity change of variables are used to transform time varying delay differential equations into autonomous delay differential equations plus small perturbations. Then Lyapunov functionals are used to relate the autonomous averaged delay differential equation to the original time varying delay differential equation.

1. INTRODUCTION

One of the most important methods of determining the behavior of periodic and almost periodic differential equations which contain a small parameter is the so-called method of averaging. By making an asymptotic expansion about a small parameter, authors [1–3] have shown that behavior of solutions to classes of time varying differential equations can be approximated by behavior of solutions to corresponding autonomous differential equations. Averaging techniques have found numerous applications in areas including adaptive control, bifurcation theory, celestial mechanics, noise control, nonlinear oscillations, stability analysis, time varying controllers and vibrational control, among many other fields.

The method of averaging is most often credited to the work of Krylov and Bogoliubov [1] and to Bogoliubov and Mitropolskii [2], and has been referred to by many authors as the KBM method of averaging (Krylov–Bogoliubov–Mitropolskii). These authors considered the time varying ODE

$$x'(\tau) = \varepsilon f(\tau, x, \varepsilon), \tag{1.1}$$

where $f: \Re \times \Re^n \times [0, \infty) \to \Re^n$ is continuous and $f(\tau, x, \varepsilon)$ is often assumed to be almost periodic in τ with respect to x in compact sets for fixed constants ε. By making the change of variables close to the identity of: $x = z + \varepsilon u(\tau, z, \varepsilon)$, (1.1) is transformed into

$$z'(\tau) = \varepsilon f_0(z) + \varepsilon g(\tau, z, \varepsilon),$$

where $f_0(z) = \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} f(s, z, 0)ds$, and u is given by

$$u(\tau, z, \varepsilon) = \int_{-\infty}^{\tau} e^{-\varepsilon(\tau-\lambda)}[f(\lambda, z, 0) - f_0(z)]d\lambda .$$

It can be shown that when f satisfies the proper conditions, $g(\tau, z, \varepsilon)$ is a Lipschitz perturbation with the property that $g(\tau, z, 0) = 0$. By using basic Lyapunov theory, it is then possible to relate stability properties of the averaged equation

$$y'(\tau) = \varepsilon f_0(y) \tag{1.2}$$

to stability properties of (1.1) when ε is sufficiently small. Additionally, if the initial conditions of (1.1) and (1.2) are chosen properly, then, for sufficiently small ε, the solutions to (1.1) and (1.2) remain arbitrarily close to each other on finite (and sometimes on infinite) time intervals.

In the 1960's, authors such as Hale [4] and Halanay [5] studied averaging for delay differential equations in forms similar to

$$x'(\tau) = \varepsilon f(\tau, x(\tau), x(\tau - r), \varepsilon), \tag{1.3}$$

where $r \geq 0$ is the constant delay. These authors [4,5] gave conditions in which, for sufficiently small ε, stability properties of solutions of (1.2) are the same as stability properties of the equilibrium points of the ODE

$$y'(\tau) = \varepsilon f_0(y(\tau), y(\tau)), \tag{1.4}$$

where once again, f_0 is an 'average value' of f, given by

$$f_0(z, z) = \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} f(s, z, z, 0)ds .$$

Recently Hale and Verduyn Lunel [6,7] and Lehman et. al. [8,9] have examined the delay differential equation

$$x'(t) = f(t/\varepsilon, x(t), x(t - r), \varepsilon), \tag{1.5}$$

and have related stability and transient properties of (1.5) to the corresponding autonomous system

$$y'(t) = f_0(y(t), y(t - r)) \tag{1.6}$$

where $f_0(z_1, z_2) = \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} f(s, z_1, z_2, 0)ds$. Here, the information on the delay is retained in the averaging process, as opposed to (1.3) and (1.4).

The main purpose of this paper is to extend the KBM method of averaging to both (1.3) and (1.5). By this, we mean that near identity transformations in $\Re^n$ will be used to relate stability properties of time varying differential delay equations to corresponding autonomous differential delay equations. An interesting consequence of this new technique is an improvement on already existing averaged models. In particular, this paper shows that a more accurate averaged equation of (1.3) is given by

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$$y'(\tau) = \varepsilon f_0(y(\tau), y(\tau - r)), \tag{1.8}$$

where once again, f_0 is an 'average value' of f, given by

$$f_0(z_1, z_2) = \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} f(s, z_1, z_2, 0)ds \ .$$

We believe that the techniques and proofs presented in this paper are simpler than standard averaging techniques and proofs for delay equations. More importantly, new averaging algorithms are proposed which provide a manner to estimate the first harmonic (ripple estimate) of the time varying system. These ripple estimates can not be obtained using the approaches to averaging suggested by [4–9]. Ripple estimates have found numerous applications in important physical problems, most recently, in dc–dc pulse width modulated voltage converters [10].

Section 2 of the paper presents preliminary lemmas and definitions. Section 3 gives the main theoretical results. Section 4 presents examples. The journal version of this paper will present averaging algorithms similar to those used in ODE's [11] and will also discuss transient behavior.

Notation. Let $C = C([-r, 0], \Re^n)$, where $r \geq 0$ is a given positive constant. Let

$$x_t = x_t(\theta) = x(t + \theta), \ -r \leq \theta \leq 0 \ .$$

We will use the notation that $|\psi|$ represents the norm of vector ψ in $\Re^n$, and $\| \psi \|_r$ is a corresponding norm on C given by $\| \psi \|_r = \sup_{-r\leq\sigma<0} |\psi(\sigma)|$. Similarly, we will sometimes use $\| \psi \|_{2r} = \sup_{-2r\leq\sigma<0} |\psi(\sigma)|$. ε_0 will always be used to denote a positive constant. All derivatives, denoted by " $'$ " or $\frac{d}{dt}$, are assumed to be right hand derivatives. Let B_h denote the closed neighborhood $B_h = \{x : |x| \leq h, h > 0\}$ and let

$$\psi \in Q_H = \{\psi \in C[-r, 0] : \| \psi \|_r < H, \ H > 0\} \ .$$

Let W be the Banach space of absolutely continuous functions on [–r,0] with absolute integrable derivative, with norm $\| \psi \|_W = [\ |\psi(0)| + \int_{-r}^{0} |\psi'(\theta)| d\theta\ ]$. Let $\bar{Q}_H \subset W$ be a ball $\psi \in \bar{Q}_H = \{\psi : \| \psi \|_W < H\}$. Functions $w_i \in \mathfrak{K}$ ($w_i \in \mathcal{K}$) are nondecreasing (increasing) functions with $w_i(0) = 0$ and $w_i(s) > 0$ for $s > 0$.

2. PRELIMINARY RESULTS

Before proceeding with the main results, it is necessary to introduce several definitions and lemmas. In what follows, we will primarily develop averaging theory for (1.5). However, the same techniques can be applied to (1.3) to obtain similar averaging results. Because of the strong similarity, the proofs for (1.3) will not be given.

Definition 2.1. The function $f(s, x_1, x_2, 0)$, where f is as defined in (1.5), is said to have mean value $f_0(x_1, x_2)$ if there exists a continuous function $\gamma(T) : [0, \infty) \to [0, \infty)$, monotonically decreasing, such that $\gamma(T) \to 0$ as $T \to \infty$ and

$$\left|\frac{1}{T}\int_t^{t+T} f(s, x_1, x_2, 0)ds - f_0(x_1, x_2)\right| \leq \gamma(T)$$

for all (t, x_1, x_2) in compact subsets of $\Re \ x \ \Re^n \ x \ \Re^n$ and for all $T \geq 0$.

Using Definition 2.1, it is possible to define a function $d : \Re \ x \ \Re^n \ x \ \Re^n \to \Re^n$, with zero mean value as

$$d(s, x_1, x_2) = f(s, x_1, x_2, 0) - f_0(x_1, x_2) \ . \tag{2.1}$$

Furthermore, if d is periodic, then it is known that $\gamma(T) = a/T$, where a is some non–negative constant.

The following Lemma is an adaptation from Lemmas 4.2.1 and 4.2.2 of Sastry and Bodson [12], which, in turn, were adaptations from the classical results of Bogoliubov and Mitropolskii [2, page 450] and Hale [3, Appendix].

LEMMA 2.1. *Suppose*

$$d(s, x_1, x_2) : \Re \ x \ \Omega \ x \ \Omega \to \Re^n$$

is a continuous function on $\Re$, *has continuous partial derivatives with respect to* x_i *and* $d(s, 0, 0) = 0$.

Suppose, further, that $d(s, x_1, x_2)$ *has zero mean value, with a convergence function* $\gamma(T)(|x_1| + |x_2|)$ *and the mean values of* $\frac{\partial d(s, x_1, x_2)}{\partial x_i}, i = 1, 2$, *are both equal to zero with convergence function* $\gamma(T)$.

Then, there exists $\xi(\varepsilon) \in \mathcal{K}$ *and a function* $u(s, x_1, x_2, \varepsilon) : \Re \ x \ \Omega \ x \ \Omega \ x \ (0, \varepsilon_0] \to \Re^n$ *such that*

(i) $|\varepsilon u(t/\varepsilon, x_1, x_2, \varepsilon)| \leq \xi(\varepsilon)[|x_1| + |x_2|]$

(ii) $$\left|\frac{\varepsilon \partial u(t/\varepsilon, x_1, x_2, \varepsilon)}{\partial t} - d(t/\varepsilon, x_1, x_2)\right| \leq \xi(\varepsilon)[|x_1| + |x_2|]$$

(iii) $$\left|\frac{\varepsilon \partial u(t/\varepsilon, x_1, x_2, \varepsilon)}{\partial x_i}\right| \leq \xi(\varepsilon) \quad , \quad i = 1, 2 \ .$$

Remark 2.1. The function u in Lemma 2.1 can be chosen as

$$u(\lambda, x_1, x_2, \varepsilon) = \int_{-\infty}^{\lambda} [e^{\varepsilon(\lambda - s)} f(s, x_1, x_2, 0) \ - f_0(x_1, x_2)]ds,$$

or in the special case when f is periodic, u can also be chosen as

$$u(\lambda, x_1, x_2) = \int [f(\lambda, x_1, x_2, 0) - f_0(x_1, x_2)]d\lambda \ - g(x_1, x_2) \tag{2.3}$$

where $g(x_1, x_2)$ is selected so that u has zero average. Note that, in (2.3), u is independent of ε. Additionally, for the periodic case with u given as in (2.3), the bound in (ii) of Lemma 2.1 is given as zero, i.e., $\left|\frac{\varepsilon \partial u(t/\varepsilon, x_1, x_2)}{\partial t} - d(t/\varepsilon, x_1, x_2)\right| = 0$.

In most applications (2.3) is used for u. In general, however, u in (2.3) is not guaranteed to be bounded, even if f is almost periodic. Therefore, for the proofs of the above Lemma, it is necessary to define u as in (2.2).

In order for us to proceed further, it is necessary to introduce the following assumptions on f in (1.5) (or equivalently (1.3)).

ASSUMPTIONS. *For some*

$$\Omega \subset \Re^n, 0 \in \Omega, \text{ and } \varepsilon_0 > 0$$

(A1) $f(\cdot,0,0,0) = 0$ *and f is Lipschitz with respect to its second and third arguments, i.e., for all* (s,x_1,x_2,ε) *in compact sets of* $\Re \times \Omega \times \Omega \times (0,\varepsilon_0]$, *there exist constants* $k_1 \geq 0$ *and* $k_2 \geq 0$ *such that*

$$|f(s,x_1,x_2,\varepsilon) - f(s,\bar{x}_1,\bar{x}_2,\varepsilon)| \leq k_1|x_1 - \bar{x}_1| + k_2|x_2 - \bar{x}_2| .$$

(A2) *f is Lipschitz in* ε, *uniformly with respect to its first argument and linearly with respect to its second and third arguments, i.e., for all* (s,x_1,x_2,ε) *in compact sets of* $\Re \times \Omega \times \Omega \times (0,\varepsilon_0]$, *there exist constants*

$$k_3 \geq 0 \text{ and } k_4 \geq 0$$

such that

$$|f(s,x_1,x_2,\varepsilon_1) - f(s,x_1,x_2,\varepsilon_2)| \leq [k_3|x_1| + k_4|x_2|] \cdot |\varepsilon_1 - \varepsilon_2| .$$

(A3) *The function* $d(s,x_1,x_2) = f(s,x_1,x_2,0) - f_0(x_1,x_2)$ *satisfies the conditions of Lemma 2.2.*

LEMMA 2.2. *Suppose f satisfies assumptions (A1)–(A3). Then, there exists a function u, as defined in Lemma 2.1, and* $\varepsilon_0 > 0$ *such that the transformation*

$$x(t) = z(t) + \varepsilon u(t/\varepsilon, z(t), z(t-r), \varepsilon) \tag{2.4}$$

is a homeomorphism in B_h *for all* $0 < \varepsilon \leq \varepsilon_0$.

Consider the two delay differential equations

$$y'(t) = g(t,y(t),y(t-r_1), \ldots ,y(t-r_m)) \tag{2.5}$$

$$z'(t) = g(t,z(t),z(t-r_1), \ldots,z(t-r_m)) + h(t,z(t),z(t-d_1), \ldots,z(t-d_n),z'(t-d_0)) \tag{2.6}$$

where

$$g : \Re \times \Omega \times \Omega \ldots \times \Omega \rightarrow \Re^n$$

$$\text{and } h : \Re \times \Omega \times \Omega \ldots \times \Omega \rightarrow \Re^n$$

are both continuous on $\Re$ and satisfy sufficient conditions such that the solutions to (2.5) and (2.6) exist. Constants r_j ($j = 1, \ldots, m$) and d_i ($i = 0, 1, \ldots, n$) are in the interval $[0,r]$, where $r \geq 0$. We will further assume that $g(t,0, \ldots ,0) = 0$ and $h(t,0, \ldots ,0) = 0$ for all t.

LEMMA 2.3. *Suppose that the trivial solution to (2.5) is exponentially stable (exponentially unstable). Suppose, further, that there exist non–negative constants K and N such that for* $(t,\chi_t) \in \Re \times \bar{Q}_H$

$$|h(t,\chi(t),\chi(t-d_1), \ldots \chi(t-d_n),\chi'(t-d_0))| \leq K(\|\chi_t\|_r + |\chi(t-d_0)|)$$

and

$$|g(t,\chi(t),\chi(t-r_1), \ldots ,\chi(t-r_m))| \leq N\|\chi_t\|_r.$$

Then there exists a positive constant, α, *sufficiently small, such that if* $K < \alpha$, *the trivial solution of (2.6) is uniformly asymptotically stable (unstable).*

3. AVERAGING OF DIFFERENTIAL DELAY EQUATIONS

We are now in a position to present the main results. We remark, once again, that the main contributions of the theorems are actually in the technique in which they are proved. These techniques permit the development of an averaging algorithm similar to those which have been developed for ODE's. This is further explained in Section 4 and Section 5.

This section will show that it is possible to introduce the change of variables $x(t) = z(t) + \varepsilon u(t/\varepsilon, z(t), z(t-r), \varepsilon)$ into (1.5) in order to transform (1.5) into (1.6) plus small perturbations (if function f satisfies specified properties). Then the lemmas of the previous section will be applied to relate the stability properties of (1.6) to (1.5).

Often [1–7], it is assumed that the function to be averaged, f, is almost periodic in t. Then the existence of the mean value is guaranteed, and Fredholm alternatives can be applied to prove existence of almost periodic orbits. In this paper, instead of assuming almost periodicity, it is assumed that the mean value of f exists and satisfies certain properties. This approach to averaging is a popular method in engineering applications (see Sastry and Bodson, Chapter 4 [12]), from which we have been greatly influenced, due to its simplicity. For results on the existence of almost periodic orbits, the reader is referred to Hale and Verduyn Lunel [6,7].

THEOREM 3.1. *Assume that f in (1.5) satisfies assumptions (A1) – (A3). If*

$$\det[sI - \frac{\partial f_0(0,0)}{\partial y(t)} - \frac{\partial f_0(0,0)}{\partial y(t-r)} e^{-rs}] = 0$$

has all solutions with real parts less than zero (where f_0 *is defined in (1.6)), then there exists an* $\varepsilon_0 > 0$ *such that, for* $0 < \varepsilon \leq \varepsilon_0$, *the trivial solution of (1.5) is uniformly asymptotically stable.*

Likewise, if

$$\det[sI - \frac{\partial f_0(0,0)}{\partial y(t)} - \frac{\partial f_0(0,0)}{\partial y(t-r)} e^{-rs} = 0$$

has at least one solution with real part greater than zero, then there exists an $\varepsilon_0 > 0$ *such that, for* $0 < \varepsilon \leq \varepsilon_0$, *the trivial solution of (1.5) is unstable.*

Proof of Theorem 3.1. Let $d(s,x_1,x_2)$ in Lemma 2.1 be given as

$$d(s,x_1,x_2) = f(s,x_1,x_2,0) - f_0(x_1,x_2)$$

and let $u(s,x_1,x_2,\varepsilon)$ be as given in Lemma 2.1 with this d. Suppose $x(t) = x(t;t_0,\phi)$ denotes the solution to (1.5) with $x(t) = \phi(t)$ on $t \in (-\infty, t_0]$, and ϕ is a continuous function in a sufficiently small neighborhood of the origin, to be defined later. (If ϕ is only continuous on the interval of $t \in [t_0 - r, t_0]$, then, it can be assumed that $\phi(t) = 0$ for $t < t_0 - r$). Introduce substitution $x(t) = z(t) + \varepsilon u(t/\varepsilon, z(t), z(t-r), \varepsilon)$ into (1.5). By Lemma 2.2, this change of variables is a homeomorphism in a neighborhood B_h of the origin for sufficiently small ε, $0 < \varepsilon \leq \varepsilon_1$. Then, (1.5) becomes

$$\frac{d}{dt}[z(t) + \varepsilon u(t/\varepsilon, z(t), z(t-r), \varepsilon)] = f(t/\varepsilon, z(t) + \varepsilon u(t/\varepsilon, z(t),$$

$$z(t-r), \varepsilon),\ z(t-r) + \varepsilon u(\tfrac{t-r}{\varepsilon},\ z(t-r), z(t-2r), \varepsilon), \varepsilon),$$

which yields

$$\begin{aligned}
&[I + \varepsilon \frac{\partial u(t/\varepsilon, z(t), z(t-r), \varepsilon)}{\partial z(t)}]z'(t) + \\
&\varepsilon[\frac{\partial u(t/\varepsilon, z(t), z(t-r), \varepsilon)}{\partial z(t-r)}]z'(t-r) \\
&= f_0(z(t), z(t-r)) + [f(t/\varepsilon, z(t) + \\
&\varepsilon u(t/\varepsilon, z(t), z(t-r), \varepsilon), z(t-r) + \\
&\varepsilon u(\tfrac{t-r}{\varepsilon}, z(t-r), z(t-2r), \varepsilon), \varepsilon) \\
&- f(t/\varepsilon, z(t), z(t-r), 0)] + \\
&\left[d(t/\varepsilon, z(t), z(t-r)) - \varepsilon \frac{\partial u(t/\varepsilon, z(t), z(t-r), \varepsilon)}{\partial t}\right] \\
&\equiv f_0(z(t), z(t-r)) + F_1(t/\varepsilon, z(t),\ z(t-r), z(t-2r), \varepsilon)\,. \qquad (3.1a)
\end{aligned}$$

For sufficiently small $\varepsilon, 0 \le \varepsilon \le \varepsilon_2$, the inverse of $[I + \varepsilon \frac{\partial u(t/\varepsilon, z(t), z(t-r), \varepsilon)}{\partial z(t)}]$ exists for all t (and is close to the identity). Therefore, we have

$$\begin{aligned}
&z'(t) + \varepsilon\left[I + \varepsilon \frac{\partial u(t/\varepsilon, z(t), z(t-r), \varepsilon)}{\partial z(t)}\right]^{-1} \\
&\cdot \left[\frac{\partial u(t/\varepsilon, z(t), z(t-r), \varepsilon)}{\partial z(t-r)}\right]z'(t-r) \\
&= \left[I + \varepsilon \frac{\partial u(t/\varepsilon, z(t), z(t-r), \varepsilon)}{\partial z(t)}\right]^{-1} \\
&[f_0(z(t), z(t-r)) + F_1(t/\varepsilon, z(t), z(t-r),\ z(t-2r), \varepsilon)] \\
&= f_0(z(t), z(t-r)) + \left[I + \varepsilon \frac{\partial u(t/\varepsilon, z(t), z(t-r), \varepsilon)}{\partial z(t)}\right]^{-1} \\
&\cdot [F_1(t/\varepsilon, z(t), z(t-r), z(t-2r), \varepsilon) \\
&- \varepsilon \frac{\partial u(t/\varepsilon, z(t), z(t-r), \varepsilon)}{\partial z(t)} f_0(z(t), z(t-r))]\,. \qquad (3.1b)
\end{aligned}$$

We now will obtain a bound on terms in both the right and left–hand side of (3.1b), which will then permit the application of Lemma 2.3.

From Lemma 2.1, we deduce for $z \in \Omega \in B_h$ and $0 < \varepsilon \le \varepsilon_3$ that

$$|\varepsilon \frac{\partial u(t/\varepsilon, z(t), z(t-r), \varepsilon)}{\partial z(t)}| \le \xi_1(\varepsilon),$$

where $\xi_1(\varepsilon) \in \mathcal{K}$. Furthermore, for $0 < \varepsilon \le \min[\varepsilon_2, \varepsilon_3]$ and for $z \in \Omega \subset B_h$,

(i) $$|[I + \varepsilon \frac{\partial u(t/\varepsilon, z(t), z(t-r), \varepsilon)}{\partial z(t)}]^{-1}| \le \frac{1}{1 - \xi_1(\varepsilon)}\,.$$

(ii) $$|d(t/\varepsilon, z(t), z(t-r)) - \varepsilon \frac{\partial u(t/\varepsilon, z(t), z(t-r), \varepsilon)}{\partial t}| \le \xi_1(\varepsilon)[|z(t)| + |z(t-r)|]\,.$$

Similarly, it is easy to show [9] that Assumption (A1) implies that, for $z \in \Omega$, $|f_0(z(t), z(t-r))| \le k_1|z(t)| + k_2|z(t-r)|$,

and

$$\begin{aligned}
&|f(t/\varepsilon,\ z(t) + \varepsilon u(t/\varepsilon, z(t), z(t-r), \varepsilon), \\
&z(t-r) + \varepsilon u(\tfrac{t-r}{\varepsilon}, z(t-r), z(t-2r), \varepsilon),\ \varepsilon) \\
&- f(t/\varepsilon, z(t), z(t-r), 0)| \\
&+ |f(t/\varepsilon,\ z(t) + \varepsilon u(t/\varepsilon, z(t), z(t-r), \varepsilon), \\
&z(t-r) + \varepsilon u(\tfrac{t-r}{\varepsilon}, z(t-r), z(t-2r), \varepsilon),\ 0) \\
&- f(t/\varepsilon, z(t), z(t-r), 0)|\,.
\end{aligned}$$

Using assumption (A2) and (i) of Lemma 2.2, for $z \in \Omega$, and for $0 < \varepsilon \le \varepsilon_3$,

$$\begin{aligned}
&|f(t/\varepsilon,\ z(t) + \varepsilon u(t/\varepsilon, z(t), z(t-r), \varepsilon), \\
&z(t-r) + \varepsilon u(\tfrac{t-r}{\varepsilon}, z(t-r),\quad z(t-2r), \varepsilon),\ \varepsilon) \\
&- f(t/\varepsilon,\ z(t) + \varepsilon u(t/\varepsilon, z(t), z(t-r), \varepsilon), \\
&z(t-r) + \varepsilon u(\tfrac{t-r}{\varepsilon}, z(t-r), z(t-2r), \varepsilon),\ 0) \\
&\le \xi_2(\varepsilon)\, \| z(t) \|_{2r}
\end{aligned}$$

where $\xi_2(\varepsilon) \in \mathcal{K}$. Likewise,

$$\begin{aligned}
&|f(t/\varepsilon,\ z(t) + \varepsilon u(t/\varepsilon, z(t), z(t-r), \varepsilon), \\
&z(t-r) + \varepsilon u(\tfrac{t-r}{\varepsilon}, z(t-r), z(t-2r), \varepsilon),\ 0) \\
&- f(t/\varepsilon, z(t), z(t-r), 0)| \quad \le \xi_3(\varepsilon)\, \| z(t) \|_{2r}
\end{aligned}$$

where $\xi_3(\varepsilon) \in \mathcal{K}$.

Hence for $z \in \Omega$ and $0 < \varepsilon \le \min[\varepsilon_1, \varepsilon_2, \varepsilon_3]$,

$$\begin{aligned}
&|[I + \varepsilon \frac{\partial u(t/\varepsilon, z(t), z(t-r), \varepsilon)}{\partial z(t)}]^{-1} \\
&\cdot [F_1(t/\varepsilon, z(t), z(t-r), z(t-2r), \varepsilon)| \\
&- \varepsilon \frac{\partial u(t/\varepsilon, z(t), z(t-r), \varepsilon)}{\partial z(t)} f_0(z(t), z(t-r))]|
\end{aligned}$$

$$\leq \frac{1}{1-\xi_1(\varepsilon)}[\xi_2(\varepsilon)\parallel z(t)\parallel_{2r} +$$

$$\xi_3(\varepsilon)\parallel z(t)\parallel_{2r} + \xi_1(\varepsilon)(|z(t)| + |z(t-r)|)$$

$$+ \xi_1(\varepsilon)(k_1|z(t)| + k_2|z(t-r)|)] \quad \leq \xi_4(\varepsilon)\parallel z(t)\parallel_{2r} \quad (3.2)$$

where $\xi_4(\varepsilon) \in \mathcal{K}$.

Likewise, we can obtain a bound on the term multiplying the derivative of the delayed state in the right had side of (3.1). Define

$$\Gamma_1(t/\varepsilon, z(t), z(t-r), \varepsilon)$$

$$\equiv \varepsilon\left[I + \varepsilon\frac{\partial u(t/\varepsilon, z(t), z(t-r), \varepsilon)}{\partial z(t)}\right]^{-1}$$

$$\left[\left[\frac{\partial u(t/\varepsilon, z(t), z(t-r), \varepsilon)}{\partial z(t-r)}\right]\right].$$

Then for $0 < \varepsilon \leq \varepsilon_3$ and for $z \in \Omega$, Γ_1 is well defined and

$$|\Gamma_1(t/\varepsilon, z(t), z(t-r), \varepsilon)| \leq \frac{1}{1-\xi_1(\varepsilon)}\xi_1(\varepsilon) = \xi_5(\varepsilon), \quad (3.3)$$

Now, suppose that

$$\det[sI - \frac{\partial f_0(0,0)}{\partial y(t)} - \frac{\partial f_0(0,0)}{\partial y(t-r)}e^{-rs}] = 0$$

has all solutions with real parts less than zero (at least one solution with positive real part). Then $y=0$ of (1.6) is exponentially stable (exponentially unstable). Hence, by Lemma 2.3, (3.1), (3.2) and (3.3) there exists a sufficiently small $\varepsilon_4, 0 < \varepsilon \leq \varepsilon_4 \leq \min[\varepsilon_1, \varepsilon_2, \varepsilon_3]$ such that (3.3) has a uniformly asymptotically stable (unstable) trivial solution. **Q.E.D.**

As previously mentioned, the above averaging techniques can be applied to (1.3). An interesting consequence of these techniques is that a new averaged model, (1.8), is obtained instead of the classical model, (1.4), introduced by [4,5]. Classical averaging techniques for (1.3) suggest that it is not necessary to consider the effects of the delay in the averaged model. However, the following theorem suggests that it is more accurate to include the delay in the averaged model:

THEOREM 3.2. *Assume that f in (1.3) satisfies assumptions (A1)–(A-3). If*

$$\det[sI - \varepsilon\frac{\partial f_0(0,0)}{\partial y(\tau)} - \varepsilon\frac{\partial f_0(0,0)}{\partial y(\tau-r)}e^{-rs}] = 0$$

has all solutions with real parts less than zero (where f_0 and is defined in (1.8)), then there exists an $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, the trivial solution of (1.3) is uniformly asymptotically stable.

Likewise, if

$$\det[sI - \varepsilon\frac{\partial f_0(0,0)}{\partial y(\tau)} - \varepsilon\frac{\partial f_0(0,0)}{\partial y(\tau-r)}e^{-rs}] = 0$$

has at least one solution with real part greater than zero, then there exists an $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, the trivial solution of (1.3) is unstable.

The proof of Theorem 3.2 is almost identical to the proof of Theorem 3.1. In this case, the near identity change of variables is given as

$$x(\tau) = z(\tau) + \varepsilon u(\tau, z(\tau), z(\tau-r), \varepsilon)$$

Remark 3.1. Theorem 3.2 suggests that a more accurate averaged model of (1.3) is given by (1.8) instead of the classical model (1.4) described in [4,5]. However, there is a strong relationship between (1.8) and (1.4). Note that the transcendental characteristic equation given in Theorem 3.2 can be rewritten as $\det[\zeta I - \frac{\partial f_0(0,0)}{\partial y(\tau)} - \frac{\partial f_0(0,0)}{\partial y(\tau-r)}e^{-\varepsilon r\zeta}] = 0$, where $\zeta \equiv \frac{s}{\varepsilon}$.

As ε becomes smaller, $e^{-\varepsilon r\zeta}$ approaches the number 1, and the transcendental characteristic equation given above approaches the non–transcendental characteristic equation given by $\det[\zeta I - \frac{\partial f_0(0,0)}{\partial y(\tau)} - \frac{\partial f_0(0,0)}{\partial y(\tau-r)}] = 0$, which is obtained when linearizing (1.4) about its zero equilibrium. This can be formalized by the following corollary, which reduces to the results of [5,6].

COROLLARY 3.1. *Assume that f in (1.3) satisfies assumptions (A1)–(A3). If $\det[sI - \frac{\partial f_0(0,0)}{\partial y(\tau)} - \frac{\partial f_0(0,0)}{\partial y(\tau-r)}] = 0$ has all solutions with real parts less than zero (where f_0 and is defined in (1.8)), then there exists an $\bar{\varepsilon}_0 > 0$ such that, for $0 < \varepsilon \leq \bar{\varepsilon}_0$, the trivial solution of (1.3) is uniformly asymptotically stable.*

Likewise, if $\det[sI - \frac{\partial f_0(0,0)}{\partial y(\tau)} - \frac{\partial f_0(0,0)}{\partial y(\tau-r)}] = 0$ has at least one solution with real part greater than zero, then there exists an $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \leq \bar{\varepsilon}_0$, the trivial solution of (1.3) is unstable.

4. EXAMPLES

EXAMPLE 4.1. Consider the non–autonomous scalar differential delay equation given by

$$x'(t) = \varepsilon[-4\cos^2(t)x(t-r) + x(t)] \equiv \varepsilon f(t, x(t), x(t-r)), \quad (4.1)$$

which, according to the classical results of [,5], has an average corresponding to (1.4) of

$$y'(t) = \varepsilon[-2y(t) + y(t)] = -\varepsilon y(t). \quad (4.2)$$

For sufficiently small $\varepsilon, 0 < \varepsilon \leq \bar{\varepsilon}_0$, the stability properties of (4.1) are the same as those of (4.2). Therefore, for $0 < \varepsilon \leq \bar{\varepsilon}_0$, the averaging theory of [13,14] predicts that the trivial solution to (4.1) is always uniformly asymptotically stable. Note that all influence of the delay has been neglected in analysis. System (4.2) is an ODE that always has asymptotically stable trivial solution, and hence, for sufficiently small ε, the trivial solution to (4.1) will be asymptotically stable. Furthermore, if $x(0) = y(0)$, then the solutions to (4.1) and (4.2) should remain close to each other provided $0 < \varepsilon \leq \bar{\varepsilon}_0$.

Using Theorem 3.2, however, the newly developed averaged model is given by

$$z'(t) = \varepsilon[-2z(t-r) + z(t)] \equiv \varepsilon f_0(z(t), z(t-r)). \quad (4.3)$$

For sufficiently small $\varepsilon, 0 < \varepsilon \leq \varepsilon_0$, one might expect that (4.3) more accurately estimates (4.1) than (4.2), due to the fact that information on the delay is maintained in the averaged equation. This, in fact, turns out to be true.

A great deal of information has been lost by averaging out the delay in (4.2). For example, stability of (4.3) is governed by the location of the roots of the transcendental characteristic equation

$$s = \varepsilon[-2e^{-rs} + 1],$$

or equivalently $z = [-2e^{-\varepsilon rz} + 1]$. It is a simple exercise (see [13]) to show that all roots of this equations all have negative real parts if and only if $\varepsilon r < \frac{\pi}{3\sqrt{3}}$. Additionally at least one of the roots will have positive real parts if $\varepsilon r > \frac{\pi}{3\sqrt{3}}$.

For the purpose of illustration, let $r = 5$ in (4.1). Then the transcendental characteristic equation predicts that, for $0 < \varepsilon \leq \varepsilon_0$, the trivial solution of the original system (4.1) is asymptotically stable if $\varepsilon < 0.121$ and is unstable when $\varepsilon > 0.121$. This prediction of instability for ranges of ε cannot be determined by analyzing (4.2), as suggested by [4,5].

Computer simulations show that the trivial solution of (4.1) is asymptotically stable for $0 < \varepsilon < 0.126$ and is unstable for $\varepsilon > 0.126$. These values closely correspond with the values predicted using the average (4.3). In summary, the classical averaging techniques [4,5] provide less information on the true behavior of the original system the newly proposed averaged model given by (1.8).

EXAMPLE 4.2. Consider the non–autonomous scalar differential delay equation given by

$$x'(t) = [-4cos^2(t/\varepsilon)x(t-r) + x(t)]. \qquad (4.4)$$

By Theorem 3.1 and 3.2, the average of (4.1) is given by

$$y'(t) = [-2y(t-r) + y(t)]. \qquad (4.5)$$

For sufficiently small $\varepsilon, 0 < \varepsilon \leq \varepsilon_0$, Theorem 3.2 guarantee that the asymptotic stability properties of (4.4) are the same as those of (4.5).

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