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APPROXIMATE CONTROLLABILITY OF SPECIAL CLASSES OF BOUNDED ABSTRACT NONAUTONOMOUS EVOLUTION EQUATIONS

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Abstract -- This paper shows that by examining the span of a priori known time invariant operators, it is possible to determine whether certain bounded abstract time-varying evolution equations are approximately controllable.

I. INTRODUCTION

This paper studies the problem of approximate controllability of the abstract nonautonomous evolution equation

$$\dot{x}(t) = A(t)x(t) + Bu(t) \quad , \ 0 \le t \le T \quad ,$$

$$A(t) = \sum_{i=1}^{m} a_i(t)A_i \,, \qquad \text{m finite} \tag{1}$$

with $x(t) \in X$, $x(0) = x_0 \in X$, $u(t) \in U$, where X and U are Banach spaces. Let the Banach space of bounded linear operators from Banach space U to Banach space X be denoted by $\mathcal{L}(U,X)$. In particular, let $\mathcal{L}(X) \equiv \mathcal{L}(X, X)$. Then, in (1), $A_i \in \mathcal{L}(X)$, $i = 1,2, \ldots ,m$, are time independent operators, and $a_i(t)$ are continuous scalar functions, $a_i \in C(0,T;\Re)$, $i = 1,2,\ldots,m$. If m is chosen as small as possible, then the A_i 's will be linearly independent and will generate an associative algebra over the field of complex numbers and a Lie algebra under commuter product $[A_i , A_j] = A_iA_j - A_jA_i = ad(A_i)A_j$. Assume further that $B \in \mathcal{L}(U,X)$ and that the control $u(t) \in L_p(0,T;U)$, where $L_p(0,T;U)$ denotes the Banach space of U-valued functions with norm $\left\{\int_0^T \| u(t) \|^p dt\right\}^{1/p}$, $1 \le p$. Let X^* denote the dual of X with element x^* , and let the range and null space of an operator $(\cdot)$ be denoted by $\Re(\cdot)$ and $\mathcal{N}(\cdot)$ respectively. The range of an operator B on U will be indicated as BU . Finally, if E_n is a sequence of subspaces, $n = 0 , 1 \ldots$, then $Sp\{E_n , n \ge 0\}$ indicates the span of these subspaces, and $\overline{Sp}\,\{E_n , n \ge 0\}$ indicates the closure of this span.

Under the conditions assumed above, it is known that there exists a unique solution to (1) given by

$$x_u(t) = x(t;x_0,u) = \Phi(t)x_0 + \Phi(t)\int_0^t \Phi^{-1}(s)Bu(s)ds, \tag{2}$$

where $\Phi(t) = \Phi(t,0) \in \mathcal{L}(X)$ is the solution to the homogenius equation

$$\Phi'(t) = A(t)\,\Phi(t) \,, \quad \Phi(0) = I \,, \tag{3}$$

and I denotes the identity in $\mathcal{L}(X)$.

Definition: Nonautonomous system (1) is:

(i) *approximately controllable on* $[0,T]$ if for any initial state $x_0 \in X$ at $t = 0$ and any final state $x_1 \in X$ there exists a control $u \in Lp(0,T;U)$ such that for any $\varepsilon > 0$, $\| x(T;x_0 , u) - x_1 \| < \varepsilon$.

It is the purpose of this paper to derive approximate controllability conditions for non-autonomous evolution equation (1). The results of this paper are extensions of the works of [1,2]. However, unlike the results in [2], and most of the known results (see [3] for a summary), controllability conditions presented in this paper do not depend on properties of the unknown linear operator $\Phi(t)$ described in (3). Instead, we derive tests in terms of properties of A_i , $a_i(t)$ and B , which are either a prior known or can constructively be determined. This technique was orginally developed by Leiva in [4,5], and has also been applied to finite dimensional systems [5,6], i.e., when $X = \Re^n$.

II. MAIN RESULTS

A. Representation of $\Phi(t)$

In Section A, we discuss the explicit representation of $\Phi(t)$ in (2) on time interval $[0,T]$, $T > 0$. In [7], it is shown that

$$\Phi(t) = \prod_{i=1}^{m} \exp[g_i(t)A_i] = e^{g_1(t)A_1} e^{g_2(t)A_2} \cdots e^{g_m(t)A_m}, 0 \le t \le T, \tag{4}$$

where the scalar functions $g_i(t)$ satisfy the set of differential equations

$$\sum_{i=1}^{m} a_i(t)\, A_i = \dot{g}_1(t)\, A_1 + \sum_{i=2}^{m} \dot{g}_i(t)\left[\prod_{j=1}^{i-1} \exp(g_j(t) ad\, A_j)\right]A_i \,,$$

$$g_i(0) = 0 \quad , \quad i = 1,2, \ldots , m \,. \tag{5}$$

The consequence of equations (4) - (5) is that operator $\Phi(t)$ is represented in terms of operators A_i , functions $a_i(t)$, and functions $g_i(t)$. Based on this representation, this paper will derive new "constructive" controllability tests, most of which are in terms of known quantities.

B. Approximate Controllability

From (2), define the continuous linear operator $G_T : L_p(0,T;U) \to X$, where

$$G_T u = \int_0^T \Phi^{-1}(s)\, Bu(s)ds \,. \tag{6}$$

The following proposition is an immediate consequence of (6) and can be found in many basic works [3].

FUNDAMENTAL PROPOSITION: System (1) is approximately controllable on [0, T] if and only if $\overline{\Re}(G_T) = X$.

We now present the results of this work:

THEOREM 1: If system (1) is approximately controllable on [0,T], then

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$$\overline{Sp}\left\{A_m^{k_m} A_{m-1}^{k_{m-1}} \cdots A_1^{k_1} BU / k_i = 0,1,2, \ldots ; i = 1,2, \ldots, m\right\} = X . \tag{7}$$

PROOF: Suppose system (1) is approximately controllable and (7) is false. Then, there exists an $x^* \in X^*$, $x^* \neq 0$, such that

$$< x^*, A_m^{k_m} \cdots A_1^{k_1} Bu >_{x^*, X} = 0, (u \in U; i = 1,2, \ldots m; k_i = 0,1,2, \ldots) . \tag{8}$$

Using (4), (6) can be rewritten as

$$\begin{aligned} G_T u &= \int_0^T \prod_{i=m}^{1} \exp\{-g_i(s)A_i\} Bu(s)ds \\ &= \int_0^T \prod_{i=m}^{1} \left[\sum_{k_i=0}^{\infty} \frac{(-g_i(s))^{k_i} A_i^{k_i}}{k_i!}\right] Bu(s)ds \\ &= \sum_{n=0}^{\infty} \sum_{k_1+k_2+\ldots k_m = m} A_m^{k_m} \cdots A_1^{k_1} B \int_0^T \prod_{i=1}^{m} \frac{(-g_i(s))^{k_i}}{k_i!} u(s)ds, \end{aligned} \tag{9}$$

which by (8) gives

$$< x^*, G_T u >_{X^*, X} = 0, \ (u \in L_p(0,T;U)) . \tag{10}$$

This implies that $\overline{\mathcal{R}}(G_T) \subsetneq X$, contradicting the assumption that system (1) is approximately controllable. **Q. E. D.**

Likewise, the following theorems are true:

THEOREM 2: System (1) is approximately controllable on [0,T] if and only if

$$B^* \prod_{i=1}^{m} \exp\left\{-g_i(t)A_i^*\right\} x^* = 0, \ \forall t \in [0,T]$$

implies $x^* = 0$.

The proof of Theorem 2 follows immediately from the proof of Theorem 3.11 of [8]. However, Theorem 2 is difficult to verify since it relies of knowledge of functions $g_i(t)$ which must be found via (5). In order to provide more easily verifiable conditions, further restrictions on $a_i(t)$ must be assumed.

THEOREM 3: If coefficients $a_i(\cdot)$ are analytic at t=0 and $a_i(0)$ are not all identically equal to zero, then condition (7) in Theorem 1 is a necessary and sufficient condition for system (1) to be approximately controllable on [0,T].

Theorem 3 provides some of the simplest conditions for approximate controllability that exist for time varying nonautonomous evolution equations. The proof of Theorem 3 is more complicated. We present only a summary:

Assume that (1) is not approximately controllable and that (7) is true. Then, by Theorem 2, there will exist an x^* such that

$$< x^*, \prod_{i=m}^{1} \exp\{-g_i(t)A_i\} BU >_{X^* X} = 0 \tag{11}$$

for all $t \in [0,T]$ and $u \in U$. Taking an infinite number of derivatives of (11), and using properties of derivatives of the products and the compositions of functions, it is possible to show that under these assumptions

$$\overline{Sp}\left\{A_m^{k_m} A_{m-1}^{k_{m-1}} \cdots A_1^{k_1} BU / k_i = 0,1,2, \ldots ; i = 1,2, \ldots, m\right\} = 0$$

and hence, a contradiction is obtained. This proves sufficiency. Theorem 1 proved necessity.

The simplicity and power of Theorem 3 is that approximate controllability of (1) can be verified by examining the span of time invariant operators which are a priori known. This is best shown through an example:

Example: Consider the following integro-differential equation of Volterra type:

$$\frac{\partial w(t,\xi)}{\partial t} = a(t) \int_0^t w(t,s)ds + v(\xi)u(t) . \tag{12}$$

where $w(t,\xi)$ is a scalar function in two scalar variables t and ξ, with $0 \le \xi \le 1$, and where $v(\xi)$ is a given scalar function. This example is discussed in [1] when $a(t) = 1$. If $v(\xi)$ is continuous, then (12) can be rewritten in abstract form $\dot{x} = a(t)Ax + bu$ by taking $x(t) = w(t,\xi)$, $b = v(\cdot)$, and A to be the Volterra integral operator. Let $X = C[0,1]$.

For $a(t) = 1$, [1] shows that condition (7) is true. Therefore, when $a(t)$ is analytic and $a(0) \neq 0$, (12) will always be approximately controllable.

As we see in this example, testing for approximate controllability of (1) can often become trivial. This is similar to the classic results found in [1], although the techniques are quite different.

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